An asymptotic refinement of the Gauss–Lucas theorem for random polynomials with i.i.d. roots

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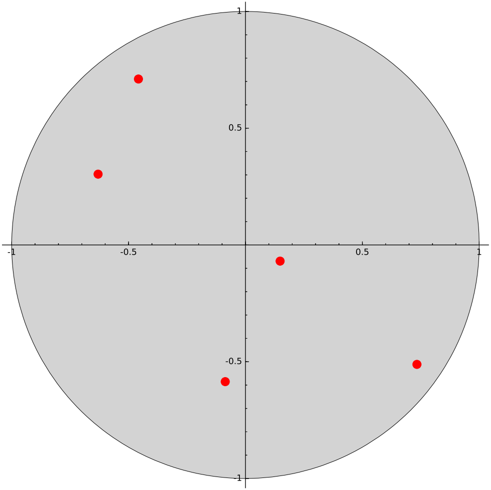
Introduction



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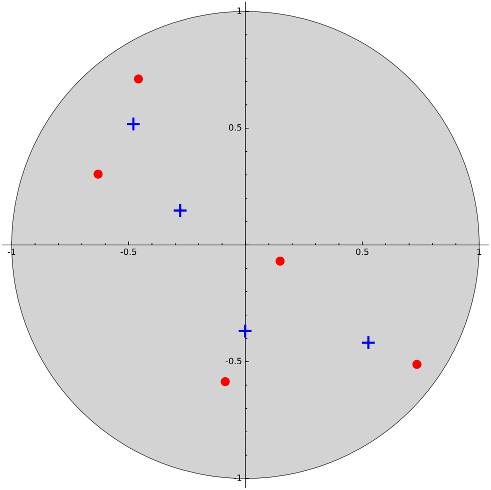

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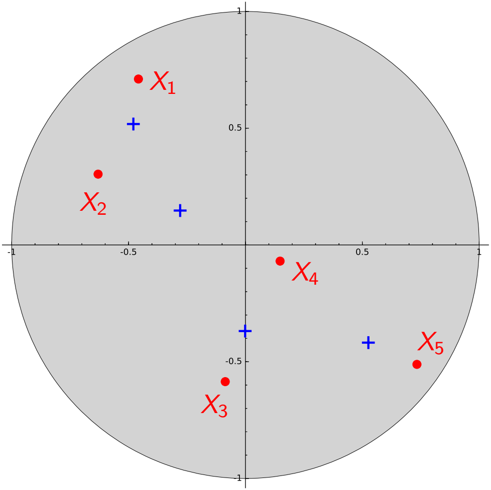
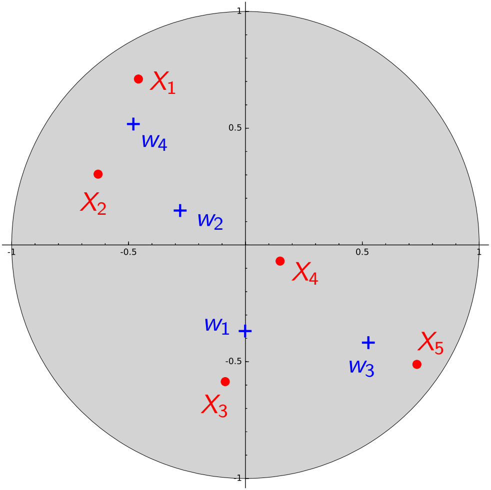


Let $p_n(z) = \prod_{j=1}^{5}(z - X_j)$.

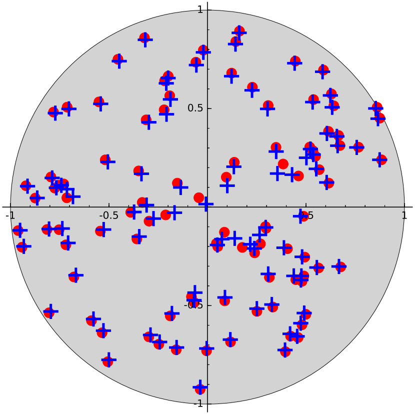


Let $p_n(z) = \prod_{j=1}^{5}(z - X_j)$.

Then, w_1, w_2, w_3, w_4 are the critical points of $p_n(z)$.

Polynomials with random roots

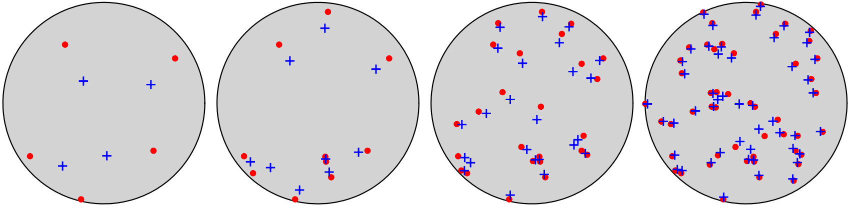
Roots and critical points of $p_{100}(z) = \prod_{j=1}^{100}(z - X_j)$.



X_j are i.i.d. uniform on the unit disk

Polynomials with random roots

Roots and critical points of $p_n(z) = \prod_{j=1}^{n}(z - X_j)$ for growing n.



$n = 5$ $n = 10$ $n = 25$ $n = 50$

X_j are i.i.d. uniform on the unit disk

Pulling a polynomial out of a hat...

Common models:

- Polynomials with random coefficients

$$p_n(z) = A_0 + A_1 z + A_2 z^2 + \cdots + A_n z^n = \sum_{j=0}^{n} A_j z^j$$

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- Characteristic polynomials of random matrices

$$p_n(z) = \det(zI - M_n)$$

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- Polynomials with random roots

$$p_n(z) = (z - X_1)(z - X_2)\cdots(z - X_n) = \prod_{j=1}^{n}(z - X_j)$$

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Polynomials with random roots - examples

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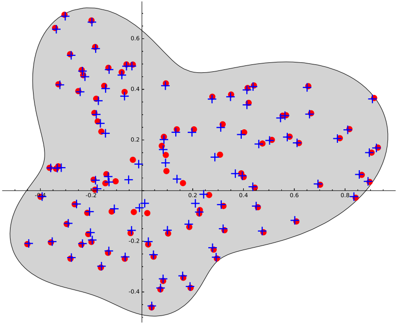
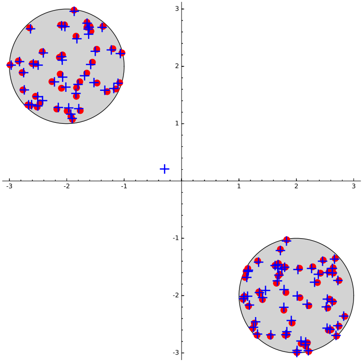


X_j are i.i.d. uniform on the blob



X_j are i.i.d. uniform on two disks

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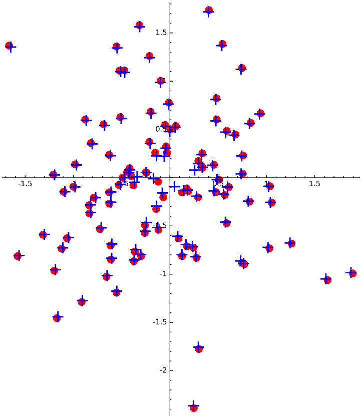
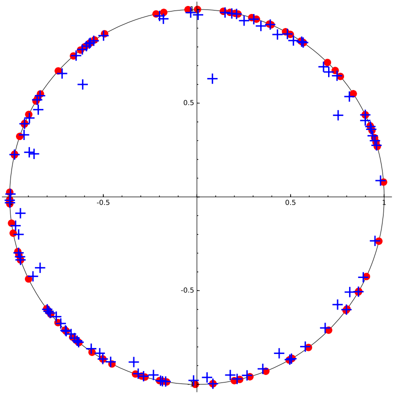
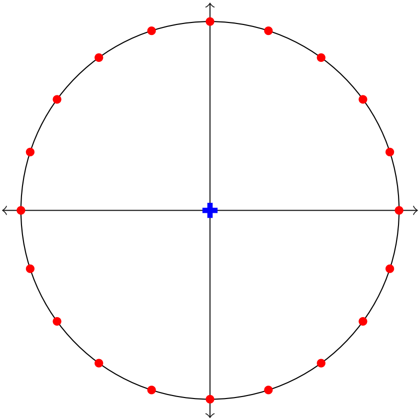




X_j are i.i.d. complex normal $\quad$ X_j are i.i.d. uniform on the circle

A counterexample

Roots and critical points of $p_{20}(z) = z^{20} - 1 = \prod_{j=1}^{20}(z - X_j)$.



X_j are the nth roots of unity

A heuristic "explanation" for pairing

Suppose $X_1, X_2, \ldots, X_n$ are iid with distribution μ. Then,

$$\frac{p_n'(z)}{p_n(z)} = \sum_{j=1}^{n} \frac{1}{z - X_j}$$

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So the zeros of p_n' are approximately the zeros of p_n provided that e.g.

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$$\frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - X_j} \approx \underbrace{\mathbb{E}\left[\frac{1}{z - X_1}\right]}_{m_\mu(z)}.$$

The Cauchy–Steiltjes transform of μ is $m_\mu(z) := \int_{\mathbb{C}} \frac{1}{z - x}\, d\mu(x)$.

What is known? Global behavior.

Pemantle–Rivin (2013), Subramanian (2012), Kabluchko (2015), Reddy (2016), O'Rourke (2016), Byun–Lee–Reddy (2018), O'Rourke–W. (2018, 2020), Angst–Malicet–Poly (2023), Michelen–Vu (2024)

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Theorem (Kabluchko, 2015)

For a bounded, continuous function $f : \mathbb{C} \to \mathbb{R}$,

$$\left(\frac{1}{n} \sum_{j=1}^{n} f(X_j) - \frac{1}{n-1} \sum_{j=1}^{n-1} f(w_j) \right) \xrightarrow{prob.} 0.$$

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Theorem (O'Rourke–W., 2020)

Suppose μ is "nice." Then, with high probability,

$$\min_{\sigma \in S_{n-1}} \frac{1}{n-1} \sum_{j=1}^{n-1} \left| X_j - w_{\sigma(j)} \right| \leq \frac{C(\ln n)^9}{n}.$$

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Suppose:

- $p_n(z) = \prod_{j=1}^{n}(z - X_j)$
  - X_j are ~~independently chosen~~ deterministic complex numbers
  - ~~X_j have distribution μ~~ $\frac{1}{n}\sum_{j=1}^{n} f(X_j) \to \int f \, d\mu$ for $f \in C_c(\mathbb{C})$
- $w_1, \ldots, w_{n-1}$ denote the critical points of p_n

Theorem (Totik, 2019)

If $\operatorname{supp}(\mu)$ *is compact with connected complement, for* $f \in C_c(\mathbb{C})$,

$$\left(\frac{1}{n} \sum_{j=1}^{n} f(X_j) - \frac{1}{n-1} \sum_{j=1}^{n-1} f(w_j) \right) \to 0.$$

What is known? Local behavior.

Hanin (2015, 2017), O'Rourke–W. (2018, 2020), Kabluchko–Seidel (2019)

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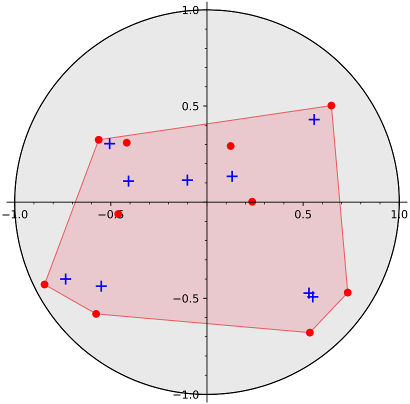
Theorem (Kabluchko–Seidel, 2019; O'Rourke–W., 2020)

For large enough n, the nearest critical point w to ξ satisfies

$$w \approx \xi - \frac{1}{m_\mu(\xi)} \frac{1}{n} + \frac{\sqrt{\pi f(\xi)}}{[m_\mu(\xi)]^2} \frac{\sqrt{\ln n}}{n^{3/2}} \cdot N,$$

where N has a complex standard normal distribution.

Extremal pairing → Refining the Gauss–Lucas theorem

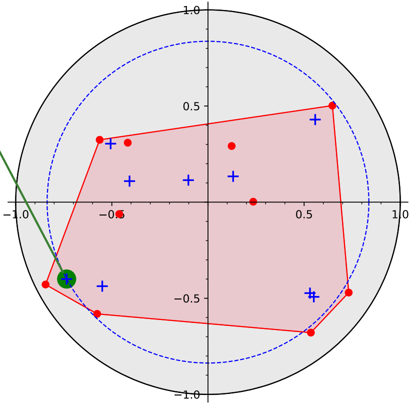


Theorem (Gauss–Lucas Theorem)

If p is a non-constant polynomial with complex coefficients, then all zeros of p' belong to the convex hull of the set of zeros of p.

Extremal pairing → Refining the Gauss–Lucas theorem

Largest Critical Point: $W^{\downarrow}_{(1)}$



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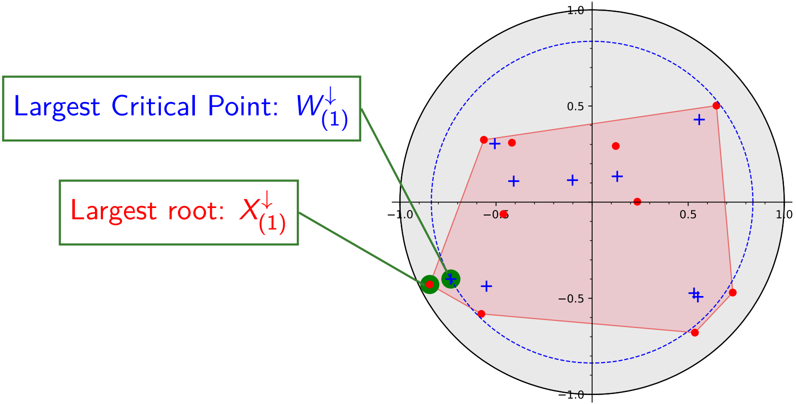

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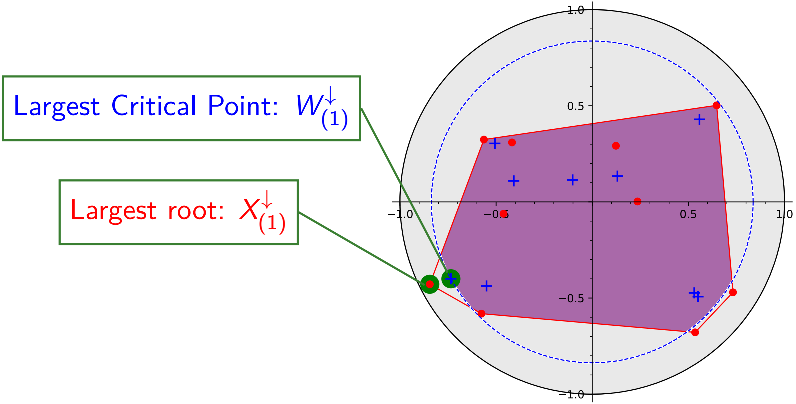

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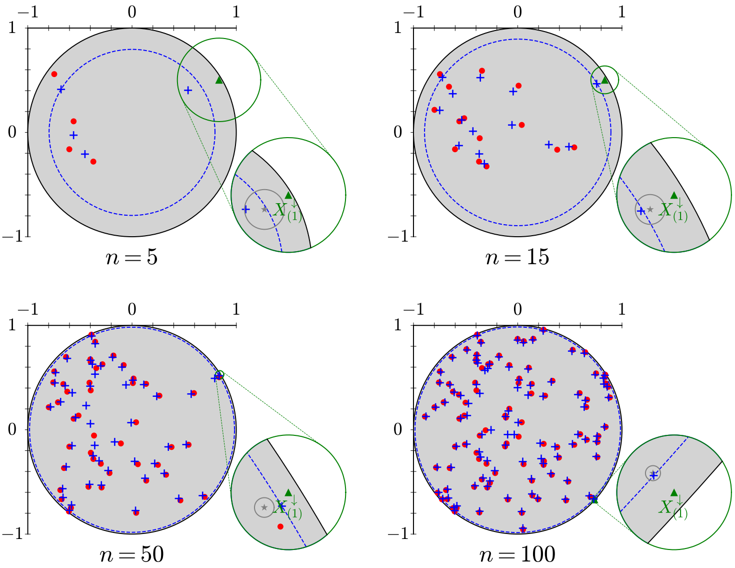
Pairing between $W^{\downarrow}_{(1)}$ and $X^{\downarrow}_{(1)}$ when X_j are uniform on disk

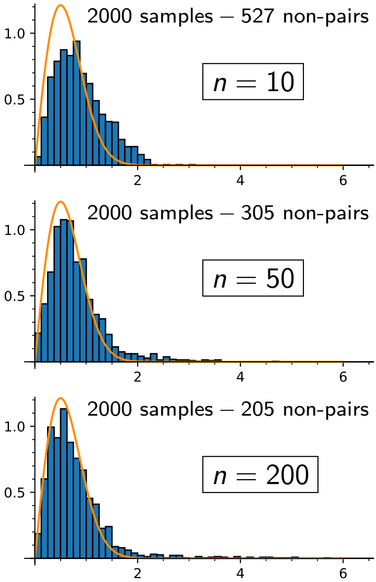


How does $W^{\downarrow}_{(1)}$ fluctuate about $(1 - n^{-1}) \cdot X^{\downarrow}_{(1)}$?

Histograms of draws from

$$\frac{n^{3/2}}{\sqrt{\ln n}} \left| W^{\downarrow}_{(1)} - \left(1 - n^{-1}\right) \cdot X^{\downarrow}_{(1)} \right| ,$$

for X_j uniform in the unit disk.



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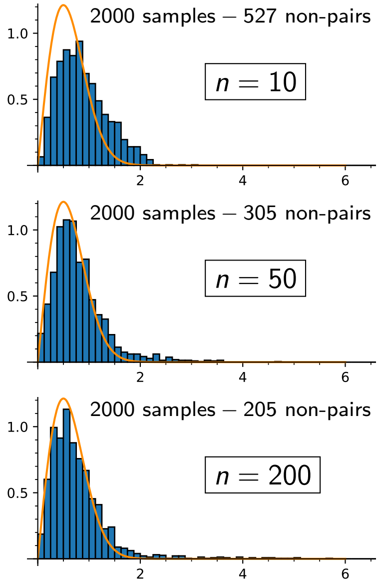
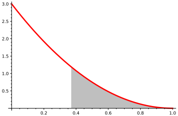
Limiting fluctuations are complex normal with modulus having a Rayleigh distribution:

$$f(x) = 4xe^{-2x^2} \longrightarrow$$

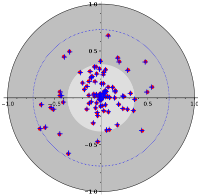


How do the fluctuations depend on the distribution of X_j?

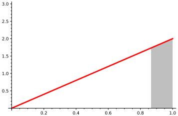
$f_R(r) = 3(1-r)^2$



$\xrightarrow{n=100}$



$f_R(r) = 2r(1-r)^0$



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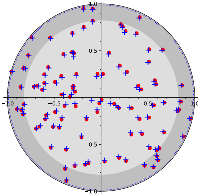
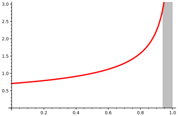


$f_R(r) = \frac{1}{2}(1-r)^{-1/2}$



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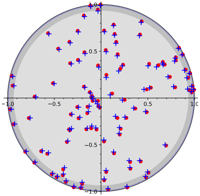

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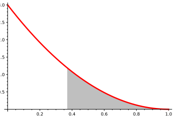

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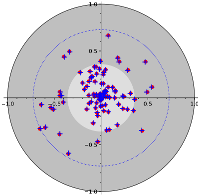
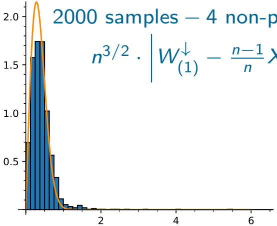


2000 samples — 4 non-pairs

$n^{3/2} \cdot \left| W^{\downarrow}_{(1)} - \frac{n-1}{n} X^{\downarrow}_{(1)} \right|$



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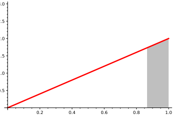

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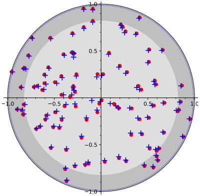
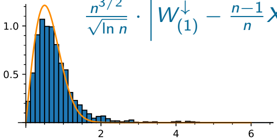
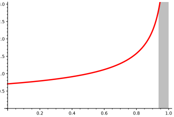


2000 samples — 256 non-pairs

$\frac{n^{3/2}}{\sqrt{\ln n}} \cdot \left| W^{\downarrow}_{(1)} - \frac{n-1}{n} X^{\downarrow}_{(1)} \right|$



$f_R(r) = \frac{1}{2}(1-r)^{-1/2}$



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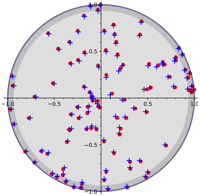
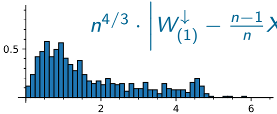


2000 samples — 1389 non-pairs

$n^{4/3} \cdot \left| W^{\downarrow}_{(1)} - \frac{n-1}{n} X^{\downarrow}_{(1)} \right|$



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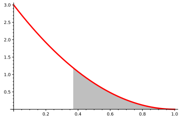
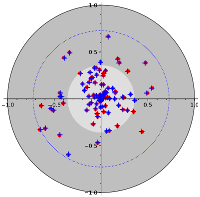
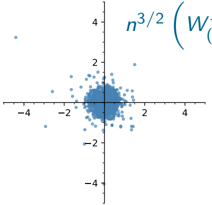
$f_R(r) = 3(1-r)^2$



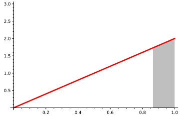
$\xrightarrow{n=100}$





$n^{3/2}\left(W_{(1)}^{\downarrow} - \frac{n-1}{n} X_{(1)}^{\downarrow}\right)$

$f_R(r) = 2r(1-r)^0$



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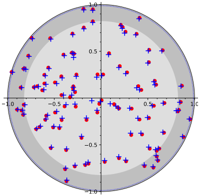
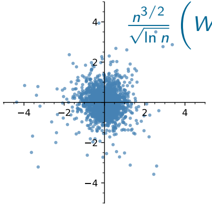
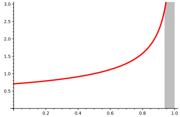




$\frac{n^{3/2}}{\sqrt{\ln n}}\left(W_{(1)}^{\downarrow} - \frac{n-1}{n} X_{(1)}^{\downarrow}\right)$

$f_R(r) = \frac{1}{2}(1-r)^{-1/2}$



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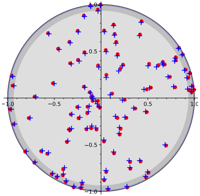

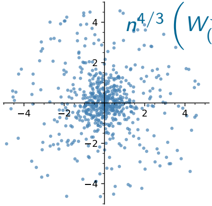

$n^{4/3}\left(W_{(1)}^{\downarrow} - \frac{n-1}{n} X_{(1)}^{\downarrow}\right)$

Highlights from "Asymptotic refinement of Gauss–Lucas"

Suppose X_j are iid with radially symmetric distribution on unit disk

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Theorem (O'Rourke–W.)

There is an $\varepsilon > 0$, so that with high probability...

1. *if X_i is among the largest n^ε roots of $p_n = \prod_{j=1}^n (z - X_j)$, there is precisely one critical point W_i with*

$$\left| W_i - (1 - n^{-1}) X_i \right| = o(1/n);$$

2. *if W is among the largest n^ε critical points of p_n, there is precisely one one root, X_{i_W} with*

$$\left| W - (1 - n^{-1}) X_{i_W} \right| = o(1/n).$$

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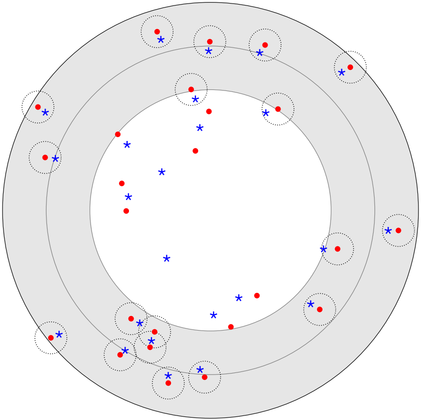

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With high probability, every c.p. W of $p_n = \prod_{j=1}^n (z - X_j)$ satisfies

$$|W| < (1 - n^{-1}) \left| X_{(1)}^{\downarrow} \right| + o(n^{-1}) < 1 - n^{-1},$$

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$$\frac{\mathfrak{a}_n}{e^{\sqrt{-1}\arg(X_{(1)}^{\downarrow})}} \cdot \left(W_{(1)}^{\downarrow} - X_{(1)}^{\downarrow}(1 - n^{-1}) \right) \xrightarrow{d} \begin{cases} N & \text{if } \alpha \geq 0 \\ \mathcal{H}_{2+\alpha} & \text{if } \alpha < 0, \end{cases}$$

where N has a complex normal distribution, and $\mathcal{H}_{2+\alpha}$ is a complex $(2+\alpha)$-stable random variable,

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where $\mathfrak{a}_n := \begin{cases} n^{\frac{3}{2}} & \text{if } \alpha > 0 \\ n^{\frac{3}{2}} / \log n & \text{if } \alpha = 0 \\ n^{\frac{3+2\alpha}{2+\alpha}} & \text{if } \alpha < 0. \end{cases}$

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With high probability, every c.p. W of $p_n = \prod_{j=1}^n (z - X_j)$ *satisfies*

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and provided $\lim_{r \to 1^-} \frac{f_R(r)}{(1-r)^\alpha}$ *exists,*

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3. Determine limiting behavior of $\frac{1}{n}\sum_{j=1}^{n} \frac{X_j}{1 - X_j}$.

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Proof sketch: Extremal asymptotic fluctuations – Step 2

Suppose $z \mapsto \frac{1}{n}\sum_{j\neq i_z}\frac{1}{z-X_j}$ is Lipschitz, and $W^{\downarrow}_{(1)}$ pairs to $X^{\downarrow}_{(1)}\ldots$

$$W^{\downarrow}_{(1)} = X^{\downarrow}_{(1)} - \frac{1}{n}\cdot\frac{1}{\frac{1}{n}\sum_{j=2}^{n}\frac{1}{W^{\downarrow}_{(1)}-X^{\downarrow}_{(j)}}} \tag{3}$$

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$$\frac{1}{n}\sum_{j=2}^{n} \frac{1}{W^{\downarrow}_{(1)} - X^{\downarrow}_{(j)}} \approx \frac{1}{n}\sum_{j=2}^{n} \frac{1}{e^{\sqrt{-1}\,\arg(X^{\downarrow}_{(1)})} - X^{\downarrow}_{(j)}}$$

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$$\begin{aligned}\frac{1}{n}\sum_{j=2}^{n} \frac{1}{W^{\downarrow}_{(1)} - X^{\downarrow}_{(j)}} &\approx \frac{1}{n}\sum_{j=2}^{n} \frac{1}{e^{\sqrt{-1}\,\arg(X^{\downarrow}_{(1)})} - X^{\downarrow}_{(j)}} \\ &= e^{-\sqrt{-1}\,\arg(X^{\downarrow}_{(1)})} \cdot \frac{1}{n}\sum_{j=2}^{n} \frac{1}{1 - X^{\downarrow}_{(j)} e^{-\sqrt{-1}\,\arg(X^{\downarrow}_{(1)})}}\end{aligned}$$

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Subtract $(1 - 1/n) \cdot X_{(1)}^{\downarrow}$ and multiply by $ne^{-\sqrt{-1}\arg(X_{(1)}^{\downarrow})}$ to obtain

$$\frac{n}{e^{\sqrt{-1}\arg(X_{(1)}^{\downarrow})}} \cdot \left(W_{(1)}^{\downarrow} - \frac{n-1}{n} \cdot X_{(1)}^{\downarrow} \right) \approx \left| X_{(1)}^{\downarrow} \right| - \frac{1}{\frac{1}{n}\sum_{j=1}^{n} \frac{1}{1-X_j}}$$

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Proof sketch: Extremal asymptotic fluctuations – Step 2

Suppose $z \mapsto \frac{1}{n}\sum_{j\neq i_z} \frac{1}{z-X_j}$ is Lipschitz, and $W^{\downarrow}_{(1)}$ pairs to $X^{\downarrow}_{(1)}$...

$$W^{\downarrow}_{(1)} \approx X^{\downarrow}_{(1)} - \frac{1}{n} \cdot \frac{1}{e^{-\sqrt{-1}\arg(X^{\downarrow}_{(1)})} \cdot \frac{1}{n}\sum_{j=1}^{n} \frac{1}{1-X_j}} \tag{3}$$

Subtract $(1-1/n)\cdot X^{\downarrow}_{(1)}$ and multiply by $ne^{-\sqrt{-1}\arg(X^{\downarrow}_{(1)})}$ to obtain

$$\frac{n}{e^{\sqrt{-1}\arg(X^{\downarrow}_{(1)})}} \cdot \left(W^{\downarrow}_{(1)} - \frac{n-1}{n}\cdot X^{\downarrow}_{(1)}\right) \approx \underbrace{\left|X^{\downarrow}_{(1)}\right|}_{\substack{\downarrow \mathrm{d} \\ 1}} - \frac{1}{\frac{1}{n}\sum_{j=1}^{n} \frac{1}{1-X_j}}$$

For large n, we have

$$\frac{n}{e^{\sqrt{-1}\arg(X^{\downarrow}_{(1)})}} \cdot \left(W^{\downarrow}_{(1)} - \frac{n-1}{n}\cdot X^{\downarrow}_{(1)}\right) \approx \frac{\frac{1}{n}\sum_{j=1}^{n} \frac{X_j}{1-X_j}}{\frac{1}{n}\sum_{j=1}^{n} \frac{1}{1-X_j}}$$

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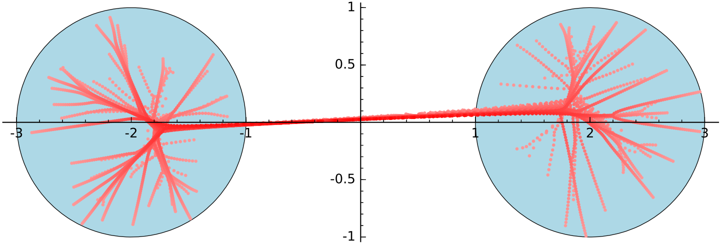
Thank you!



Thank you!