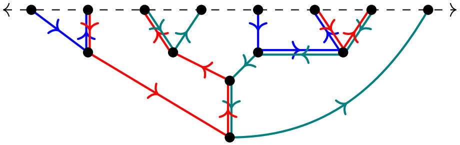
Quantum representations on webs

Julianna Tymoczko (Smith College)
with H. Russell (University of Richmond), and others



October 23, 2025

This talk has four goals:

1. Give context and history about the big picture.
2. Introduce webs and web vectors in quantum representation theory.
3. Describe a new construction of web vectors.
4. Discuss applications to geometry and combinatorics.

Diagrammatic representation theory

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- Algebraic geometers: Springer fibers, generalized Schubert calculus (Fung 2003)

Diagrammatic representation theory

Core concerns:

- Explicit computations (e.g. link invariant, invariant vector)
- Deciding if two objects (bases, graphs, etc.) are equivalent
- Counting how many, parametrizing sets
- Tools for calculating
- Equivariant maps preserving something, categorification

Background: $\mathfrak{sl}_n$ representations (vibes only)

The usual set-up of group representations considers group G acting on vector space V. For instance, take $G = SL_n(\mathfrak{C})$ acting on $\mathfrak{C}^n$.

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} = F_1 \vec{v}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = K_1 \vec{v}$$

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Upshot:

- $E_1, E_2, \ldots, E_{n-1}$ move the standard basis vectors $x_1, x_2, \ldots$ up one
- $F_1, F_2, \ldots, F_{n-1}$ move the standard basis vectors $x_1, x_2, \ldots$ down one
- $K_1, K_2, \ldots, K_{n-1}$ scale two standard basis vectors simultaneously and "oppositely"

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- The same thing works if V is any wedge product of copies of $\mathbb{C}^n$. The fundamental representations are the vector spaces $V_k = \bigwedge^k \mathbb{C}^n$. For each $S = \{j_1 > \cdots > j_k\}$ there is a basis element $x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_k}$

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- It's customary to encode each basis vector and its subset S using a binary string. For instance, use 0110 to represent $x_3 \wedge x_2$ and $\{3, 2\} \subset \{4, 3, 2, 1\}$.
- The operators induced by E_i and F_i send one basis vector to another based on whether a single 1 in one binary string can be moved one step in the appropriate direction to get to the other binary string.

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- The same basic shorthand works for representations

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k}$$

$\mathcal{U}_q(\mathfrak{sl}_n)$ representations (vibes only)

Denote $\mathbb{C}(q)$ by $\mathbb{C}_q$. Pretend that $\mathcal{U}_q(\mathfrak{sl}_n)$ consists of matrices with entries in $\mathbb{C}_q$. Then $\mathcal{U}_q(\mathfrak{sl}_n)$ acts on $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k}$ like $\mathfrak{sl}_n$ does but with extra factors of $q^{\pm 1}$.

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Let $S = \{i_1 > \cdots > i_k\} \subset \{1, \ldots, n\}$.

- Basis vectors for $\bigwedge_q^k(\mathbb{C}_q^n)$: $x_S = x_{i_1} \wedge_q \cdots \wedge_q x_{i_k}$
- Basis vectors for $(\bigwedge_q^k(\mathbb{C}_q^n))^*$: $(x_S)^*$
- Quantum exterior products work like ordinary exterior products but with an extra factor of $(-q)^{\pm 1}$ depending on whether basis vector e_i with bigger index moves left or right.

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Question: What are the $\mathcal{U}_q(\mathfrak{sl}_n)$-invariant vectors?

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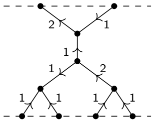
- Web edges carry weights and orientations that depend on the choice of n.
- Edge decorations at the boundary dictate source and target spaces.
- The web interior describes a specific homomorphism between representations.



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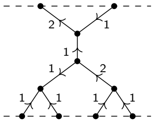

$$\Lambda_q^2(\mathbb{C}_q^3) \otimes \left(\mathbb{C}_q^3\right)^*$$

$$\uparrow$$

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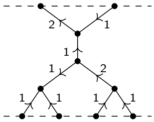

$$\mathbb{C}_q^3 \otimes \mathbb{C}_q^3 \otimes \mathbb{C}_q^3 \otimes \mathbb{C}_q^3 \longrightarrow \Lambda_q^2(\mathbb{C}_q^3) \otimes \left(\mathbb{C}_q^3\right)^*$$

$$V_1 \otimes V_1 \otimes V_1 \otimes V_1 \longrightarrow V_2 \otimes V_1^*$$

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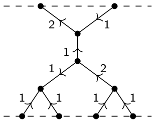

$$\begin{array}{c} \Lambda^2_q(\mathbb{C}^3_q) \otimes \left(\mathbb{C}^3_q\right)^* \\ \uparrow \\ \mathbb{C}^3_q \otimes \mathbb{C}^3_q \otimes \mathbb{C}^3_q \otimes \mathbb{C}^3_q \end{array} \qquad \begin{array}{rc} \mathcal{U}_q(\mathfrak{sl}_3) \curvearrowright & V_2 \otimes V_1^* \\ & \uparrow \\ \mathcal{U}_q(\mathfrak{sl}_3) \curvearrowright & V_1 \otimes V_1 \otimes V_1 \otimes V_1 \end{array}$$

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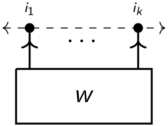
When all vertices of a web lie on one axis, this setup specifies a $\mathcal{U}_q(\mathfrak{sl}_n)$-invariant web vector.



$$\mathcal{U}_q(\mathfrak{sl}_n) \curvearrowright V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_k}$$

$$\uparrow$$

$$\mathcal{U}_q(\mathfrak{sl}_n) \curvearrowright V_0 = \mathbb{C}_q$$

$$\rightsquigarrow v_w \in \mathrm{Inv}_{\mathcal{U}_q(\mathfrak{sl}_n)}(V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_k})$$

Computing with webs: web bases and web relations

The web vector associated to a web graph is the invariant vector lying inside some tensor product of copies of V_i or V_i^*. The vector space spanned by all $\mathfrak{sl}_n$ web vectors is the invariant space for $\mathcal{U}_q(\mathfrak{sl}_n)$. A web basis is a set of webs that gives a basis for an invariant space.

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Good properties for a web basis:

- *Identifiability:* Is this particular web vector an element of the web basis?
- *Decomposition:* Given a web vector, how do I write it in terms of basis vectors?
- *Rotation invariance:* Is the web basis preserved under operations like rotation of graphs?

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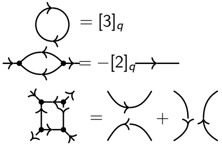
When do web graphs represent the same web vector? Web relations are skein-theoretic equivalence relations that encode algebraic operations preserving an invariant vector.

$\mathfrak{sl}_3$ web relations:



Computing with webs:

- For $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, the web relations yield web bases called reduced webs that have all three "good properties."
  - The reduced web basis for $\mathfrak{sl}_2$ "is" noncrossing matchings (or Temperley-Lieb diagrams).
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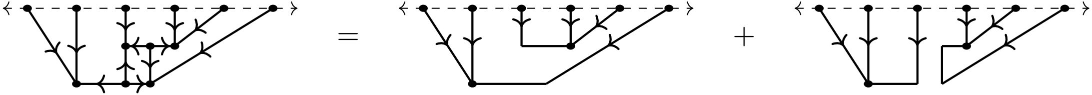
- **Web relations when** $n = 2$ **and** $n = 3$ **always simplify the web graph.** Edge labels can be omitted. Relations reduce the number of faces and interior vertices. Boundary depth of the web graph is a partial order preserved by web relations.



Computing with webs

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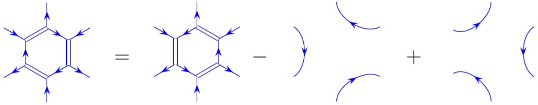
- **Why does it get harder when n gets bigger?**

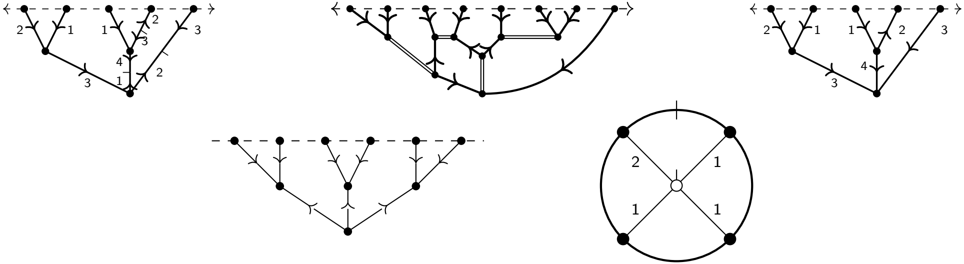
  Non-reducing $\mathfrak{sl}_4$ web relation:
  (from Kim's 2003 thesis)



- Also the construction of webs for $n > 4$ makes it easy to accidentally scale and hard to identify the coefficient of a web vector.

A big issue: constructing webs and relations

There are many different conventions for drawing webs and constructing web vectors.



Top Row: Cautis-Kamnitzer-Morrison, Kim, Fontaine,
Bottom Row: Kuperberg, Fraser-Lam-Le

But most don't give a complete set of relations when $n > 4$.

Exception: CKM webs

Cautis-Kamnitzer-Morrison (2014) have a complete setup for computing with webs. They give explicit formulas for web vectors and a complete set of local relations.

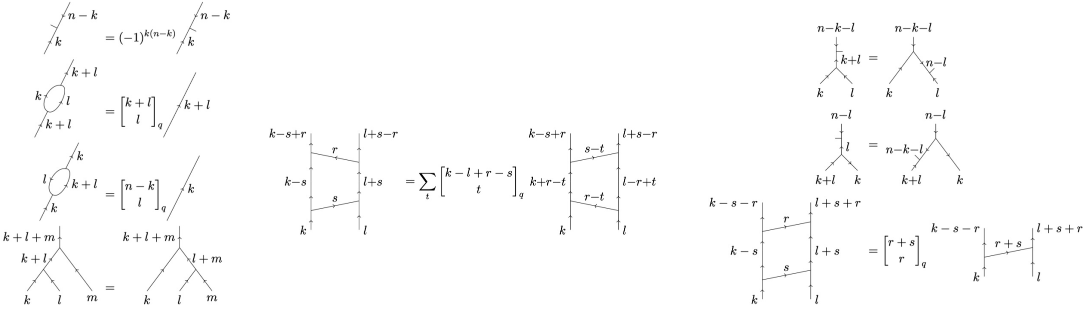


Figure: Web relations from 2014 CKM paper

Getting an invariant vector from a CKM web

To construct a web vector from the CKM web graph:

1. Morsify the graph.
2. Find all binary labelings.
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4. The web vector is the state sum over all binary labelings.

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- If you are a topologist, you don't mind Morsifying.
- If you want a diagrammatic categorification that preserves information about duals $V \leftrightarrow V^*$ then you want something like tags.

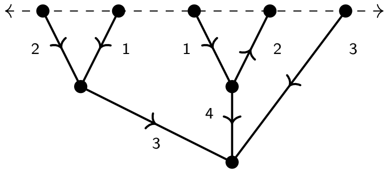
Fontaine web conventions

An $\mathfrak{sl}_n$ web is a plane graph with

- univalent boundary vertices along a top axis and
- trivalent internal vertices.
- Each edge is oriented and weighted with an element of $\{1, \ldots, n-1\}$.
- At each trivalent vertex, the sum of incoming edge weights minus the sum of outgoing edge weights is divisible by n:

flow is preserved mod n at each vertex

An $\mathfrak{sl}_5$ web:



Relations for Fontaine webs: strandings

Definition: Given a directed edge $u \mapsto v$ labeled k, a (valid) stranding of the edge is a choice of $0 < i_1 < i_2 < i_3 < \cdots < i_j < n$ with

$$i_j - i_{j-1} + i_{j-2} - i_{j-3} + \cdots + (-1)^{j-1} i_1 \in \{k, n-k\}$$

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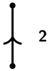
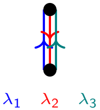
The strands are denoted $\lambda_{i_1}, \ldots, \lambda_{i_j}$ and drawn as colored directed paths on $u \mapsto v$ to distinguish them from each other and from the edge $u \mapsto v$.

- If the alternating sum is k then the strands directed *with* the edge $u \mapsto v$ are associated to *positive* coefficients (respectively against and negative).
- If the alternating sum is $n-k$ then these signs are reversed: strands directed *against* the edge are associated to *positive* coefficients (respectively with and negative).

Example:



$= 1 - 2 + 3$



Relations for Fontaine webs: strandings

Definition: A (valid) stranding of a web graph is a collection of directed paths with labels in $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$ (called strands) with the properties that:

- There is a valid stranding on each edge.
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Relations for Fontaine webs: strandings

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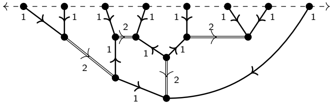
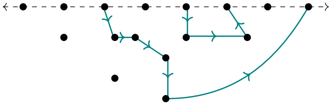
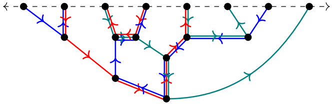
This means the set of all strands labeled λ_i form a directed noncrossing matching on a subset of the web's boundary vertices (possibly with closed, oriented loops in the interior of the web).

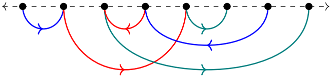




Relations for Fontaine webs: strandings

Together, strands give a multicolored, directed noncrossing matching that "saturates" each edge of the graph.





Strandings and web vectors: the boundary vector

- Given a valid stranding of a web graph, label each boundary vertex v with the binary vector whose $i, i+1$ entries are given by how λ_i runs along the edge incident to v:

$$\begin{cases} 01 & \text{if } \lambda_i \text{ is directed with the edge} \\ 10 & \text{if } \lambda_i \text{ is directed opposite the edge} \\ \text{equal} & \text{if } \lambda_i \text{ is not on the edge} \end{cases}$$

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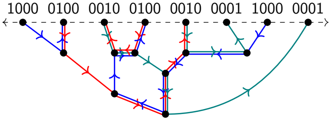

$$\mapsto x_1 \otimes x_2 \otimes x_3 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_1 \otimes x_4$$

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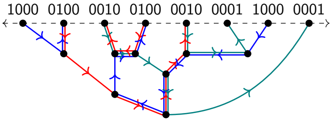

$$\mapsto x_1 \otimes x_2 \otimes x_3 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_1 \otimes x_4$$

Fact: This uniquely specifies a binary vector with the same weight as the edge and a basis vector in the desired quantum tensor space. This is the boundary vector of the stranding.

Strandings and web vectors: the coefficient

Definition: An edge in a stranded web graph is an (i,j) flow edge if the edge has an odd number of strands λ_ℓ satisfying $i \leq \ell < j$.

Example: The $(i, i+1)$ flow edges are exactly the edges with strand λ_i.

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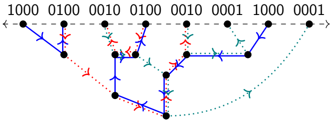
Definition: The (i,j) flows in a stranded web graph are the connected components of the subgraph formed by only considering (i, j) flow edges.

Fact: All (i, j) flows are directed paths in the web graph that either start and end at the boundary, or form a closed loop in the interior of the graph.

Strandings and web vectors: the coefficient

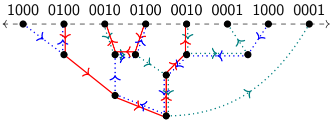
$(1,2)$ flows

1 clockwise,
1 counter



$(2,3)$ flows

1 clockwise
1 counter



$(3,4)$ flows

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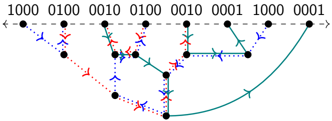
2 counter



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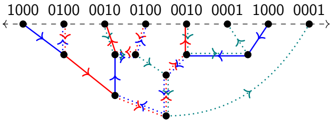

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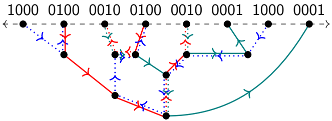

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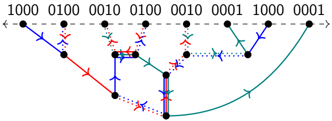

Strandings and web vectors

Theorem [Russell–T]: Fix a web graph G and let $\mathcal{S}$ be the set of all valid strandings. Denote the boundary vector of the stranding $S \in \mathcal{S}$ by $\vec{b}_S$. Let $a(S)$ be the number of closed clockwise flows and $b(S)$ be the number of counterclockwise flows. Then the web vector corresponding to G is

$$\vec{v}_G = \sum_{S \in \mathcal{S}} (-q)^{a(S)-b(S)} \vec{b}_S$$

Previous example: $(-q)^{-8} \left(x_1 \otimes x_2 \otimes x_3 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_1 \otimes x_4\right)$

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Takeaway: Web graphs admit a global structure of strands. Web vectors can be read from web graphs by the number and direction of strands in each stranding. Relations are a natural consequence of the global structure of strands.

Some connections to other work

- **Khovanov-Kuperberg**: Use a state-sum over flows to generate $\mathfrak{sl}_3$ web vectors. Their flows are our $(1, 3)$ flows.

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- **Gaetz-Pechenik-Pfannerer-Striker-Swanson**: Give a web vector state sum over web labelings identifying the leading term using trip permutations. Trip permutations have $n-1$ *trip strands*, but they are not the same as our strands.

Applications

Using strandings, we prove:

- Nonvanishing/Positivity: If a web graph G has a valid stranding by S then the associated web vector w_G is nonvanishing in term x_S.
- Every web graph has a valid stranding/Base stranding: If G is a web graph, then there is a straightforward algorithm to produce a valid stranding of G by a particular S_0.
- Basis Criteria: We provide a condition of Fontaine's for a set of webs to form a web basis, without relying on Fontaine's notion of "coherent webs."
- $\mathfrak{sl}_n$ Web Bases: We produce a collection of web bases for $\mathfrak{sl}_n$ webs, extending Fontaine's web basis.
- Complete Set of Relations: We prove a complete, concise set of relations.
- Geometry of Springer fibers: A family of non-reduced webs parametrizes the *entries* of top-dimensional Springer Schubert cells.
- Evacuation "is" Reflection of Web Graphs: The tableau operation of evacuation corresponds to reflecting a web graph.

Application: Webs in geometry

The flag variety is G/B

If $G = GL_n(\mathbb{C})$ and B is upper-triangular matrices then each flag is

- … a coset gB
- … a nested subspace $V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$

- … a matrix with zeros to the right and below a permutation

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$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

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Each gB has a representative in exactly one of the following:

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These are the **Schubert cells** BwB/B. They are parametrized by permutation matrices w.

Application: Webs in geometry

Fix a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$

The Springer fiber of X consists of flags $gB = V_\bullet$ for which

- ... $g^{-1}Xg$ is upper-triangular
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For example: The Springer fiber of $X = 0$ is the full flag variety.

Springer fibers

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- ....$X^m = 0$ for some m
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Jordan blocks partition n. The conjugacy class of X has a unique representative in Jordan canonical form with blocks arranged in nondecreasing order.

Example: If $\lambda(X)$ has 2 rows then the Springer fiber is a 2-row Springer fiber.

Springer fibers

Theorem: (Springer's Representation)

- S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$
- The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible
- In fact $H^{top}(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$
- The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n

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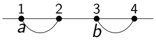
One consequence: The *number of components* of the Springer fiber $\lambda(X)$ equals the *dimension* of the irreducible representation of type $\lambda(X)$.

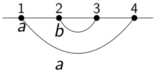
Concretely: If $\lambda(X)$ is a rectangle then the components of the Springer fiber are indexed by reduced webs.

2-row Springer fibers and webs

- **Theorem: [Goldwasser, Nadeem, Sun, T]** Reduced webs parametrize the cells of 2-row Springer fibers.

$$\left(\begin{array}{cccc} a & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{cccc} a & b & 1 & 0 \\ 0 & a & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$





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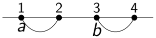
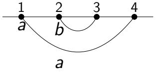




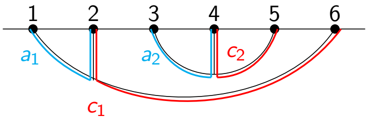
- **More Theorem:** In fact, closures of cells can be identified by cutting arcs appropriately.

3-row Springer fibers and strandings

We wanted to say the same for 3-row Springer fibers.

$$\left(\begin{array}{cccccc} b_1 & c_1 & b_2 & c_2 & 1 & 0 \\ 0 & 0 & b_1 + \star & c_1 & 0 & 1 \\ \hdashline a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\star = a_1 c_1 - a_2 c_1$$



3-row Springer fibers and strandings

We realized that a purple noncrossing matching accounts for the remaining variables.

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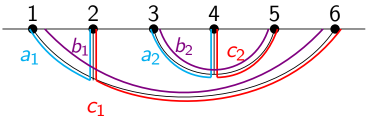

3-row Springer fibers and strandings

Emily Hafken, Veronica Lang, and Orit Tashman figured out that

- if we pull blue arcs below red arcs,
- then we can read $\star$ from how the purple arcs cross the blue-red combos

$$\begin{pmatrix} b_1 & c_1 & b_2 & c_2 & 1 & 0 \\ 0 & 0 & b_1+\star & c_1 & 0 & 1 \\ \hline a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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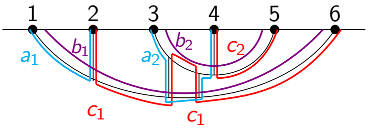

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Theorem [Hafken, Lang, Tashman, T-]: The top-dimensional cells of 3-row Springer fibers are parametrized by a family of (unreduced) webs, together with a stranding from the corresponding reduced web.

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- A similar result appears to hold for smaller-dimensional cells — but not all are affine.
- We believe that if this web is expressed as a linear combination of reduced webs, the reduced web corresponding to the component is the leading term.

Combinatorial connections: promotion and evacuation

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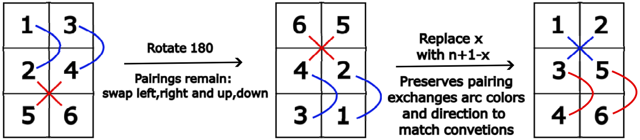
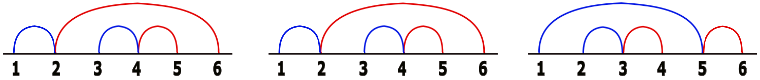
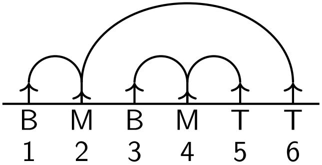
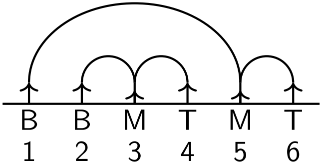
Theorem [Cowen-Seekamp-Spaulding-T] If T is a rectangular SYT and w_T is its corresponding stranded web then the stranded web corresponding to the evacuation of T is the reflection of w_T.

Example of evacuation









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- Strands also seem to encode geometric information. ¯_(ツ)_/¯

Overall takeaways

- Strands describe a global structure on web graphs that is more natural from a graph-theoretic point of view and more useful for computations.
- Strands also seem to encode geometric information. ¯_(ツ)_/¯
- The right technical tools can greatly expand our understanding of a problem.

Thank you for your attention!

Preprint available for Russell-T: arXiv:2510.12035

Thanks to the organizers!

Also to NSF and Budapest Semesters in Mathematics for supporting this work.