Quantum loop algebras and simple modules

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Quantum affine / loop algebras

- Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. We associate to it

  1. the quantum affine algebra

  $$U_q(\widehat{\mathfrak{g}})|_{c=1} = \mathbb{C}\Big\langle e_i, f_i, h_i \Big\rangle_{i \in I \sqcup \{0\}} \Big/ \Big(\text{relations}\Big)$$

  by cleverly adding a vertex 0 to the Dynkin diagram of $\mathfrak{g}$.

  2. the quantum loop algebra

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- The two algebras are isomorphic (Drinfeld and Beck, Damiani).

- But only $U_q(L\mathfrak{g})$ is defined when $\mathfrak{g}$ is a Kac-Moody Lie algebra.

Representations of quantum affine algebras

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$$\boldsymbol{\psi} = \left(\psi_i(z) = \sum_{k=0}^{\infty} \frac{\psi_{i,k}}{z^k} \right)_{i \in I} \in \mathcal{R}^I$$

where $\mathcal{R}$ is the set of power series expansions of rational functions (over $\mathbb{C}$) that are regular and non-zero at $z = 0$ and ∞.

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- Hernandez-Jimbo generalized this by defining an irreducible

$$U_q(L\mathfrak{g}) \curvearrowright L(\boldsymbol{\psi})$$

for any $\boldsymbol{\psi} \in \mathcal{R}^I$, which might be infinite-dimensional.

Decompositions into generalized eigenspaces

- Frenkel-Reshetikhin and Frenkel-Mukhin considered decompositions

$$L(\psi) = \bigoplus_{\boldsymbol{x}} L(\psi)_{\boldsymbol{x}}$$

as $\boldsymbol{x} = (x_{i1}, \dots, x_{in_i})_{i \in I} \subset \mathbb{C}^*$ goes over all multisets of non-zero complex numbers, and the integers $(n_i \geq 0)_{i \in I}$ are arbitrary.

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- Specifically, $L(\psi)_{\boldsymbol{x}}$ is the generalized eigenspace of the power series

$$\left(\sum_{k=0}^{\infty} \frac{h_{i,k}}{z^k} \right)_{i \in I} \in U_q(L\mathfrak{g})[[z^{-1}]]^I$$

acting on $L(\psi)$ according to the eigenvalues

$$\left(\psi_i(z) \prod_{j \in I} \prod_{a=1}^{n_j} \frac{z - x_{ja} q^{d_{ij}}}{z q^{d_{ij}} - x_{ja}} \right)_{i \in I}$$

$L(\psi)$ and q-characters

- The generating series of dim $L(\psi)_{\boldsymbol{x}}$ is called a q-character. These [1] form a very rich subject, involving deep connections to many fields:

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  - cluster algebras: Hernandez-Leclerc observed that the q-characters of the various $L(\psi)$ generate a cluster algebra;
  - monoidal categorification that implies the above cluster algebra statement (Kang, Kashiwara, Kim, Oh, Park, Qin).

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- Though elegant and useful, it is not very explicit and doesn't readily generalize [2] to the case of Kac-Moody Lie algebras $\mathfrak{g}$.
- In what follows, we propose an explicit construction of $L(\psi)$, for which we will present geometric and categorical interpretations.

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- We will now emulate this procedure for quantum loop algebras.

Definition of quantum loop algebras

- For a Kac-Moody algebra $\mathfrak{g}$ with Cartan matrix $\left(\frac{2d_{ij}}{d_{ii}} \in \mathbb{Z}\right)_{i,j \in I}$, let

$$U_q(L\mathfrak{g}) = \mathbb{C}\Big\langle e_{i,k}, f_{i,k}, h_{i,k} \Big\rangle_{i \in I, k \in \mathbb{Z}} \Big/ \Big(h\text{'s commute and relations below}\Big)$$

$$\begin{cases} e_i(x)e_j(y)\zeta_{ji}\left(\frac{y}{x}\right) = e_j(y)e_i(x)\zeta_{ij}\left(\frac{x}{y}\right) \\ e_i(x)h_j^{\pm}(y)\zeta_{ji}\left(\frac{y}{x}\right) = h_j^{\pm}(y)e_i(x)\zeta_{ij}\left(\frac{x}{y}\right) \\ \text{higher order relations among } e\text{'s, known for } \mathfrak{g} \begin{cases} \text{finite type (Drinfeld)} \\ \text{simply-laced (Neguț)} \end{cases} \\ \text{opposite relations with } e\text{'s replaced by } f\text{'s} \\ \Big[e_i(x), f_j(y)\Big] = \delta_{ij}\delta(x/y)\Big(h_i^{+}(x) - h_i^{-}(y)\Big) \end{cases}$$

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- Above, set $\zeta_{ij}(x) = \frac{x - q^{-d_{ij}}}{x-1}$, $\delta(x) = \sum_{k\in\mathbb{Z}} x^k$ and write $\forall i \in I$

$$e_i(x) = \sum_{k\in\mathbb{Z}} \frac{e_{i,k}}{x^k}, \qquad f_i(x) = \sum_{k\in\mathbb{Z}} \frac{f_{i,k}}{x^k}, \qquad h_i^{\pm}(x) = h_{i,0}^{\pm1} + \sum_{k=1}^{\infty} \frac{h_{i,\pm k}}{x^{\pm k}}$$

Shuffle algebras

- The subalgebra $U_q^+(L\mathfrak{g}) \subset U_q(L\mathfrak{g})$ generated by the $e_{i,d}$'s has a shuffle algebra model, introduced by Feigin-Odesskii and Enriquez:

$$\mathcal{V} = \bigoplus_{(n_i \geq 0)_{i \in I}} \frac{\mathbb{C}[z_{i1}^{\pm 1}, z_{i2}^{\pm 1}, \dots, z_{in_i}^{\pm 1}]_{i \in I}^{\text{symmetric}}}{\prod_{(i,a),(j,b) \text{ with } i \neq j}^{\text{unordered pairs}} (z_{ia} - z_{jb})}$$

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- The algebra structure on $\mathcal{V}$ is given by the shuffle product

$$E(z_{i1}, \dots, z_{in_i})_{i \in I} * E'(z_{i1}, \dots, z_{in'_i})_{i \in I} =$$

$$\text{Sym}\left[E(z_{i1}, \dots, z_{in_i})_{i \in I} E'(z_{i,n_i+1}, \dots, z_{i,n_i+n'_i})_{i \in I} \prod_{a \leq n_i, b > n_j}^{i,j \in I} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right) \right]$$

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- Sending $e_{i,k} \rightsquigarrow z_{i1}^k$ yields a monomorphism $\Upsilon : U_q^+(L\mathfrak{g}) \hookrightarrow \mathcal{V}$. Let

$$\text{Im } \Upsilon =: \boxed{\mathcal{S} \cong U_q^+(L\mathfrak{g})}$$

The construction of $L(\psi)$

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- Then we show that the following yields a "Shapovalov form"

$$\mathcal{S} \otimes \mathcal{S} \xrightarrow{\langle \cdot, \cdot \rangle_{\boldsymbol{\psi}}} \mathbb{C}$$

$$\Big\langle E, E' \Big\rangle_{\boldsymbol{\psi}} = \oint \frac{E(z_{i1}, \dots, z_{in_i})_{i \in I} E'(z_{i1}, \dots, z_{in_i})_{i \in I}}{\prod_{(i,a) \neq (j,b)} \zeta_{ij} \left(\frac{z_{ia}}{z_{jb}} \right)} \prod_{(i,a)} \psi_i(z_{ia})$$

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where $\oint$ denotes a certain explicit contour integral.

- **Theorem** (Neguț): For any ℓ-weight ψ,

$$L(\psi) = \mathcal{S} \Big/ \text{Ker } \langle \cdot, \cdot \rangle_{\psi}$$

is an irreducible representation of $U_q(L\mathfrak{g})$.

Connection with geometry, following Hernandez-Leclerc

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- In a seminal result, Hernandez-Leclerc proved that for so-called Kirillov-Reshetikhin modules $L(\psi)$, we have for all $\mathbf{x}$

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- $N^{\text{stab}}_{\boldsymbol{x},\psi}$ is defined using the quiver with vertex set $I \times \mathbb{C}^*$ and arrows

$$(i, z) \mapsto (j, zq^{-d_{ij}}), \qquad \forall i, j \in I, \ z \in \mathbb{C}^*$$

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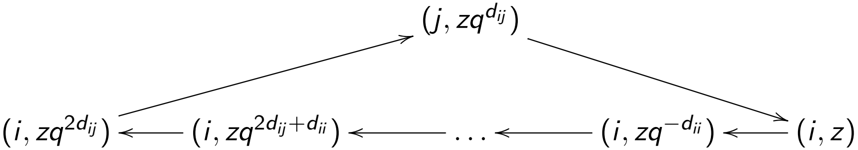
- We have the function $f : M^{\text{stab}}_{\boldsymbol{x},\boldsymbol{\psi}} \to \mathbb{C}$ obtained by summing the traces of the compositions of quiver arrows ($\forall i \neq j \in I,\ z \in \mathbb{C}^*$)



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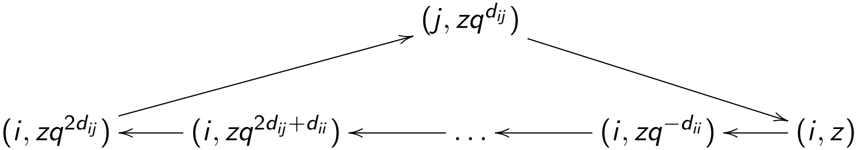

- **Conjecture** (Neguț): for any ℓ-weight $\boldsymbol{\psi}$ and all $\boldsymbol{x}$, we have

$$L(\boldsymbol{\psi})_{\boldsymbol{x}} \cong \left(\text{Critical } K\text{-theory of } (M^{\text{stab}}_{\boldsymbol{x},\boldsymbol{\psi}}, f)\right)$$

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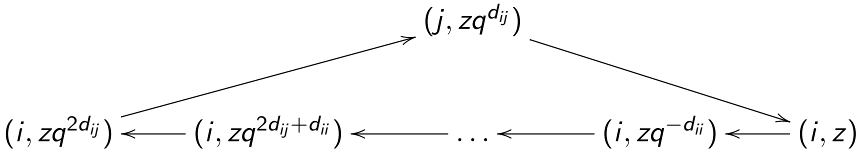

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- This is inspired by the case of Kirillov-Reshetikhin $L(\boldsymbol{\psi})$ studied by Varagnolo-Vasserot, plus the fact that $\text{Crit}(M^{\text{stab}}_{\boldsymbol{x},\boldsymbol{\psi}}, f) = N^{\text{stab}}_{\boldsymbol{x},\boldsymbol{\psi}}$.

Connection with categorification

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- If one developed an explicit (ideally diagrammatic) presentation of these categories, one would accomplish for $U_q(L\mathfrak{g})$ what Khovanov, Lauda, Rouquier and Webster did for $U_q(\mathfrak{g})$ and its representations.

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$$\Big(\text{the derived factorization category of } (M^{\text{stab}}_{\boldsymbol{x},\psi}, f)\Big) \qquad (1)$$

- Following Varagnolo-Vasserot, $U_q(L\mathfrak{g})$ is also similarly categorified (the so-called derived Hall algebra, introduced by Pădurariu).

- If one developed an explicit (ideally diagrammatic) presentation of these categories, one would accomplish for $U_q(L\mathfrak{g})$ what Khovanov, Lauda, Rouquier and Webster did for $U_q(\mathfrak{g})$ and its representations.

- As shown above, the Grothendieck groups of the categories (1) admit explicit descriptions as shuffle algebras. This is one of our main motivations for wanting to understand the latter explicitly.