

# The Temperley-Lieb tower and the Weyl algebra

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# Overview: Two parallel stories

## ① The Heisenberg category $\mathcal{H}$ :

- Definition is diagrammatic: [Kho '14], [LSav '13], [BSavW '20]
- $\mathcal{H}$  is monoidal, generated by  $P, Q$ , acts on  $\oplus_n H_n\text{-mod}$
- $K_0(\mathcal{H})$  is the Heisenberg algebra, [Kho '14], [BSavW '23]
- $Tr(\mathcal{H})$  is the elliptic Hall algebra  $\mathcal{E}_{q,q}$ , [CLLSamSus18], [MSav22]
- $Tr(\mathcal{H})$  acts on  $Sym \cong Tr(\oplus_n H_n\text{-mod}) \cong \oplus_n K^T(\text{Hilb}_n(\mathbb{C}^2))$

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## ② The Weyl category $\mathcal{W}$ :

- Definition is diagrammatic: [Harper-S.], building on [Quinn]
- $\mathcal{W}$  is monoidal, generated by  $P, Q$ , acts on  $\oplus_n TL_n\text{-mod}$
- Conjecture for  $K_0(\mathcal{W})$ : not Weyl algebra, but similar flavor
- Guess for  $Tr(\mathcal{W})$ : not  $\mathfrak{sl}_2$  spherical DAHA, but similar
- $\Sigma(\mathcal{W})$ : ??

# The definition and action of $\mathcal{H}$

$$\mathcal{H} \rightarrow \text{End}_{\mathbf{Cat}}(\oplus_n H_n\text{-mod})$$

objects  $\mapsto$  functors

$$\otimes \mapsto \circ$$

morphisms  $\mapsto$  natural transformations

- Objects monoidally generated by  $P \mapsto \text{Ind}$  and  $Q \mapsto \text{Res}$
- Morphisms generated by (oriented) cups, caps, and crossings (up to isotopy, and modulo local relations on next slide)
- cups and caps map to units and counits of the biadjunctions  
 $\text{RCap} \mapsto [x \otimes y \mapsto xy] : {}_n(n)_{n-1}(n)_n \rightarrow {}_n(n)_n$ ,  
 $\text{RCup} \mapsto [x \mapsto x] : {}_n(n)_n \rightarrow {}_n(n+1)_{n+1}(n+1)_n$
- crossing maps to  $[x \mapsto x\sigma_{n+1}] : {}_n(n+2)_n \rightarrow {}_n(n+2)_n$

# The relations

$$\begin{array}{c} \text{Diagram: two strands crossing, top-right over bottom-left, with arrows pointing right.} = q \\ \text{Diagram: two strands crossing, top-left over bottom-right, with arrows pointing right.} + (q-1) \\ \text{Diagram: two strands crossing, top-right over bottom-left, with arrows pointing left.} = q \\ \text{Diagram: two strands crossing, top-left over bottom-right, with arrows pointing left.} - q \\ \text{Diagram: a circle with a clockwise arrow.} = \text{id} \\ \text{Diagram: a circle with a clockwise arrow.} = 0 \end{array}$$

Theorem (Kho, LSav, BSavW)

*With these relations, the algebra map  $Heis \rightarrow K_0(\mathcal{H})$  is injective.*

# Some lies

First slide claimed  $\mathcal{W} \rightarrow \text{End}_{\mathbf{Cat}}(\oplus_n TL_n\text{-mod})$ :

- $\mathcal{W}$  is monoidally gen. by  $P, Q$ , morphisms gen. by (oriented) cups and caps, dis-oriented cup-cap,  $\mathbb{Z}$ -labelled boxes
- **2Weyl** is a 2-category, objects  $\mathbb{Z}_{\geq 0}$ , Hom cats quotients of  $\mathcal{W}$
- $\mathcal{W}^\infty$  an “asymptotic,  $n \rightarrow \infty$ ” quotient of  $\mathcal{W}$

Why? New features?

- relations unavoidably involve  $\frac{[n+k+1]_q}{[n+k+2]_q}$ , need region labels
- some isomorphisms in  $TL_n\text{-mod}$  “only hold for large  $n$ ”
- New:  $\mathcal{Z}(\mathcal{H}) = \text{Sym}$ , but  $\mathcal{Z}(\mathcal{W})$  spanned by idempotent bubbles

Questions:

- Is there an enlargement  $\mathcal{H} \subset \tilde{\mathcal{H}}$  containing idempotent bubbles? geometrically in  $\text{Hilb}_n(\mathbb{C}^2)$ , “take the stalk at a fixed point”
- Can we describe all of  $\text{End}_{\mathbf{Cat}}(H_n\text{-mod})$  diagrammatically?
- Is there a categorification of  $\mathcal{H}$  whose “positive part” is affine Soergel stuff??

# Reminders about $TL_n$ -mod

- $TL_n \curvearrowright (\mathbb{C}^2)^{\otimes n} \in \mathcal{U}_q(\mathfrak{sl}_2)\text{-mod}$
- $\{\text{Irreps of } TL_n\} \longleftrightarrow \{W_m^n \mid 0 \leq m \leq n, n - m \equiv 0 \pmod{2}\}$   
(convention:  $W_n^n$  is the trivial rep.,  $C_k$  is “project onto  $W_{n-k}^n$ ”)
- $\text{Ind}(W_m^n) \cong W_{m+1}^{n+1} \oplus W_{m-1}^{n+1}$   
 $\text{Res}(W_m^n) \cong W_{m+1}^{n-1} \oplus W_{m-1}^{n-1}$
- $\text{Res} \circ \text{Ind} \cong (\text{Ind} \circ \text{Res}) \oplus C_0$
- Example isomorphism for small  $n$  vs big  $n$ :

$$n > 2k : \quad C_{k+1}Q_+C_{k+1}Q_+C_k \cong C_{k+1}Q_+C_kQ_+C_k$$

$$n = 2k : \quad 0 \not\cong C_{k+1}Q_+C_kQ_+C_k$$

$$n < 2k : \quad 0 \cong 0$$

# The definition and action of $\mathcal{W}$

Theorem (Harper-S.)

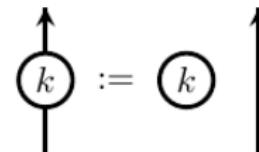
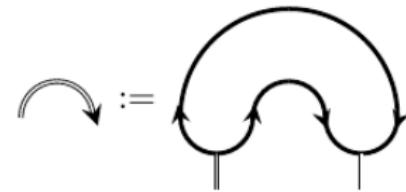
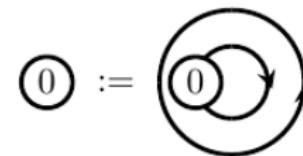
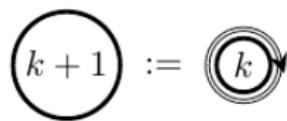
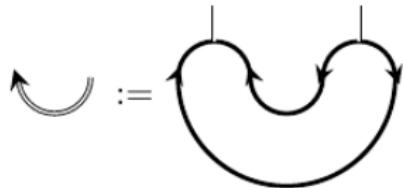
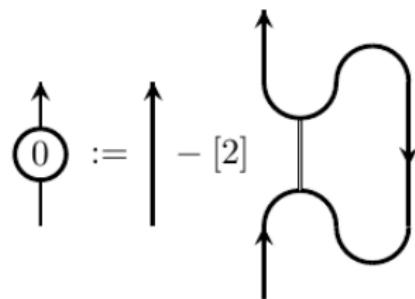
**2Weyl**  $\rightarrow \text{End}_{\mathbf{Cat}}(TL_n\text{-mod})$

## Definition

$\mathcal{W}$  is monoidally generated by  $P, Q$ :

- $P \mapsto \text{Ind}$ ,  $Q \mapsto \text{Res}$ ,  $\otimes \mapsto \circ$
- Morphisms gen. by (oriented) cups, caps, and **disoriented cup-cap** (modulo local relations on next slides)
- cups and caps map to units and counits of the biadjunctions  
 $\text{RCap} \mapsto [x \otimes y \mapsto xy] : {}_n(n)_{n-1}(n)_n \rightarrow {}_n(n)_n$ ,  
 $\text{RCup} \mapsto [x \mapsto x] : {}_n(n)_n \rightarrow {}_n(n+1)_{n+1}(n+1)_n$
- **cup-cap** maps to  $[x \mapsto x e_{n+1}] : {}_n(n+2)_n \rightarrow {}_n(n+2)_n$

## More generators



## Relations

$$\begin{array}{c} \text{[2]} \\ \text{[2]} \end{array} \begin{array}{c} \text{[2]} \\ \text{[2]} \end{array} = \begin{array}{c} \text{[2]} \\ \text{[2]} \end{array}$$

$$\begin{array}{c} \text{[2]} \\ \text{[2]} \end{array} = (1 - \text{①}) \begin{array}{c} \text{[2]} \\ \text{[2]} \end{array}$$

$$\text{○} = [2]$$

$$\text{○} = \boxed{-2k+1} \text{○} + (\boxed{-2k})^\dagger \text{○}$$

$$\text{○} = [2] - \text{①} (\boxed{0})^\dagger$$

$$\text{○} = \boxed{-2k-1} \text{○} + (\boxed{-2k-2})^\dagger \text{○}$$

$$\text{○} = \text{○} \uparrow (\text{○} + \text{○})$$

$$\uparrow \text{○} = (\text{○} + \text{○}) \uparrow \text{○}$$

## More relations

$$\begin{array}{c} \text{Diagram 1: Two strands with arrows pointing up, crossing, and then merging.} \\ \text{Diagram 2: Two strands with arrows pointing up, crossing, and then merging.} \\ = \\ \text{Diagram 3: Two strands with arrows pointing up, crossing, and then merging.} \end{array}$$

$$\text{Diagram 4: A single strand with a circle containing '0' and an arrow pointing up.} = 0$$

$$\text{Diagram 5: A circle with 'k' inside.} \text{ } \text{Diagram 6: A circle with 'j' inside.} = \delta_{kj} \text{ } \text{Diagram 7: A circle with 'k' inside.}$$

$$\boxed{k} \uparrow = \boxed{k+1}$$

$$\left( \boxed{k} \right)^\dagger = \left( \boxed{k} \right)^{-1} \text{ for } k \geq 0$$

$$\text{Diagram 8: A circle with 'k' inside.} = 0 \text{ for } k < 0$$

## Three more relations

$$\begin{array}{c} \text{Diagram: } \text{Three circles } k, k, k \text{ in a row. The first } k \text{ has a vertical arrow down, the second } k \text{ has a vertical arrow up, the third } k \text{ has a vertical arrow down.} \\ \text{Diagram: } \text{Two circles } k, k \text{ in a row. The first } k \text{ has a vertical arrow up, the second } k \text{ has a vertical arrow up. Below them is a box labeled } -2k. \end{array} =$$

$$\begin{array}{c} \text{Diagram: } \text{Two circles } k-1, k-1 \text{ in a row. The first } k-1 \text{ has a vertical arrow down, the second } k-1 \text{ has a vertical arrow up.} \\ \text{Diagram: } \text{Two circles } k, k-1 \text{ in a row. The first } k \text{ has a vertical arrow up, the second } k-1 \text{ has a vertical arrow up. Below them is a box labeled } -2k+1 \text{ with a } \dagger \text{ symbol.} \end{array} =$$

$$\begin{array}{c} \text{Diagram: } \text{A circle } k \text{ with a vertical arrow up, followed by a box } [2]. \\ \text{Diagram: } \text{A circle } k+1 \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow down, followed by a box } [2]. \\ \text{Diagram: } \text{A circle } k-1 \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow down, followed by a box } [2]. \end{array} = [2] =$$

$$\begin{array}{c} \text{Diagram: } \text{A circle } k+1 \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow down, followed by a circle } k \text{ with a vertical arrow up, followed by a box } -2k \text{ with a } \dagger \text{ symbol.} \\ \text{Diagram: } \text{A circle } k+1 \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow up, followed by a box } -2k \text{ with a } \dagger \text{ symbol.} \\ \text{Diagram: } \text{A circle } k \text{ with a vertical arrow up, followed by a circle } k-1 \text{ with a vertical arrow up, followed by a box } -2k+1. \\ \text{Diagram: } \text{A circle } k+1 \text{ with a vertical arrow up, followed by a circle } k+1 \text{ with a vertical arrow up, followed by a circle } k \text{ with a vertical arrow up, followed by a box } -2k-1. \\ \text{Diagram: } \text{A circle } k \text{ with a vertical arrow up, followed by a circle } k-1 \text{ with a vertical arrow up, followed by a circle } k-1 \text{ with a vertical arrow up, followed by a box } -2k+2 \text{ with a } \dagger \text{ symbol.} \end{array}$$

# The Grothendieck group

## Definition

$\mathcal{W}^\infty$  is the “boxes are invertible” quotient of  $\mathcal{W}$ .

Let  $A$  be the algebra generated by  $x, y$  and  $c_k$  for  $k \geq 0$  with relations

$$\begin{array}{ll} yx = xy + c_0 & c_k c_\ell = \delta_{k\ell} c_k \\ c_k x = c_k x(c_k + c_{k-1}) & xc_k = (c_k + c_{k+1}) xc_k \\ c_k y = c_k y(c_k + c_{k+1}) & yc_k = (c_k + c_{k-1}) yc_k \\ c_{k+1} xc_k xc_k = c_{k+1} xc_{k+1} xc_k & 0 = -c_k yc_{k+1} yc_{k+1} + c_k yc_k yc_{k+1} \\ c_k yc_k xc_k = c_k & c_{k-1} = c_{k-1} yc_k xc_{k-1} \\ c_k xc_k yc_k = c_k & c_k = c_k xc_{k-1} yc_k \\ c_{-1} = 0 & xc_k y = c_k + c_{k+1} + c_k xy c_{k+1} + c_{k+1} xy c_k \\ yc_0 x = c_0 & yc_k x = c_k + c_{k-1} + c_k xy c_{k-1} + c_{k-1} xy c_k \end{array}$$

**Proposition:** There is an algebra map  $A \rightarrow K_0(\mathcal{W}^\infty)$ .

**Conjecture:** This map is injective.

**Lemma:** The subalgebra  $c_k A c_k$  is a Laurent polynomial ring.



# Asymptotic actions

## Goal

*Prove injectivity of  $A \rightarrow K_0(\mathcal{W}^\infty)$  using a faithful action.*

## Proposition (Harper-S.)

*$A$  acts asymptotically on  $TL := K_0(\bigoplus_n TL_n\text{-mod}) \cong \mathbb{Z}\langle W_m^n \mid n \geq m \rangle$*

## Definition

An *asymptotic action* of a filtered algebra  $A_n \subset A_{n+1}$  on a decreasingly filtered v.s.  $V_j \supset V_{j+1}$  is collection of maps

$$f_{n,j} : A_n \otimes V_j \rightarrow V_{j-n}$$

which are “associative when defined.”

## Question

$\exists$  categorical notion of an asymptotic action  $\mathcal{W}^\infty \curvearrowright \bigoplus_n TL_n\text{-mod}?$

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