Real roots of non-centered random polynomials

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Introduction

Random polynomials

A random (algebraic) polynomial is typically given by

$$P_n(x) := \sum_{j=0}^{n} \omega_j c_j x^j,$$

where c_j are deterministic constants and ω_j are independent real-valued random variables with unit variance.

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where c_j are deterministic constants and ω_j are independent real-valued random variables with unit variance.

Examples

Kac polynomials: $c_j = 1$;

Hyperbolic polynomials: $c_j = \sqrt{\frac{L(L+1)\cdots(L+j-1)}{j!}}$ for $L > 0$;

Elliptic polynomials: $c_j = \sqrt{\binom{n}{j}}$;

Weyl polynomials: $c_j = \frac{1}{\sqrt{j!}}$.

The number of real roots of random polynomials

The number of random roots

For $I \subset \mathbb{R}$, the number of real roots $N_{P_n}(I)$ of $P_n(x)$ in I (counting multiplicities) is a random variable with possible values in $\{0, 1, ..., n\}$.

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Some key problems

As $n \to \infty$, study:

1. The asymptotics of $\mathbb{E}[N_{P_n}(I)]$.
2. The asymptotics of $\mathrm{Var}[N_{P_n}(I)]$.
3. Asymptotic normality (CLT): $\dfrac{N_{P_n}(I) - \mathbb{E}[N_{P_n}(I)]}{\sqrt{\mathrm{Var}[N_{P_n}(I)]}} \xrightarrow{d} \mathcal{N}(0,1)$?
4. Concentration: $\mathbb{P}(|N_{P_n}(I) - \mathbb{E}[N_{P_n}(I)]| > \varepsilon \mathbb{E}[N_{P_n}(I)]) \leq$?
5. Almost sure asymptotics (SLLN): $\dfrac{N_{P_n}(I)}{\mathbb{E}[N_{P_n}(I)]} \xrightarrow{a.s.} 1$?

Known results for centered Kac polynomials

- Bloch and Pólya (1932) showed that if ω_j take values in $\{-1, 0, 1\}$ with equal probability, N_{P_n} is at most $O(\sqrt{n})$ with high probability.
- Littlewood and Offord (1943, 1945, 1948) refined this bound to powers of $\log n$.
- Kac (1943, 1949) proved that for random polynomials with i.i.d. centered Gaussian or uniform coefficients,

$$\mathbb{E}[N_{P_n}] = \left(\frac{2}{\pi} + o(1)\right) \log n.$$

- Erdős and Offord (1956) established this asymptotic result for Rademacher coefficients.
- Ibragimov and Maslova (1968, 1971) generalized it to i.i.d. coefficients in the domain of attraction of the normal law.
- Universality methods have advanced significantly, extending to non-i.i.d. settings with bounded $(2 + \varepsilon_0)$-moments (see Do-Nguyen-Nguyen-Vu 2016, Nguyen-Nguyen-Vu 2016, etc.).

- In the 1970s, Maslova introduced two fundamental themes that remain central to the field: the variance asymptotics,

$$\mathrm{Var}[N_{P_n}] = \left[\frac{4}{\pi}\left(1 - \frac{2}{\pi}\right) + o(1)\right] \log n,$$

  and the asymptotic normality (or CLT) of N_{P_n},

$$\frac{N_{P_n} - \mathbb{E}[N_{P_n}]}{\sqrt{\mathrm{Var}[N_{P_n}]}} \xrightarrow{d} \mathcal{N}(0,1),$$

  where $\xrightarrow{d}$ denotes convergence in distribution.
- Concentration inequalities were established by Can-Nguyen (2025), while SLLNs were studied by Do (2025).
- Despite these advances, much less is known about the behavior of N_{P_n} when the coefficients ω_j are non-centered, a case that naturally arises in many settings.

Why should we consider non-centered random polynomials?

- One approach is to begin with a deterministic polynomial $M_n(x)$, add a random noise polynomial $R_n(x)$, and study the perturbed polynomial

$$P_n(x) = M_n(x) + R_n(x).$$

This fits naturally into **perturbation theory**, where deterministic coefficients often arise from data or computation and may be truncated due to limited precision, introducing inherent noise into the system.

- An analogous situation arises in **regression analysis**. Consider, for example, a regression model of the form

$$M_n(x) + \mathcal{E},$$

where $M_n(x)$ is the true polynomial regression curve and $\mathcal{E}$ is the error term assumed to follow a centered Gaussian distribution. The least-squares estimate of $M_n(x)$ gives rise to a non-centered random polynomial $P_n(x)$ that satisfies the property $\mathbb{E}[P_n(x)] = M_n(x)$.

Non-centered Kac polynomials and open questions

Theorem (Ibragimov and Maslova, 1971)

Let $P_n(x) = \sum_{j=0}^{n} \omega_j x^j$, where ω_j are i.i.d. random variables belonging to the domain of attraction of the normal law such that $\mathbb{P}(\omega_j = 0) = 0$ and $\mathbb{E}[\omega_j] = \mu \neq 0$. Then, as $n \to \infty$,

$$\mathbb{E}[N_{P_n}] = \frac{1}{\pi} \log n + o(\log n).$$

- Do-Nguyen-Vu (2018) and Do (2021) extended and refined this result for a broader class of non-centered generalized Kac polynomials.

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Open questions

1. What is the asymptotic behavior of $\mathrm{Var}[N_{P_n}]$?
2. Does N_{P_n} satisfy a CLT?
3. Can similar results be obtained for other classes of random polynomials?

Main results (Do, N., O'Rourke, 2025)

Theorem (Real roots of non-centered Kac polynomials)

Let $P_n(x) = \sum_{j=0}^n \omega_j x^j$, where ω_j are independent real-valued random variables, with

$$\mathbb{E}[\omega_j] = \mu, \quad \mathrm{Var}[\omega_j] = 1, \quad \text{and} \quad \mathbb{E}[|\omega_j|^{2+\varepsilon_0}] < C_0$$

for all j and some constants $\mu \neq 0$, $\varepsilon_0 > 0$, and $C_0 > 0$. Then, as $n \to \infty$,

$$\mathrm{Var}[N_{P_n}] = \frac{2}{\pi}\left(1 - \frac{2}{\pi}\right)\log n + o(\log n).$$

Furthermore, N_{P_n} satisfies the CLT; that is, as $n \to \infty$, we have the following convergence in distribution:

$$\frac{N_{P_n} - \mathbb{E}[N_{P_n}]}{\sqrt{\mathrm{Var}[N_{P_n}]}} \xrightarrow{d} \mathcal{N}(0,1).$$

Real roots of Kac polynomials with non-centered i.i.d. coefficients

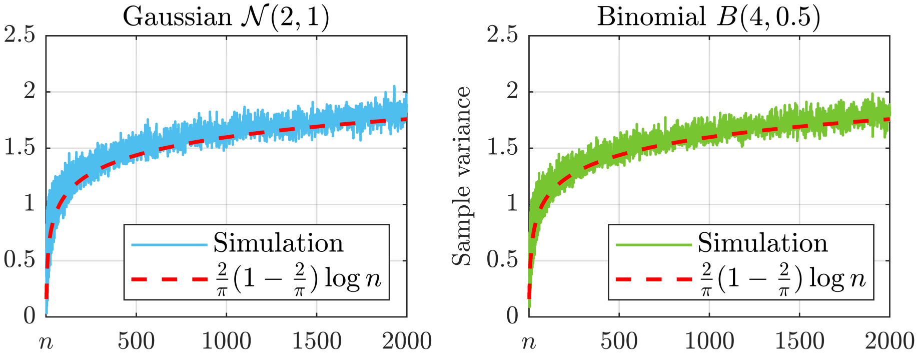


Figure: Plots of sample variances of N_{P_n} versus the degree n.

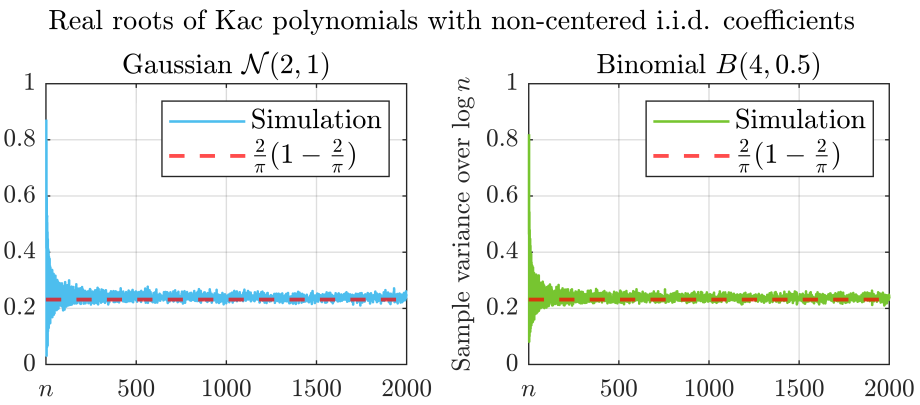


Figure: The sample variances of N_{P_n}, normalized by $\log n$, are approaching the limit $\frac{2}{\pi}\left(1-\frac{2}{\pi}\right)$.

Real roots of Kac polynomials with non-centered i.i.d. coefficients

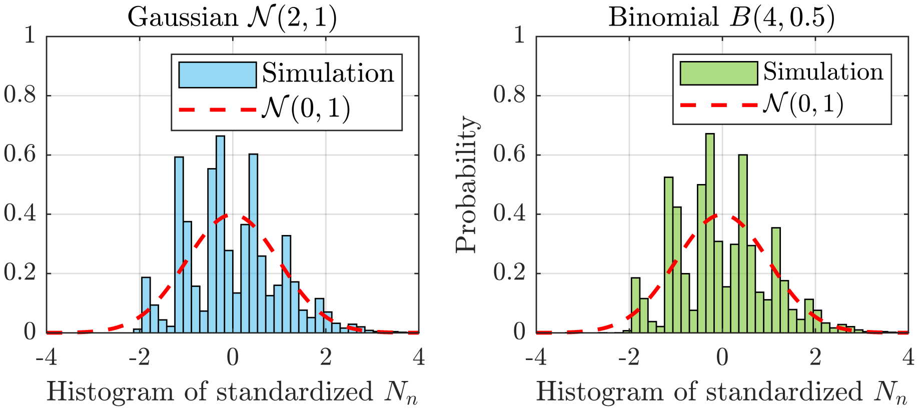


Figure: The histograms of the standardized N_{P_n} are plotted alongside the probability density function of the standard normal distribution.

Theorem (Non-centered hyperbolic polynomials and their derivatives)

Fix $L > 0$. Consider the random hyperbolic polynomials $P_{n,L}$ defined by

$$P_{n,L}(x) := \omega_0 + \sqrt{L}\omega_1 x + \cdots + \sqrt{\frac{L(L+1)\cdots(L+n-1)}{n!}}\omega_n x^n,$$

where ω_j are independent real-valued random variables with

$$\mathbb{E}[\omega_j] = \mu, \quad \mathrm{Var}[\omega_j] = 1, \quad \text{and} \quad \mathbb{E}[|\omega_j|^{2+\varepsilon_0}] < C_0$$

for some constants $\mu \neq 0$, $\varepsilon_0 > 0$, and $C_0 > 0$. For any integer $\ell \geq 0$, let $N_{n,\ell}$ represent the number of real roots of the ℓth derivative of $P_{n,L}$. Then, there exists a constant $\kappa = \kappa(L, \ell)$ such that

$$\mathrm{Var}[N_{n,\ell}] = \left[\kappa + \frac{1}{\pi}\left(1 - \frac{2}{\pi}\right) + o(1)\right] \log n. \tag{1}$$

Moreover, $N_{n,\ell}$ satisfies the CLT.

The general setting: Technical ingredients

Let
$$P_n(x) = \sum_{j=0}^{n} \omega_j x^j.$$

We say that the coefficients ω_j of P_n have **polynomial growth** of order τ if we can write (for each j)
$$\omega_j = m_j + v_j \xi_j,$$

where m_j, v_j are deterministic and ξ_j are independent real-valued random variables with zero mean and unit variance, such that the following holds for some constants $\varepsilon_0 > 0$, $C_0 > 0$, and $N_0 > 0$:

A1) $\mathbb{E}[|\xi_j|^{2+\varepsilon_0}] < C_0$ for all $0 \le j \le n$;

A2) $|m_j|, |v_j| \le C_0(1+j)^\tau$ for all $0 \le j \le n$; and

A3) $|v_j| \ge \frac{1}{C_0}(1+j)^\tau$ for $N_0 \le j \le n$.

Note that we do not require the exact values of the mean m_j and variance v_j^2 of ω_j.

- We decompose the polynomial $P_n(x)$ into its deterministic and random components:

$$P_n(x) = M_n(x) + R_n(x),$$

where

$$M_n(x) := \mathbb{E}[P_n(x)] = \sum_{j=0}^{n} m_j x^j \quad \text{and} \quad R_n(x) := \sum_{j=0}^{n} v_j \xi_j x^j.$$

- Here, $R_n(x)$ is the centered counterpart of $P_n(x)$, whose real roots are well understood (see Do-Nguyen-Vu 2018, Do 2021, Nguyen-Vu 2022, and Do-N. 2025).

- For $|x| \geq 1$, $R_n(x)$ diverges as $n \to \infty$. In this regime, we instead consider the reciprocal polynomial $P_n^*(x) := x^n P_n(1/x)$, and compare its deterministic part $M_n^*(x) = \mathbb{E}[P_n^*(x)]$ with the random part $R_n^*(x) = P_n^*(x) - M_n^*(x)$.

Domination

- Let $k \geq 0$ be an integer. We say that M_n dominates R_n on a set $J \subset \mathbb{R}$ up to order k with factor function ϕ if, for all $i = 0, 1, \ldots, k$,

$$|M_n^{(i)}(x)| \gg \phi(1 - |x| + 1/n)\sqrt{\mathrm{Var}[R_n^{(i)}(x)]}, \quad x \in J \cap [-1, 1],$$
$$|M_n^{*\,(i)}(x)| \gg \phi(1 - |x| + 1/n)\sqrt{\mathrm{Var}[R_n^{*\,(i)}(x)]}, \quad x \in J \cap [-1, 1].$$

  If these inequalities are reversed, we say that M_n is dominated by R_n on J up to order k with factor function ϕ.

- We define J as an enlargement of the interval $I = (a, b)$ if it results from a slight extension of I to both the left and right, i.e., there exists a constant $c > 0$ such that the right extension is of length at least $c(|1 - |b|| + 1/n)$, and the left extension is of length at least $c(|1 - |a|| + 1/n)$.

Comparison principles (Do, N., O'Rourke, 2025)

Theorem (Comparison principles for variance)

Assume the coefficients of P_n have polynomial growth of order $\tau > -1/2$. There is a constant $C > 0$ such that the following holds for any $0 < d < 1$. Let $I_n \subset \mathbb{R}$ be an interval and let J_n be an enlargement of I_n.

1. If M_n dominates R_n on J_n with factor function $C\sqrt{|\log(x)|}$, then
$$\mathrm{Var}[N_{P_n}(I_n)] = O(\log^d n).$$
2. If M_n is dominated by R_n on J up to order 2 with a suitable factor function $\phi : [0,1] \to [0,1]$, then
$$\mathrm{Var}[N_{P_n}(I_n)] = \mathrm{Var}[N_{R_n}(I_n)] + O(\log^d n).$$

In applications, we consider intervals I_n of length $o(1)$ and choose $\phi(x) = x^\epsilon$ for some $\epsilon > 0$. In this case, we obtain $\phi_{R_n}(I_n) = o(\log n)$.

Comparison principles (Do, N., O'Rourke, 2025)

Theorem (Comparison principles for CLT)

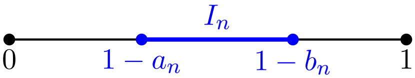
Assume the coefficients of P_n have polynomial growth of order $\tau > -1/2$. Let $I_n \subset \mathbb{R}$ be an interval and let J_n be an enlargement of I_n. If M_n is dominated by R_n on J up to order n with a suitable factor function $\phi : [0,1] \to [0,1]$ and $\mathrm{Var}[N_{R_n}(I_n)] \geq \epsilon \log n$ for some constant $\epsilon > 0$, then $N_{P_n}(I_n)$ satisfies the CLT; that is, as $n \to \infty$,

$$\frac{N_{P_n}(I_n) - \mathbb{E}[N_{P_n}(I_n)]}{\sqrt{\mathrm{Var}[N_{P_n}(I_n)]}} \xrightarrow{d} \mathcal{N}(0,1).$$

These comparison principles apply to Kac and hyperbolic polynomials, their derivatives, and any linear combinations thereof.

Universality: Reduction to the Gaussian case in core regions

- Let $0 \leq b_n < a_n = o((\log n)^{-A}),\ \forall A > 0.$



$\mathcal{I}_n := I_n \cup (-I_n) \cup I_n^{-1} \cup (-I_n^{-1})$

- We extend the universality results of Nguyen and Vu (2022) to the non-centered setting.

Theorem (Universality: a general result)

Assume either M_n dominates R_n on J with a factor function of $\sqrt{\log x}$, or M_n is dominated by R_n on J up to order 2 with a factor function of 1. There exist constants $C, c > 0$ such that for all sufficiently large n, the following holds for any function $\varphi : \mathbb{R} \to \mathbb{R}$ whose derivatives up to order 3 are bounded by 1:

$$|\mathbb{E}[\varphi(N_{P_n}(I \cap \mathcal{I}_n))] - \mathbb{E}[\varphi(N_{P_{n,G}}(I \cap \mathcal{I}_n))]| \leq C\left(a_n^c + n^{-c}\right), \qquad (2)$$

where $P_{n,G}$ denotes the Gaussian counterpart of P_n.

Corollary (Universality for moments and variance)

Let $k \geq 1$ be an integer. There exist positive constants C and c such that for all sufficiently large n,

$$|\mathbb{E}[N^k_{P_n}(I \cap \mathcal{I}_n)] - \mathbb{E}[N^k_{P_{n,G}}(I \cap \mathcal{I}_n)]| \leq C\left(a_n^c + n^{-c}\right). \quad (3)$$

Consequently, for $k = 2$,

$$|\operatorname{Var}[N_{P_n}(I \cap \mathcal{I}_n)] - \operatorname{Var}[N_{P_{n,G}}(I \cap \mathcal{I}_n)]| \leq C\left(a_n^c + n^{-c}\right). \quad (4)$$

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Consequently, for $k = 2$,

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Theorem (Real roots outside core intervals)

Assume that there is domination for all order. Then for every $k \geq 2$, there exists a finite positive constant C_k such that

$$\mathbb{E}[N^k_{P_n}(I \backslash \mathcal{I}_n)] \leq C_k \left(|\log a_n|^{2k} + \log_+^k(nb_n)\right),$$

where $\log_+(x) := \max(0, \log(x))$. Furthermore, if there is only domination up to order 2, the above estimate holds for $k = 1$ and $k = 2$.

Kac–Rice formulas for non-centered Gaussian processes

- Let $\mathcal{G} = \{G(x)\}_{x \in I}$ be a Gaussian process on an interval $I \subset \mathbb{R}$ with mean
$$m(x) := \mathbb{E}[G(x)] \tag{5}$$
and covariance
$$r(x,y) := \mathrm{Cov}[G(x), G(y)]. \tag{6}$$
We normalize $G(x)$ so that $r(x,x) = \mathrm{Var}[G(x)] = 1$ for every $x \in I$.

- Additionally, we assume that $\mathcal{G}$ satisfies the following hypotheses:
  - G1) C^1-paths: $\mathcal{G}$ has C^1-paths, with $m \in C^1(I)$ and $r \in C^2(I^2)$.
  - G2) Nondegeneracy: The joint distribution of $G(x)$, $G(y)$, $G'(x)$, and $G'(y)$ is nondegenerate for distinct $x, y \in I$.
  - G3) Gaussianity: The joint distribution of $G(x)$ and $G'(x)$ follows a Gaussian distribution for all $x \in I$.

- Let $N_G(I)$ denote the number of real zeros of G inside $I \subset \mathbb{R}$.
- Let $\rho_1(x)$ and $\rho_2(x,y)$ be the 1-point and 2-point correlation functions of the real roots of G. Then

$$\mathbb{E}[N_G(I)] = \int_I \rho_1(x)dx$$

and

$$\mathbb{E}[N_G(I)(N_G(I)-1)] = \iint_{I^2} \rho_2(x,y)dxdy.$$

- This gives

$$\mathrm{Var}[N_G(I)] = \iint_{I^2} [\rho_2(x,y) - \rho_1(x)\rho_1(y)]dxdy + \int_I \rho_1(x)dx.$$

- Using the Kac–Rice formulas, we establish exact formulas for $\rho_1(x)$ and $\rho_2(x,y)$.

Theorem (2-point correlation function)

Let $\{G(x)\}_{x\in I}$ be a Gaussian process satisfy hypotheses (G1)–(G3). Define

$$\mu_1 := m'(x) + \frac{r(x,y)r_{10}(x,y)}{1-r^2(x,y)}m(x) - \frac{r_{10}(x,y)}{1-r^2(x,y)}m(y), \tag{7}$$

$$\sigma_1 := \sqrt{r_{11}(x,x) - \frac{r_{10}^2(x,y)}{1-r^2(x,y)}}, \tag{8}$$

$$\mu_2 := m'(y) + \frac{r(x,y)r_{01}(x,y)}{1-r^2(x,y)}m(y) - \frac{r_{01}(x,y)}{1-r^2(x,y)}m(x), \tag{9}$$

$$\sigma_2 := \sqrt{r_{11}(y,y) - \frac{r_{01}^2(x,y)}{1-r^2(x,y)}}, \tag{10}$$

$$\delta(x,y) := \frac{1}{\sigma_1\sigma_2}\left[r_{11}(x,y) + \frac{r(x,y)r_{10}(x,y)r_{01}(x,y)}{1-r^2(x,y)}\right], \tag{11}$$

and let $\nu_1 = \mu_1/\sigma_1$ and $\nu_2 = \mu_2/\sigma_2$.

Then, we obtain the following formula for the 2-point correlation function $\rho_2(x, y)$ of the real roots of G when $x \neq y$:

$$\rho_2(x,y) = \frac{E(x,y)}{\pi^2\sqrt{1-r^2(x,y)}}\left(\sum_{i=1}^{5}\rho_{2,i}(x,y)\right), \tag{12}$$

where

$$\begin{aligned}
E(x,y) &= \exp\left(-\frac{m^2(x)-2r(x,y)m(x)m(y)+m^2(y)}{2(1-r^2(x,y))}\right),\\
\rho_{2,1} &= \sigma_1\sigma_2\left(\sqrt{1-\delta^2}+\delta\arcsin\delta\right),\\
\rho_{2,2} &= \mu_1\mu_2\arcsin\delta,\\
\rho_{2,3} &= \sigma_1\sigma_2\sqrt{1-\delta^2}\sum_{j=1}^{2}\int_0^1\frac{(1-t)\nu_j^2}{\exp(\nu_j^2t^2/(2-2\delta^2))}dt\\
&\quad+\sqrt{\frac{\pi}{2}}\sigma_1\sigma_2|\delta|\sum_{j=1}^{2}\nu_j^2\int_0^1\frac{(1-t)|\nu_j t|}{\exp(\nu_j^2t^2/2)}\operatorname{erf}\left(\frac{|\delta\nu_j t|}{\sqrt{2(1-\delta^2)}}\right)dt,
\end{aligned}$$

$$\rho_{2,4} = \sqrt{\frac{\pi}{2}}\mu_1\mu_2 \sum_{j=1}^{2} \int_0^1 \frac{(t-1)\nu_j}{\exp(\nu_j^2 t^2/2)} \operatorname{erf}\left(\frac{\delta\nu_j t}{\sqrt{2(1-\delta^2)}}\right) dt,$$

$$\rho_{2,5} = \frac{|\mu_1\mu_2\nu_1\nu_2|}{\sqrt{1-\delta^2}} \int_0^1 \int_0^1 (1-t)(1-s) \exp\left(-\frac{\nu_1^2 t^2 - 2\delta\nu_1\nu_2 ts + \nu_2^2 s^2}{2(1-\delta^2)}\right) dtds.$$

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Lemma (1-point correlation function, see also Leadbetter-Cryer, 1965)

The density $\rho_1(x)$ of real roots of $G(x)$ is given by

$$\rho_1(x) = \rho_{1,1}(x) + \rho_{1,2}(x), \tag{13}$$

where

$$\rho_{1,1}(x) = \frac{1}{\pi}\sqrt{r_{11}(x,x)} \exp\left(-\frac{1}{2}\left[m^2(x) + \frac{(m'(x))^2}{r_{11}(x,x)}\right]\right), \tag{14}$$

$$\rho_{1,2}(x) = \frac{1}{\sqrt{2\pi}} m'(x) \exp\left(-\frac{1}{2}m^2(x)\right) \operatorname{erf}\left(\frac{m'(x)}{\sqrt{2r_{11}(x,x)}}\right). \tag{15}$$

The Gaussian cases

- Let $P_n(x) = \sum_{j=0}^{n}(m_j + v_j\xi_j)x^j$ and $P_n^*(x) = x^n P_n(1/x)$, where ξ_j are i.i.d. $\mathcal{N}(0,1)$ random variables.
- We apply the previous Kac–Rice formulas to the Gaussian processes

$$G(x) = \frac{P_n(x)}{\sqrt{\mathrm{Var}[P_n(x)]}} \quad \text{and} \quad G^*(x) = \frac{P_n^*(x)}{\sqrt{\mathrm{Var}[P_n^*(x)]}}.$$

- Let $m(x) = \mathbb{E}[G(x)]$ and $m^*(x) = \mathbb{E}[G^*(x)]$. Let $a_n = e^{-\log^d n} = \frac{1}{nb_n}$, $I_n = [1-a_n, 1-b_n)$, $U_n = I_n \cup (-I_n)$, and $U_n^* = I_n^{-1} \cup (-I_n^{-1})$.

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Theorem (Real roots inside $[-1, 1]$)

There exists a positive constant C such that if

$$|m(x)| > C|\log(1 - |x|)|^{1/2}, \quad x \in U_n, \tag{16}$$

then

$$\mathrm{Var}[N_{P_n}(U_n)] = o(1). \tag{17}$$

There exists a function $\phi : [0,1] \to [0,1]$ such that the following holds. Assume that for each $i = 0, 1, 2$, we have

$$|m^{(i)}(x)| \ll \frac{\phi(1-|x|)}{(1-|x|)^i} \text{ uniformly on } U_n. \tag{18}$$

Then for any interval $I \subset [-1,1]$, we have

$$\mathrm{Var}[N_{P_{n,G}}(I \cap U_n)] = \mathrm{Var}[N_{R_n}(I \cap U_n)] + O(\phi_{R_n}(I \cap U_n)), \tag{19}$$

where

$$\phi_{R_n}(I \cap U_n) = \left(\int_{I \cap U_n} \frac{\phi^2(1-|x|)}{1-|x|} dx \right) \mathbb{E}[N_{R_n}(I \cap U_n)].$$

Moreover, if $\mathrm{Var}[N_{R_n}(I \cap U_n)] \geq \epsilon \log n$ for some constant $\epsilon > 0$ then $N_{P_n}(I \cap U_n)$ satisfies the CLT.

Theorem (Real roots outside $[-1,1]$)

There exists a positive constant C such that if

$$|m^*(1/y)| > C|\log(1-|1/y|)|^{1/2}, \quad y \in U_n^*,$$

then

$$\mathrm{Var}[N_{P_n}(U_n^*)] = o(1). \tag{20}$$

Assume that for $i = 0, 1, 2$, we have the following uniform estimates:

$$(m^*)^{(i)}(1/y) \ll \frac{\phi(1-|1/y|)}{(1-|1/y|)^i}, \quad y \in U_n^*. \tag{21}$$

Then, for any interval $I \subset \mathbb{R} \setminus [-1,1]$, we have

$$\mathrm{Var}[N_{P_n}(I \cap U_n^*)] = \mathrm{Var}[N_{R_n}(I \cap U_n^*)] + O(\phi_{R_n}(I \cap U_n^*)). \tag{22}$$

Furthermore, if $\mathrm{Var}[N_{R_n}(I \cap U_n^*)] \geq \epsilon \log n$ for some constant $\epsilon > 0$ then $N_{P_n}(I \cap U_n^*)$ satisfies the CLT.

Open problems for non-centered polynomials

1. **Non-centered Kac polynomials:**
   - Concentration:

$$\mathbb{P}(|N_{P_n}(I) - \mathbb{E}[N_{P_n}(I)]| > \varepsilon\mathbb{E}[N_{P_n}(I)]) \leq ?$$

   - Almost sure asymptotics (SLLN):

$$\frac{N_{P_n}(I)}{\mathbb{E}[N_{P_n}(I)]} \xrightarrow{a.s.} 1?$$

2. **Extend to other models:** elliptic, Weyl, trigonometric, orthogonal, etc.
3. **Applications:** diffusion equations with random data, limit cycles in random vector fields.

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Thank you for your time and attention!
I welcome any questions.

Appendix: Asymptotic constants

For any real number $\tau > -1/2$, define

$$f_\tau(u) := \left(\sqrt{1-\Delta_\tau^2(u)} + \Delta_\tau(u) \arcsin \Delta_\tau(u)\right) \Sigma_\tau(u) - 1,$$

where

$$\Delta_\tau(u) = u^{\tau+1/2} \frac{u(1-u^{2\tau+1}) - (2\tau+1)(1-u)}{1-u^{2\tau+1} - (2\tau+1)u^{2\tau+1}(1-u)},$$

$$\Sigma_\tau(u) = \frac{1-u^{2\tau+1} - (2\tau+1)(1-u)u^{2\tau+1}}{(1-u^{2\tau+1})^{3/2}},$$

and let

$$\kappa_\tau := \left(\frac{2\tau+1}{\pi} \int_0^\infty f_\tau(\operatorname{sech}^2 v) dv + \frac{\sqrt{2\tau+1}}{2}\right) \frac{1}{\pi}. \qquad (23)$$

In particular, $\kappa_0 = \frac{1}{\pi}(1-\frac{2}{\pi})$. For hyperbolic polynomials, we have

$$\kappa = \kappa_\tau, \quad \text{where } \tau = \ell + \frac{L-1}{2}.$$