

How to compute many decomposition numbers for symmetric groups

Geordie Williamson

Diagrammatic Categorification, ICERM, Oct 2025

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(... in principle, on a big computer, using code that doesn't yet exist)

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Basic Questions: (appear attached to much deep mathematics)

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(what is it?)

$\dim D_\lambda$??

(mod p dimensions)

$[S_\lambda \otimes_{\mathbb{Z}} \mathbb{F}_p : D_\mu]$??

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$\Leftarrow [S_\lambda \otimes_{\mathbb{Z}} \mathbb{F}_p : D_\mu] ??$

(decomposition numbers)

(Question with most structure)

$$[S_\lambda \otimes_{\mathbb{Z}} \mathbb{F}_p : \mathcal{O}_\mu] = ??$$

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is false for homogenous m .

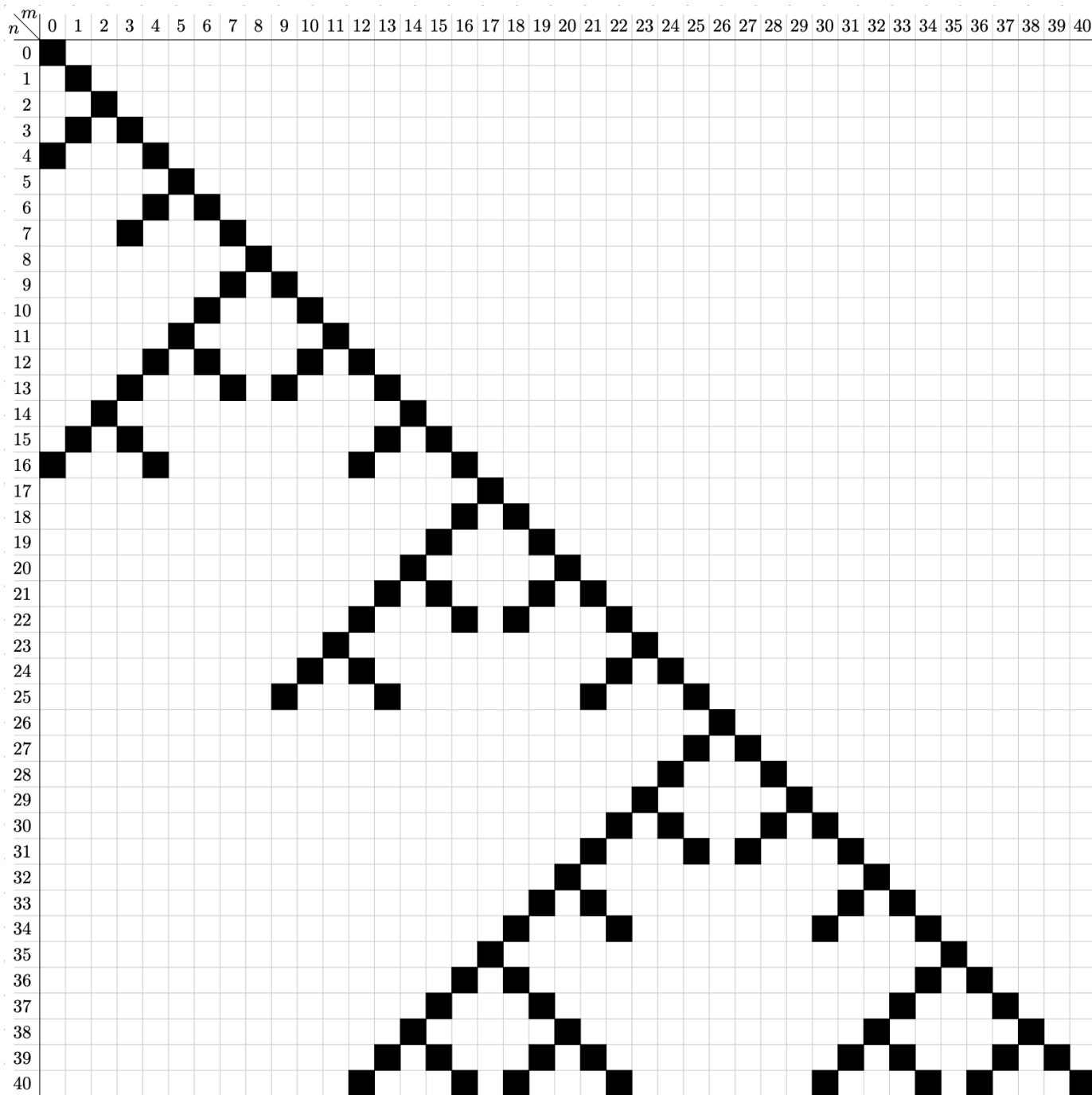
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E.g. If F_{2n+1} (Fibonacci number) has prime factors
larger than n^2 then JC is false.

$F_{571} = \underbrace{960 \dots}_{119 \text{ digits}}$ is prime.



(some encoding of)
2 row
decomposition
numbers for
 $p=3$

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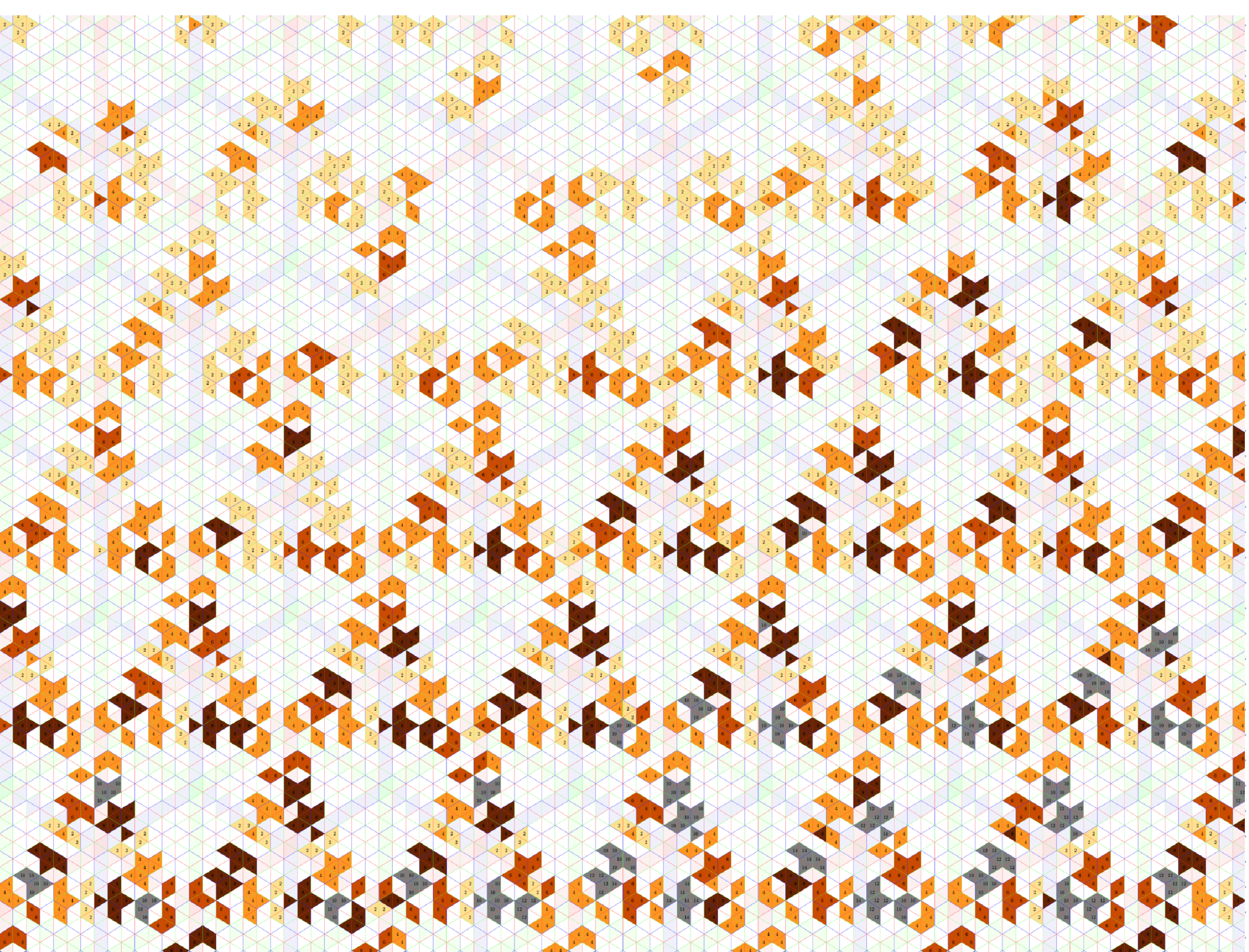
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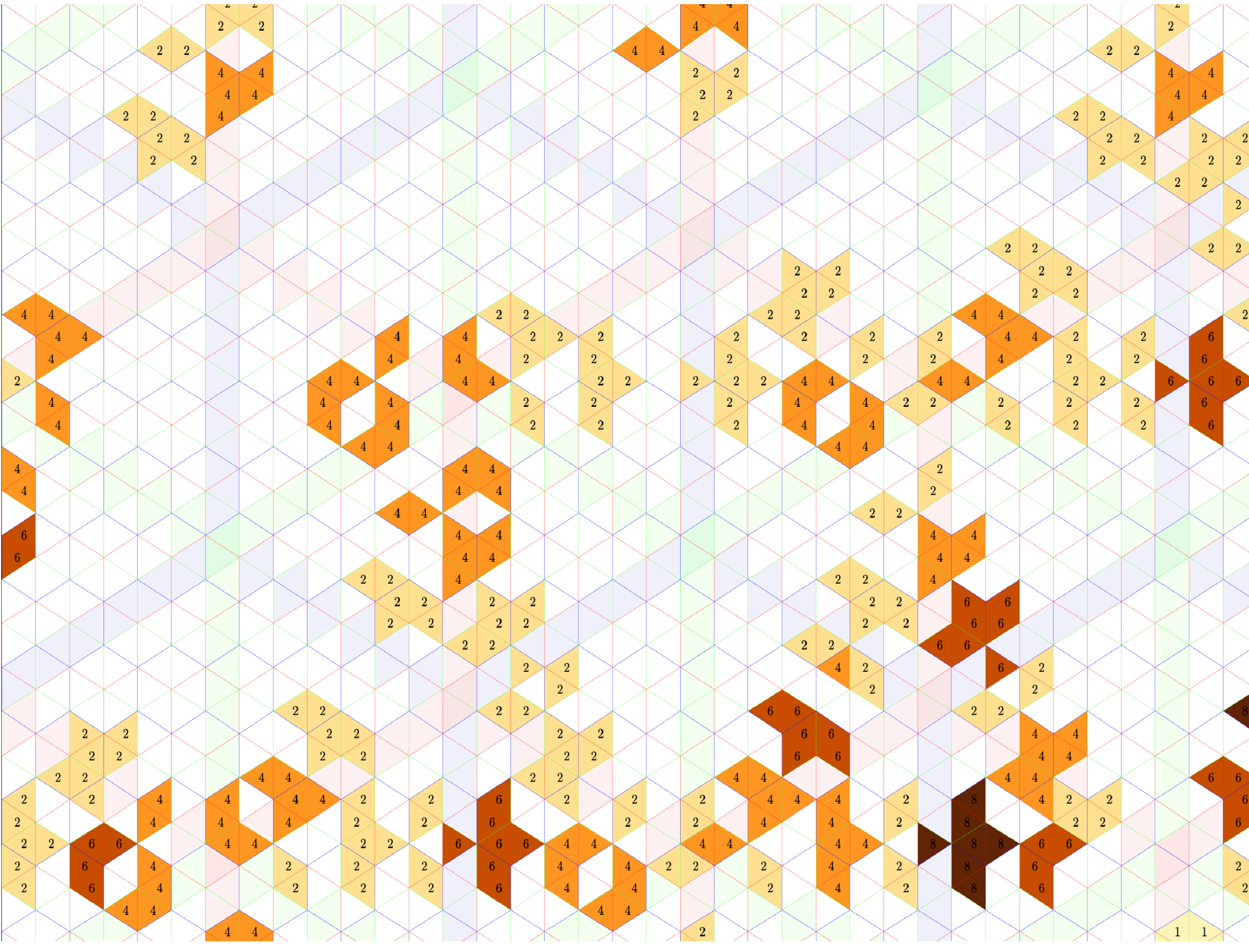
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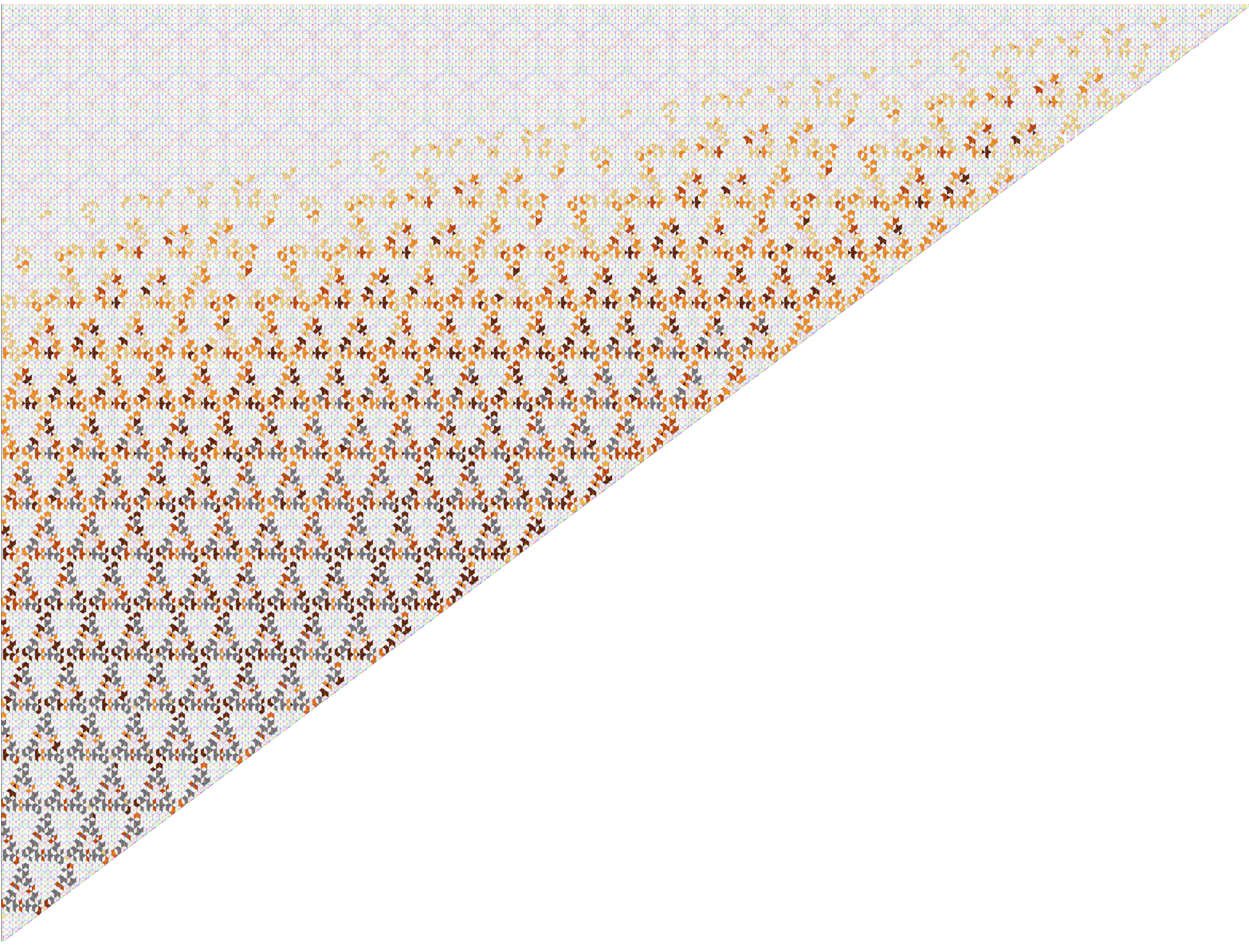
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Thought experiment: What would answer look like if
 we could step back and see all decomposition numbers at once?







Some Philosophy + a prediction

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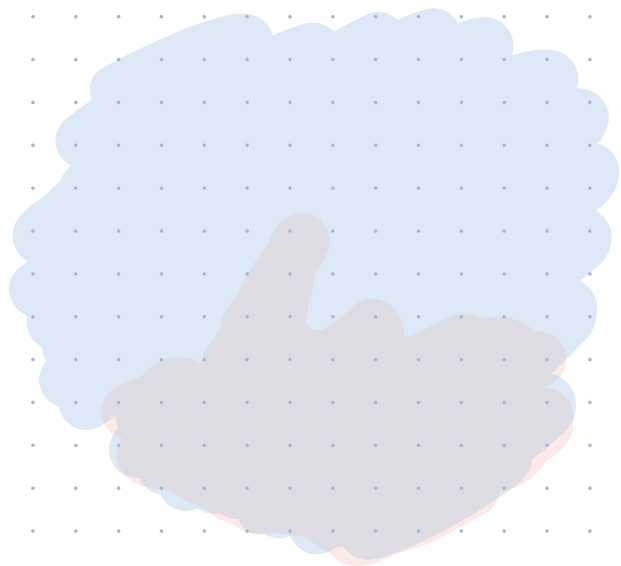
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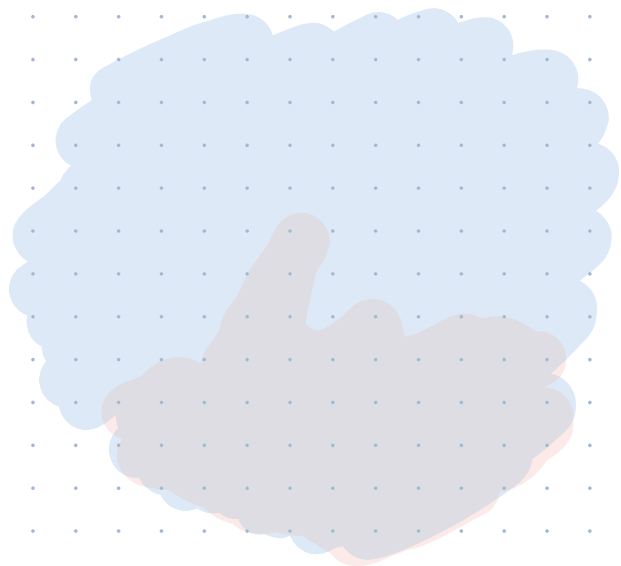
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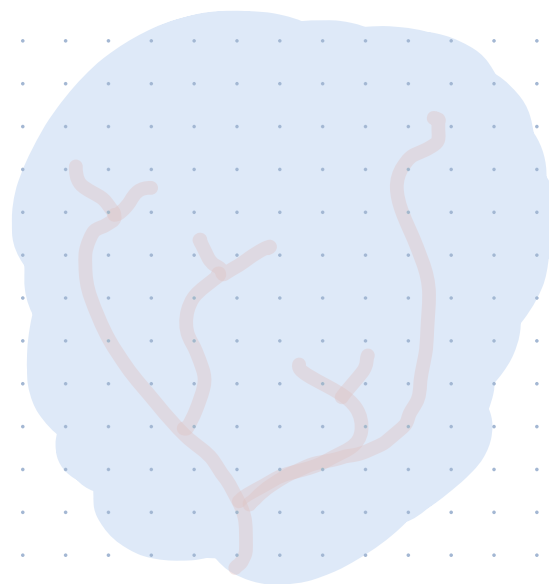
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New mental picture



● accessible to reasoning

Goal For Rest of Talk:

- Representations of S_n and tilting modules
- Principal block of tilting modules and anti-spherical module
- $\mathbb{Q}$ -algorithm + billiards
- $\mathbb{F}_p$ -algorithm + boats

Main Point: One can explain actions of monoidal categories to computers, and doing so may allow us to explore regions previously inaccessible.

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Indecomposable tilting modules admit a highest weight classification $\{T_\lambda\}$

and Tilt is closed under $\otimes$ (Donkin).

(James, Doukin, Erdmann)

$$(T_\lambda : \Delta_\mu) = [S_\mu : D_\lambda]$$

(λ p -regular)

↑ multiplicity of Δ_μ in any
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Ind. tilting modules for SL_2 are known $\Rightarrow$ 2-row dec. numbers.

Ind. tilting modules for SL_m for $m \geq 3$ are unknown.

(Apparently much harder task than simple modules.)

No finite list of cases which answers this question.

(James, Doukin, Erdmann)

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$(\lambda \text{ p-regular})$

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Example:

$$\left[S \begin{array}{|c|} \hline \square \\ \hline \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} : D \begin{array}{|c|} \hline \square \\ \hline \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right],$$

$(\dim S = 1925)$

Instead we could try to compute understand $T_{\text{[diagram]}} \subset V^{\otimes 12}$

for SL_3 .

Principal Block and Anti-spherical Module

Rep GL_n , Rep kS_m decompose into blocks.

Working inside a block offers great condensation of difficulty.

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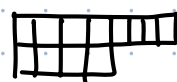
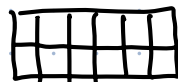
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SL_2 : 0 1 2 3 4 5

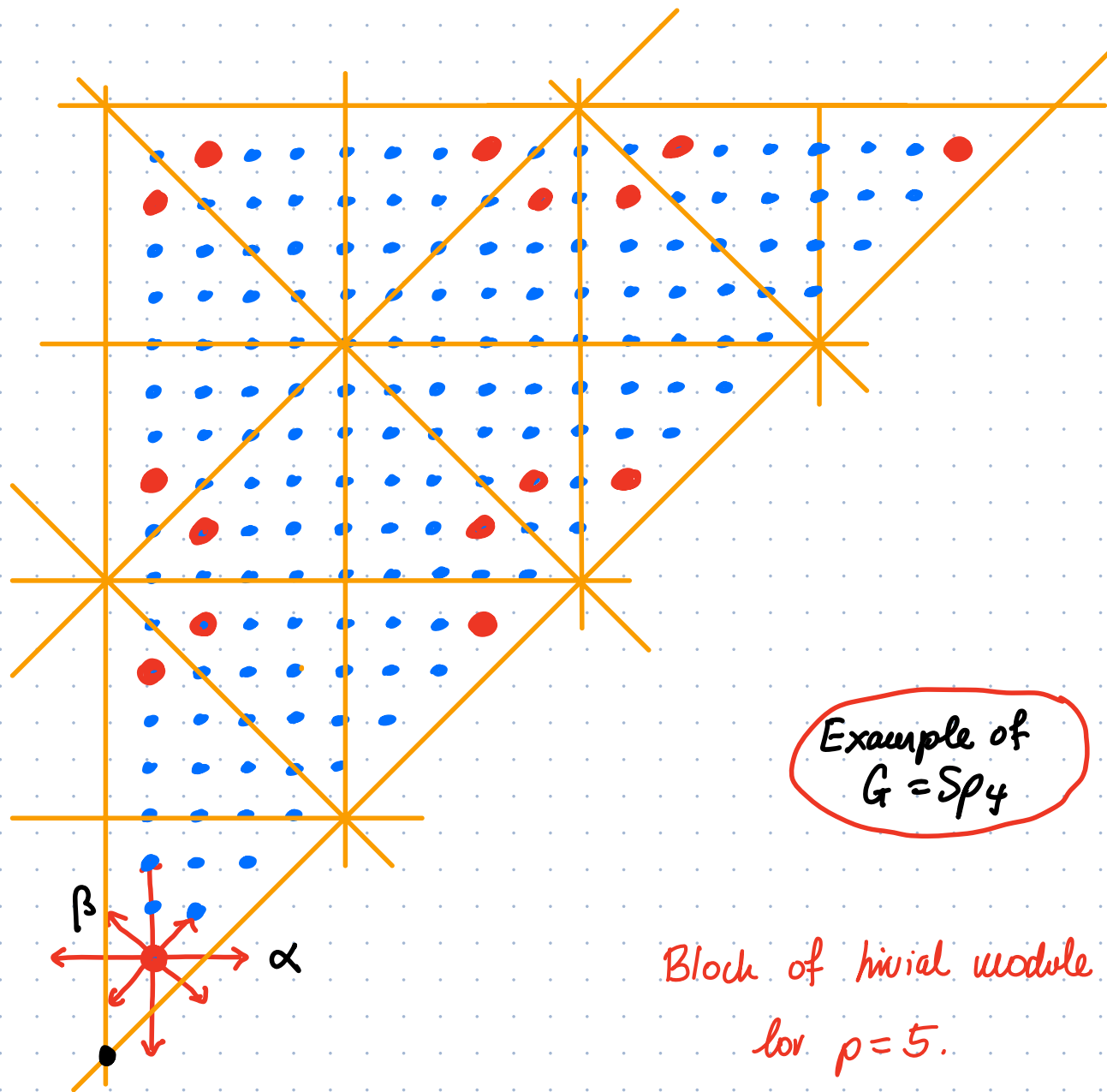
Ex: Block of trivial module for $p=3$ for SL_2 :

0 . . . 4 . 6 . . 10 . 12 . .

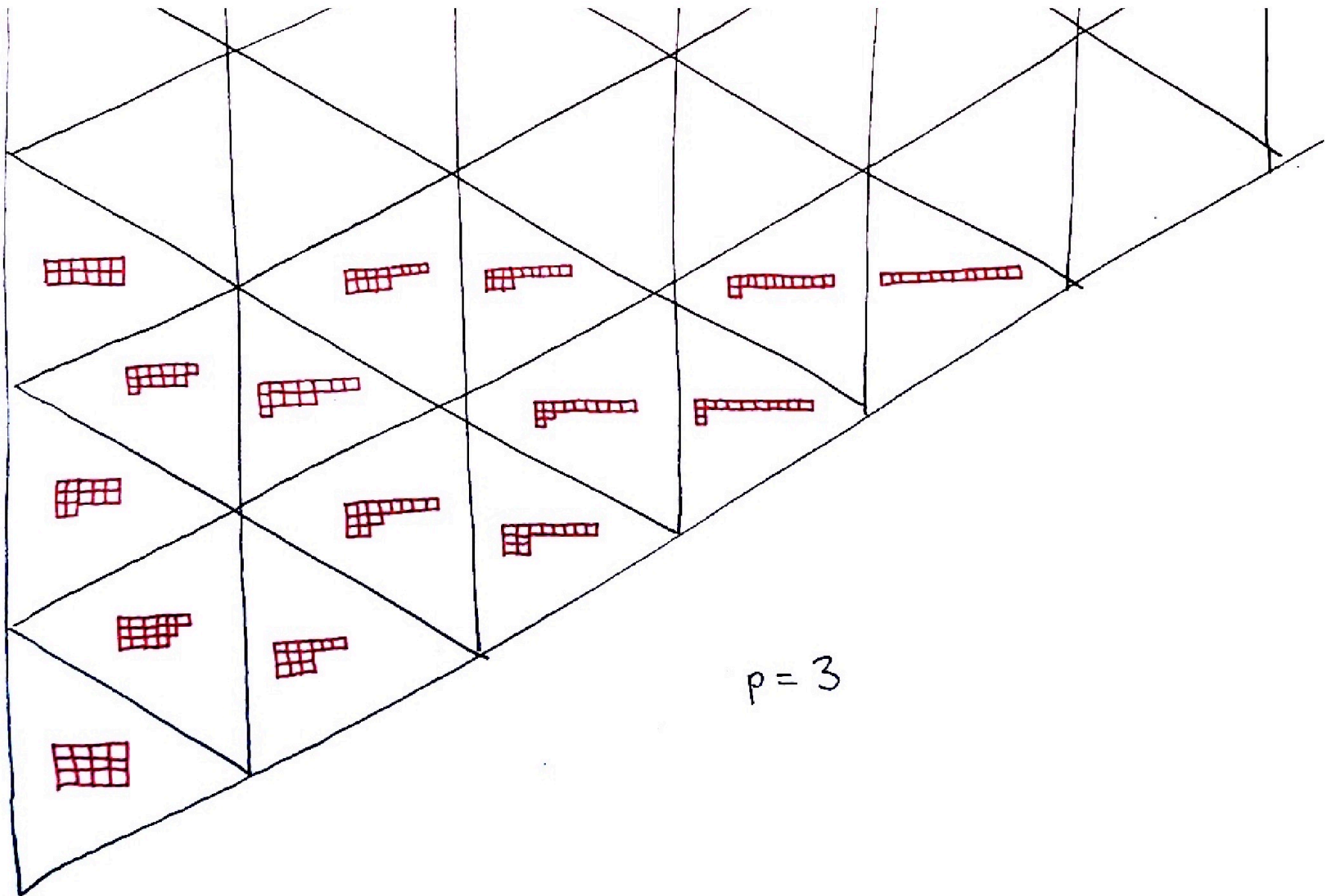
(GL_2
version)



Ex:



$\leq n$ row partitions in principal block $\subset$ dominant alcoves for GL_n
(often we work with SL_n instead).



Lievis by

Joel Gibson.

(jgibson.id.au/lievis/)

G reductive algebraic group

$\text{Rep } G \quad \hookrightarrow \quad (\text{Tilt } G, \otimes)$

$\text{Rep}_0 G \quad \hookrightarrow \quad \text{Translation Functors } \mathcal{V}_s$
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Conjecture now well-understood: $\rightarrow$ RW for GL_n via Khovanov-Lauda-Rouquier cat.

$\rightarrow$ Bezrukavnikov-Riche via coherent sheaves $\rightarrow$ Elias-Losev via unicity.

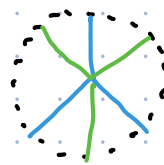
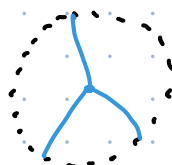
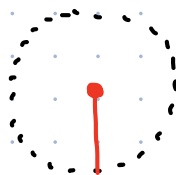
$\rightarrow$ Giappara via Smith theory $\rightarrow$ Bowman-Cox-Hazi via KLR.

Anti-Spherical Category:

$\mathcal{H}^v$ = diagrammatic Soergel bimodules = $\langle B_s, B_t, B_u, \dots \rangle$
associated to W



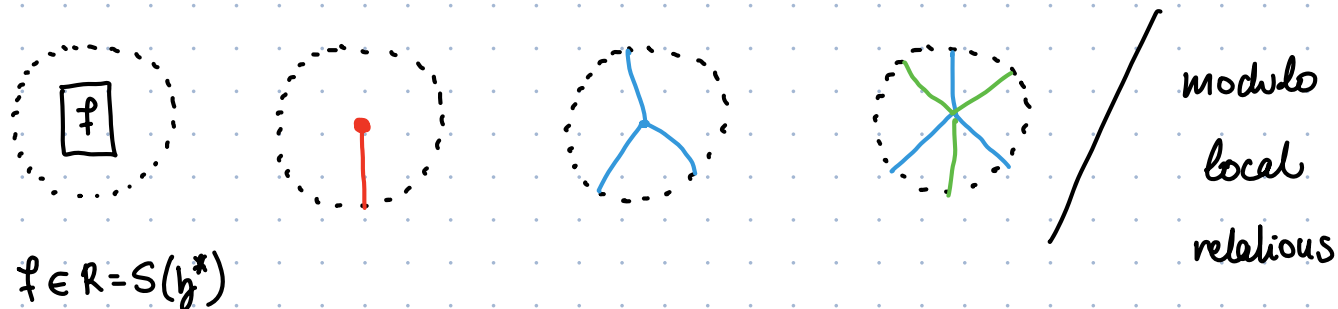
$f \in R = S(\mathfrak{h}^*)$



modulo
local
relations

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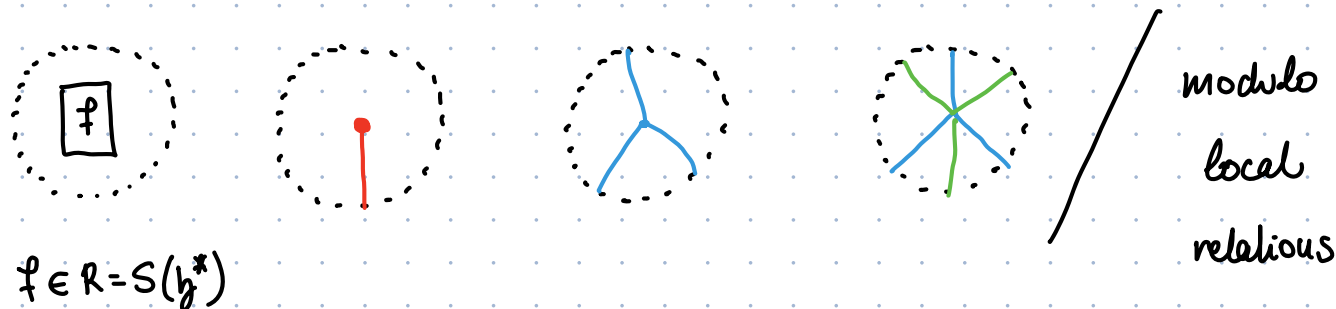
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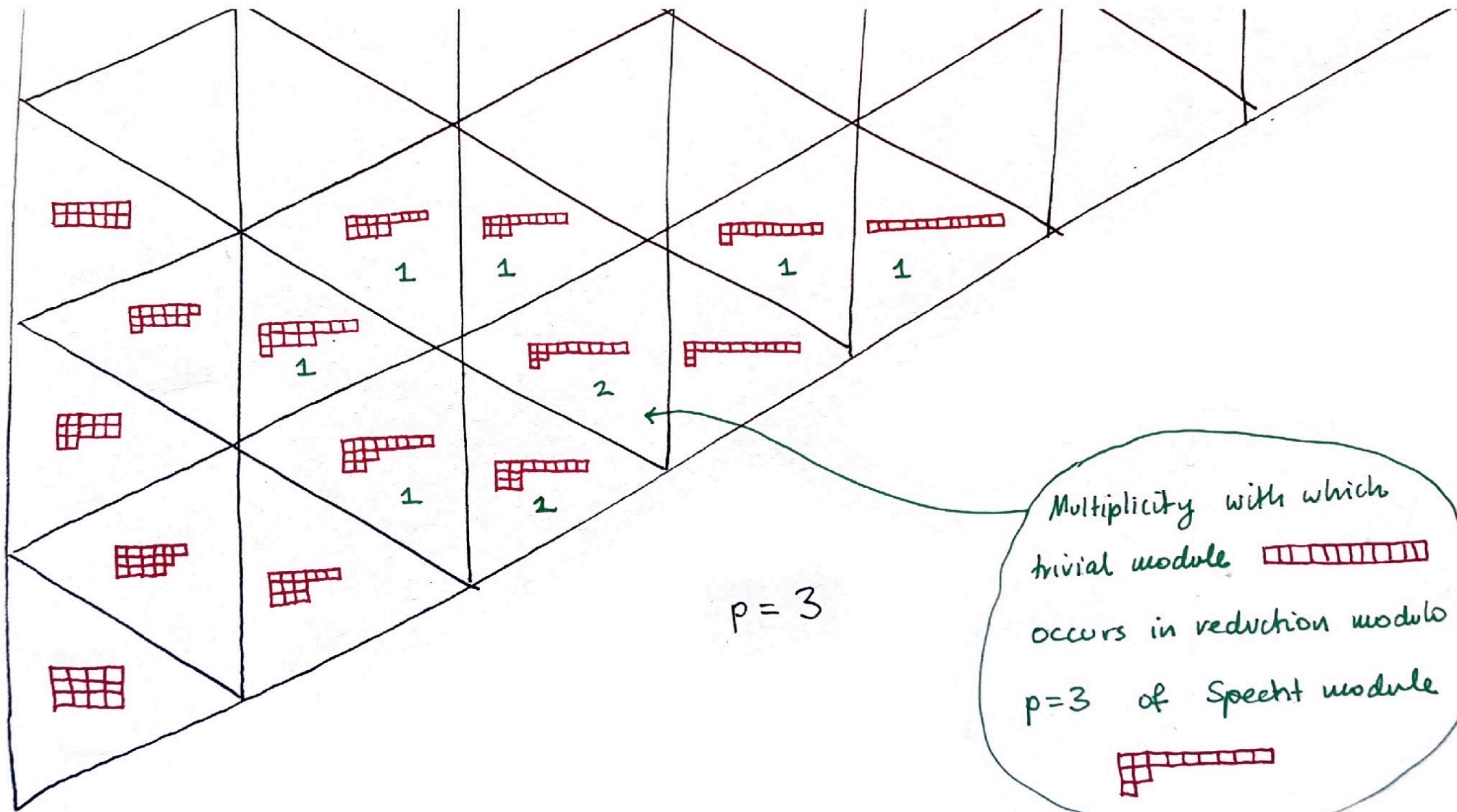
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RW Conjecture $\Rightarrow k \otimes_{k[\mathfrak{h}]} AS \simeq \text{Tilt}_0$ as modules over translation functors / Hecke category.

↑
to be explained in a moment.



$$p=3$$

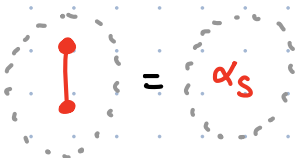
Multiplicity with which
trivial module 
occurs in reduction modulo
 $p=3$ of Specht module



Input to $\mathcal{H}^v$ is a realization of the Coxeter system (W, S) .

It is convenient for us to take $\mathfrak{h} = \mathfrak{h}_{\text{finite}} \oplus k\delta$ ← imaginary root

$$\alpha_0 = -\alpha_{\text{highest}} + \delta$$

Because of the relation  , $\mathfrak{h}_{\text{finite}}$ acts

by zero on the left of $AS \Rightarrow k[\delta]$ -linear.

Thm (Lbedinsky-W.): Suppose characteristic of k is 0.

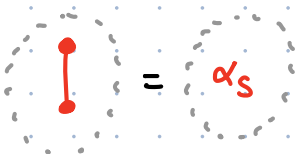
- Hom spaces in AS are free over $k[\delta]$.

- $k(\delta) \otimes_{k[\delta]} AS$ is semi-simple with simple objects $\Delta_x, x \in {}^\neq W$.

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See:
Calculating the
 p -canonical basis
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arXiv

$\mathbb{Q}$ -algorithm (W. v1, Jensen + W. v2, Gibson v3)

Goal: Work over $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid p \nmid b \right\} \subset \mathbb{Q}$.

Describe $AS_{\mathbb{Z}_{(p)}}$ inside $\mathbb{Q}(s) \otimes_{\mathbb{Z}_{(p)}} AS$

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Step 2: Now use that

$$AS_{\mathbb{Z}_{(p)}} = \mathbb{1} \cdot \mathcal{H}^\vee \subset \mathbb{Q}(\delta) \otimes_{\mathbb{Z}_{(p)}} AS$$

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- (Unclear whether I'm being silly or an arithmetic reality!)

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Consequence:

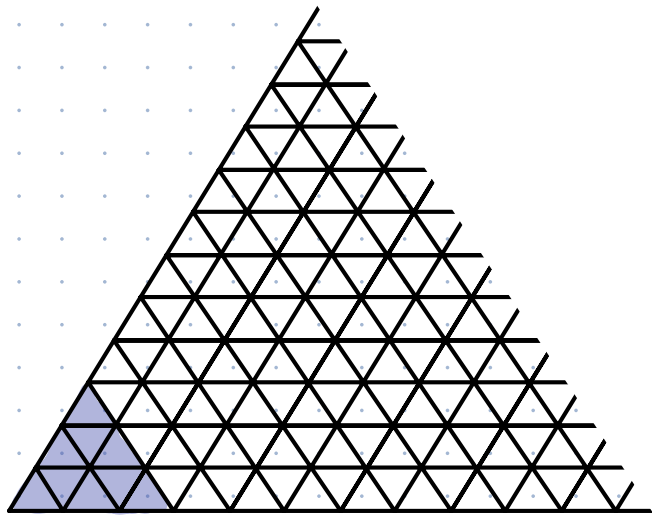
$$\mathbb{F}_p[\delta^{\pm 1}] \otimes AS \simeq \bigoplus_{x \in W/W_p} \mathbb{F}_p[\delta^{\pm 1}] \otimes_{\mathbb{F}_p} \mathbb{F}_p \otimes_{\mathbb{F}_p[\delta]} AS^x$$

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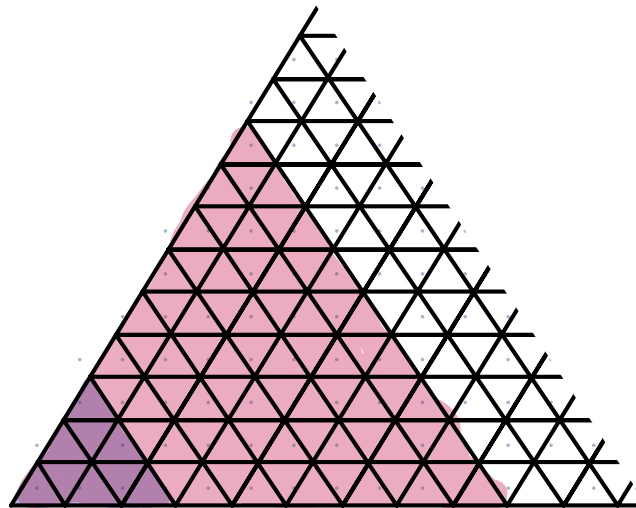
$\mathbb{F}_p[\delta^{\pm 1}] \otimes AS$ sits inside p^{rank} copies of itself".

Consequence: $\mathbb{F}_p[\delta^{\pm 1}] \otimes AS \simeq \bigoplus_{x \in W/W_p} \mathbb{F}_p[\delta^{\pm 1}] \otimes_{\mathbb{F}_p} \mathbb{F}_p \otimes_{\mathbb{F}_p[\delta]} AS^x$

Should allow for a matryoshka-like description of $\text{Rep}_0 G$.



Step 1



Step 2



Some illustrations in the mixed basis.



Thank You!
(I'll report back in 4 years...)