# The Temperley-Lieb tower and the Weyl algebra

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# Overview: Two parallel stories

1. The Heisenberg category $\mathcal{H}$:
    - Definition is diagrammatic: [Kho '14], [LSav '13], [BSavW '20]
    - $\mathcal{H}$ is monoidal, generated by $P, Q$, acts on $\oplus_n H_n$-mod
    - $K_0(\mathcal{H})$ is the Heisenberg algebra, [Kho '14], [BSavW '23]
    - $Tr(\mathcal{H})$ is the elliptic Hall algebra $\mathcal{E}_{q,q}$, [CLLSamSus18], [MSav22]
    - $Tr(\mathcal{H})$ acts on $Sym \cong Tr(\oplus_n H_n\text{-mod}) \cong \oplus_n K^T(\mathrm{Hilb}_n(\mathbb{C}^2))$

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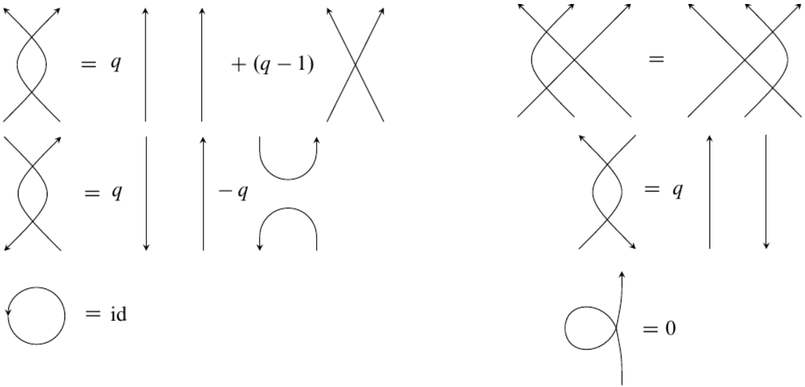
2. The Weyl category $\mathcal{W}$:
    - Definition is diagrammatic: [Harper-S.], building on [Quinn]
    - $\mathcal{W}$ is monoidal, generated by $P, Q$, acts on $\oplus_n TL_n$-mod
    - Conjecture for $K_0(\mathcal{W})$: not Weyl algebra, but similar flavor
    - Guess for $Tr(\mathcal{W})$: not $\mathfrak{sl}_2$ spherical DAHA, but similar
    - $\Sigma(\mathcal{W})$: ??

# The definition and action of $\mathcal{H}$

$$
\begin{aligned}
\mathcal{H} &\to \mathrm{End}_{\mathbf{Cat}}(\oplus_n H_n\text{-mod}) \\
\text{objects} &\mapsto \text{functors} \\
\otimes &\mapsto \circ \\
\text{morphisms} &\mapsto \text{natural transformations}
\end{aligned}
$$

- Objects monoidally generated by $P \mapsto \mathrm{Ind}$ and $Q \mapsto \mathrm{Res}$
- Morphisms generated by (oriented) cups, caps, and crossings (up to isotopy, and modulo local relations on next slide)
- cups and caps map to units and counits of the biadjunctions
  $\mathrm{RCap} \mapsto [x \otimes y \mapsto xy] : {}_n(n)_{n-1}(n)_n \to {}_n(n)_n$,
  $\mathrm{RCup} \mapsto [x \mapsto x] : {}_n(n)_n \to {}_n(n+1)_{n+1}(n+1)_n$
- crossing maps to $[x \mapsto x\sigma_{n+1}] : {}_n(n+2)_n \to {}_n(n+2)_n$

# The relations




**Theorem (Kho, LSav, BSavW)**

*With these relations, the algebra map* $Heis \to K_0(\mathcal{H})$ *is injective.*

# Some lies

First slide claimed $\mathcal{W} \to \mathrm{End}_{\mathbf{Cat}}(\oplus_n TL_n\text{-mod})$:

- $\mathcal{W}$ is monoidally gen. by $P, Q$, morphisms gen. by (oriented) cups and caps, dis-oriented cup-cap, $\mathbb{Z}$-labelled boxes
- **2Weyl** is a 2-category, objects $\mathbb{Z}_{\geq 0}$, Hom cats quotients of $\mathcal{W}$
- $\mathcal{W}^\infty$ an "asymptotic, $n \to \infty$" quotient of $\mathcal{W}$

Why? New features?

- relations unavoidably involve $\frac{[n+k+1]_q}{[n+k+2]_q}$, need region labels
- some isomorphisms in $TL_n$-mod "only hold for large $n$"
- New: $\mathcal{Z}(\mathcal{H}) = Sym$, but $\mathcal{Z}(\mathcal{W})$ spanned by idempotent bubbles

Questions:

- Is there an enlargement $\mathcal{H} \subset \tilde{\mathcal{H}}$ containing idempotent bubbles? geometrically in $\mathrm{Hilb}_n(\mathbb{C}^2)$, "take the stalk at a fixed point"
- Can we describe all of $\mathrm{End}_{\mathbf{Cat}}(H_n\text{-mod})$ diagrammatically?
- Is there a categorification of $\mathcal{H}$ whose "positive part" is affine Soergel stuff??

# Reminders about $TL_n$-mod

- $TL_n \curvearrowright (\mathbb{C}^2)^{\otimes n} \in \mathcal{U}_q(\mathfrak{sl}_2)$-mod
- $\{\text{Irreps of } TL_n\} \longleftrightarrow \{W^n_m \mid 0 \le m \le n,\ n - m \equiv 0 \pmod 2\}$
(convention: $W^n_n$ is the trivial rep., $C_k$ is "project onto $W^n_{n-k}$)
- $\mathrm{Ind}(W^n_m) \cong W^{n+1}_{m+1} \oplus W^{n+1}_{m-1}$
$\mathrm{Res}(W^n_m) \cong W^{n-1}_{m+1} \oplus W^{n-1}_{m-1}$
- $\mathrm{Res} \circ \mathrm{Ind} \cong (\mathrm{Ind} \circ \mathrm{Res}) \oplus C_0$
- Example isomorphism for small $n$ vs big $n$:

$$\begin{aligned}
n > 2k : &\qquad C_{k+1}Q_+C_{k+1}Q_+C_k \cong C_{k+1}Q_+C_kQ_+C_k \\
n = 2k : &\qquad 0 \ncong C_{k+1}Q_+C_kQ_+C_k \\
n < 2k : &\qquad 0 \cong 0
\end{aligned}$$

# The definition and action of $\mathcal{W}$

**Theorem (Harper-S.)**

$\mathbf{2Weyl} \to \mathrm{End}_{\mathbf{Cat}}(TL_n\text{-mod})$

**Definition**

$\mathcal{W}$ is monoidally generated by $P, Q$:

- $P \mapsto \mathrm{Ind}$, $Q \mapsto \mathrm{Res}$, $\otimes \mapsto \circ$
- Morphisms gen. by (oriented) cups, caps, and disoriented cup-cap (modulo local relations on next slides)
- cups and caps map to units and counits of the biadjunctions
  RCap $\mapsto [x \otimes y \mapsto xy] : {}_n(n)_{n-1}(n)_n \to {}_n(n)_n$,
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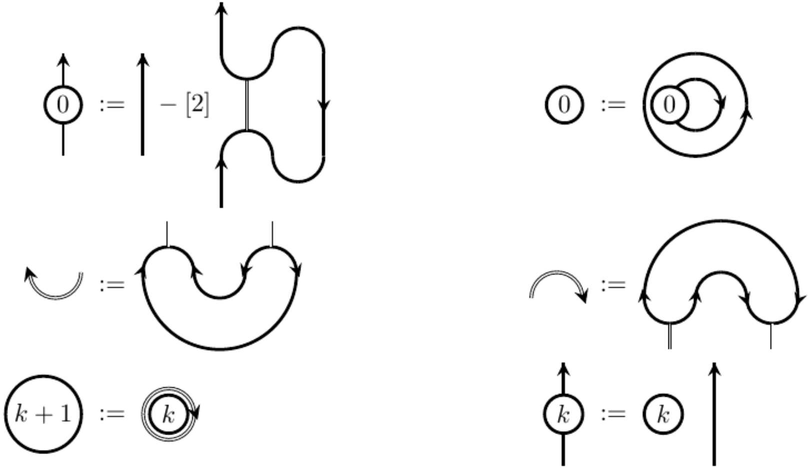
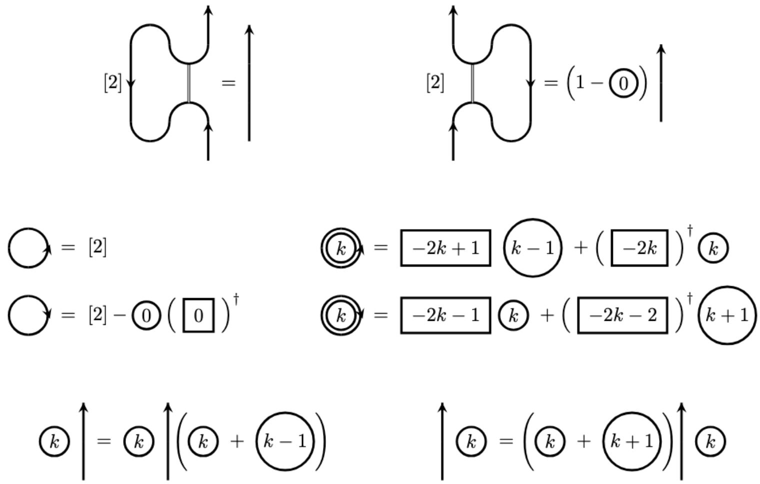
- cup-cap maps to $[x \mapsto xe_{n+1}] : {}_n(n+2)_n \to {}_n(n+2)_n$

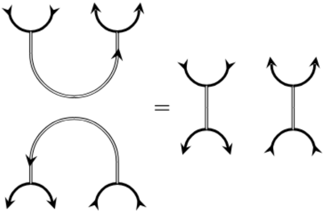
# More generators

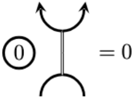


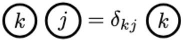
# Relations

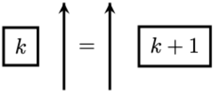


# More relations

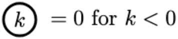


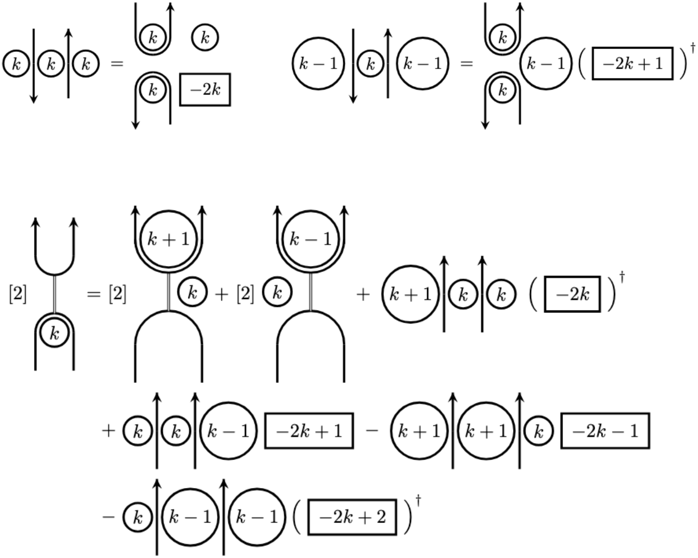







# Three more relations



# The Grothendieck group

**Definition**

$\mathcal{W}^\infty$ is the "boxes are invertible" quotient of $\mathcal{W}$.

Let $A$ be the algebra generated by $x$, $y$ and $c_k$ for $k \geq 0$ with relations

$$
\begin{aligned}
yx &= xy + c_0 & c_k c_\ell &= \delta_{k\ell} c_k \\
c_k x &= c_k x(c_k + c_{k-1}) & x c_k &= (c_k + c_{k+1}) x c_k \\
c_k y &= c_k y(c_k + c_{k+1}) & y c_k &= (c_k + c_{k-1}) y c_k \\
c_{k+1} x c_k x c_k &= c_{k+1} x c_{k+1} x c_k & 0 &= -c_k y c_{k+1} y c_{k+1} + c_k y c_k y c_{k+1} \\
c_k y c_k x c_k &= c_k & c_{k-1} &= c_{k-1} y c_k x c_{k-1} \\
c_k x c_k y c_k &= c_k & c_k &= c_k x c_{k-1} y c_k \\
c_{-1} &= 0 & x c_k y &= c_k + c_{k+1} + c_k x y c_{k+1} + c_{k+1} x y c_k \\
y c_0 x &= c_0 & y c_k x &= c_k + c_{k-1} + c_k x y c_{k-1} + c_{k-1} x y c_k
\end{aligned}
$$

**Proposition:** There is an algebra map $A \to K_0(\mathcal{W}^\infty)$.
**Conjecture:** This map is injective.
**Lemma:** The subalgebra $c_k A c_k$ is a Laurent polynomial ring.

# Asymptotic actions

## Goal

*Prove injectivity of* $A \to K_0(\mathcal{W}^\infty)$ *using a faithful action.*

## Proposition (Harper-S.)

*$A$ acts asymptotically on* $TL := K_0(\oplus_n TL_n\text{-}mod) \cong \mathbb{Z}\langle W^n_m \mid n \geq m\rangle$

## Definition

An *asymptotic action* of a filtered algebra $A_n \subset A_{n+1}$ on a decreasingly filtered v.s. $V_j \supset V_{j+1}$ is collection of maps

$$f_{n,j} : A_n \otimes V_j \to V_{j-n}$$

which are "associative when defined."

## Question

*$\exists$ categorical notion of an asymptotic action* $\mathcal{W}^\infty \curvearrowright \oplus_n TL_n\text{-}mod$*?*

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