Asymptotic root distribution and free probability

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Random Polynomials and their Applications, ICERM

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For $p \in \mathcal{P}_d$ we use the notation:

$$p(x) \quad = \prod_{k=1}^{d}(x - \lambda_k(p)) \quad = \sum_{k=0}^{d} x^{d-k}(-1)^k \binom{d}{k} \mathsf{e}_k\,(p).$$

$$\textbf{Roots:} \quad \lambda_1(p), \dots, \lambda_d(p).$$

$$\textbf{Coefficients:} \quad \mathsf{e}_k\,(p) := \frac{1}{\binom{d}{k}} \sum_{1 \le i_1 < \cdots < i_k \le d} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p).$$

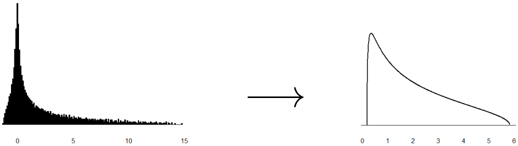
$$\textbf{Empirical root distribution:} \quad \mu\,[\![p]\!] := \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i(p)}$$

Notice that

$$\mathsf{e}_k\left(\tfrac{1}{d}p'\right) = \mathsf{e}_k\,(p)$$

Motivation

Framework: $(p_d)_{d=1}^{\infty}$ is a sequence of polynomials with $p_d \in \mathcal{P}_d(\mathbb{R})$, μ is a compactly supported measure in $\mathbb{R}$, and $\mu[\![p_d]\!]$ converges weakly to μ.



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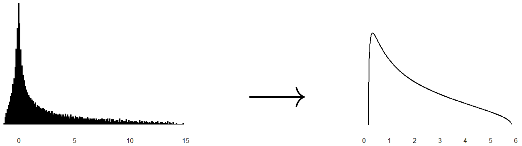

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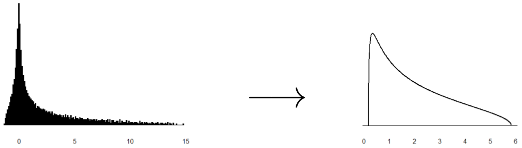

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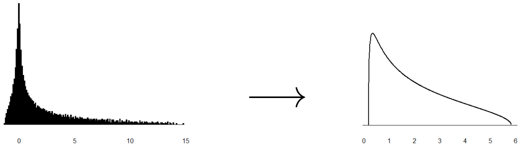

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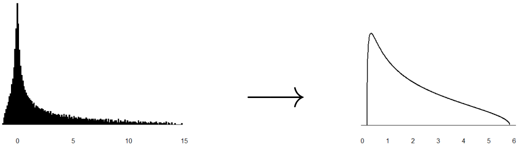

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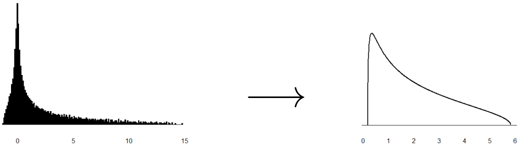

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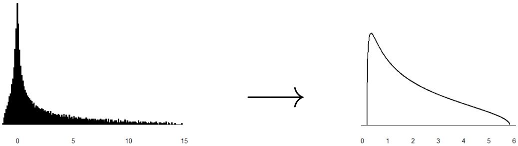

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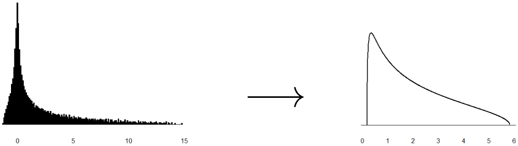

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Independence (in the classical setting) can be viewed a universal rule to compute the mixed moments in terms of the moments of the individual variables:
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E.g. if $a, b \in \mathcal{A}$ are free random variables, then also
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Partitions of a set and cumulants

$\pi = \{V_1, \ldots, V_k\}$ is a **partition** of the set $\{1, \ldots, n\}$ if we have the disjoint union $V_1 \cup V_2 \cup \cdots \cup V_k = \{1, \ldots, n\}$, we say $\pi \in \mathcal{P}(n)$.

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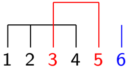

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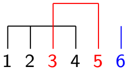

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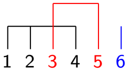

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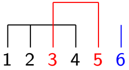

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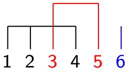

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Non-crossing partitions and free cumulants

$\pi \in \mathcal{P}(n)$ is a **non-crossing partition** if blocks do not cross.

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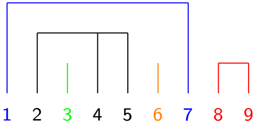

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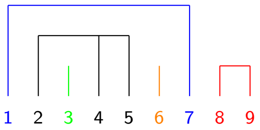

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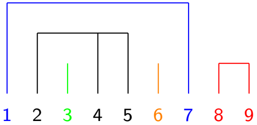

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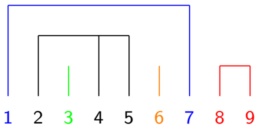

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$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) \quad = \sum_{n=0}^{\infty} z^{-n-1} m_n(\mu)$$ **Cauchy-Stieltjes transform**

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Transforms and Free convolutions

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Finite Free Probability

Definition

Given $p, q \in \mathcal{P}_d$, their **multiplicative and additive convolutions** are the polynomials $p \boxtimes_d q \in \mathcal{P}_d$ and $p \boxplus_d q \in \mathcal{P}_d$ with coefficients

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Basic properties: interpretation in terms of differential operators, bilinear, commutative, associative, identity (x^d for $\boxplus_d$, $(x-1)^d$ for $\boxtimes_d$), preserve interlacing, preserve root separation.

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Since $\mathrm{e}_j\left(p'\right) = d\mathrm{e}_j\left(p\right)$, then

$$d(p \boxplus_d q)' = p' \boxplus_{d-1} q' \qquad \text{and} \qquad d(p \boxtimes_d q)' = p' \boxtimes_{d-1} q'$$

Connection to Random Matrices

[Marcus, Spielman, Srivastava '15] Let A and B be $d \times d$ matrices with characteristic polynomials $p = \chi(A)$ and $q = \chi(B)$. Then

$$p \boxplus_d q = \mathbb{E}_Q[\chi(A + QBQ^*)] \qquad \text{and} \qquad p \boxtimes_d q = \mathbb{E}_Q[\chi(AQBQ^*)]$$

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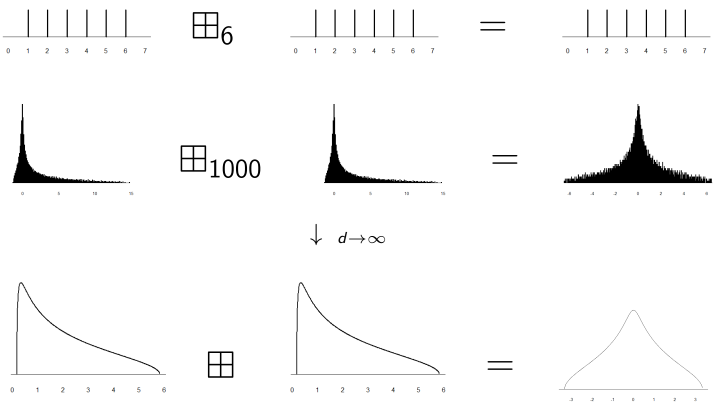
[Marcus '16] Establishes a connection with free probability:
When $d \to \infty$, finite free convolutions should tend to free convolutions ($\boxplus_d \to \boxplus$).
Studies *Finite Free Probability* and gives basic examples like LLN, CLT, Poisson limit.

Finite free convolutions tend to free convolutions

Consider sequences $\mathfrak{p} = (p_d)_{d=1}^{\infty}$ and $\mathfrak{q} = (q_d)_{d=1}^{\infty}$ with $p_d, q_d \in \mathcal{P}_d(\mathbb{R})$ and limiting measures $\mu, \nu \in \mathcal{M}_c(\mathbb{R})$: $\quad \mu\left[\!\left[p_d\right]\!\right] \longrightarrow \mu \quad$ and $\quad \mu\left[\!\left[q_d\right]\!\right] \longrightarrow \nu.$

[Marcus '16, Arizmendi, P '16] Then $\mu\left[\!\left[p_d \boxplus_d q_d\right]\!\right] \longrightarrow \mu \boxplus \nu$

[Arizmendi, Garza-Vargas, P '21] Then $\mu\left[\!\left[p_d \boxtimes_d q_d\right]\!\right] \longrightarrow \mu \boxtimes \nu.$



Main idea behind proof

- Define finite free cumulants

$$\kappa_n^{(d)}(p) := \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in P(n)} (-1)^{\#\pi-1}(\#\pi-1)!\ \mathrm{e}_\pi\,(p) \qquad \text{for } n = 1, 2, \ldots, d,$$

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- This gives a proof of $\mu [\![p_d \boxplus_d q_d]\!] \longrightarrow \mu \boxplus \nu$.
- For the multiplicative case one can use formulas for the multiplicative convolution in terms of cumulants.

Probability vs Free Probability vs Finite Free Probability

Classic Probability	**Free Probability**	**Finite Free**
$*$	$\boxplus$	$\boxplus_d$
$\log \mathbb{E}\left[e^{tX}\right]$	R-transform	finite R-transform
δ_a	δ_a	$(x-a)^d$
Normal/Gaussian	Semicircular	Hermitte polynomials
Poisson	Marchenko-Pastur	Laguerre polynomials
Beta	Free Beta	Jacobi polynomials

Applications

Behavior of roots after repeated differentiation

Let $(p_d)_{d=1}^{\infty}$ with $p_d \in \mathcal{P}_d(\mathbb{R})$, and $\mu [\![p_d]\!] \longrightarrow \mu$.

Question 1. Fix $t \in (0,1)$ and differentiate $\lfloor (1-t)d \rfloor$ times each polynomial, $q_d := p_d^{(\lfloor (1-t)d \rfloor)}$. What is the limiting measure ν of the sequence $(q_d)_{d=1}^{\infty}$?

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For m a positive integer, $\mu^{\boxplus m} = \mu \boxplus \cdots \boxplus \mu$ is the free convolution of m copies of μ. (Nica, Speicher '96) There is a continuous interpolation: for every real $s \geq 1$ there exists a probability measure $\mu^{\boxplus s}$ determined by

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Proof. For $j \approx td$, we have

$$\kappa_n^{(j)}(p^{(d-j)}) = j^{n-1} \sum_{\pi \in P(n)} c(\pi)\, \mathrm{e}_\pi(p) = \left(\frac{j}{d}\right)^{n-1} \kappa_n^{(d)}(p) = \left(\frac{j}{d}\right)^{n} \left(\frac{d}{j}\right) \kappa_n^{(d)}(p).$$

Q2. Let $(p_d)_{d=1}^{\infty}$ with $p_d \in \mathcal{P}_d([0,\infty))$. What is the limiting behavior of $\mathsf{e}_k\left(p_d\right)$?

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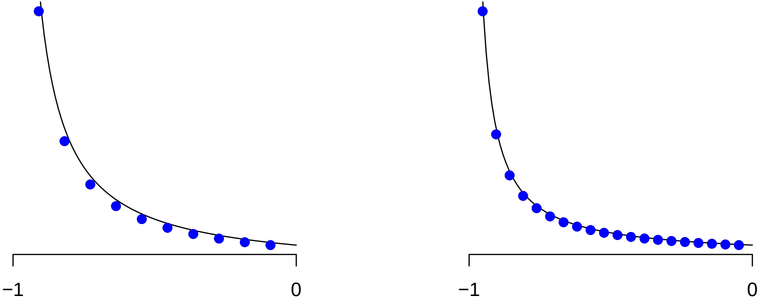
Theorem (Arizmendi, Fujie, P, Ueda '24)

$(p_d)_{d\in\mathbb{N}}$ sequence with $p_d \in \mathcal{P}_d([0,\infty))$ and $\nu \in \mathcal{M}([0,\infty))$. Then

$$\mu [\![p_d]\!] \to \nu \qquad \Leftrightarrow \qquad \lim_{\substack{d\to\infty \\ \frac{k}{d}\to t}} \frac{\mathrm{e}_{k-1}(p_d)}{\mathrm{e}_k(p_d)} = S_\nu(-t). \qquad \textit{for } t \in (0, 1-\mu(\{0\}))$$

Corollary. Assume $\mathrm{e}_1(p_d) \to m_1(\mu) \in (0,\infty)$, then

$$\frac{1}{d}\log(\mathrm{e}_k(p_d)) \quad \longrightarrow \quad -\int_0^t \log S_\nu(-x)dx$$



Intuition: Multiplicative LLN

Given $\mu \in \mathcal{M}(\mathbb{R}_{>0})$, the *(shifted) T-transform* is the function $T_\mu : (0,1) \to \mathbb{R}_{\geq 0}$ with

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[Tucci '10, Haagerup and Möller '13]
there exists a limiting measure

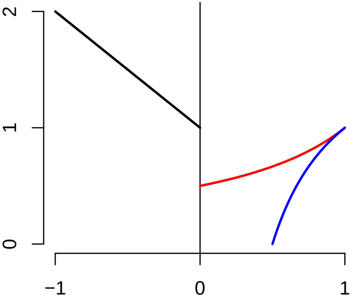
$$\Phi(\mu) := \lim_{m\to\infty} (\mu^{\boxtimes m})^{\langle 1/m \rangle}, \qquad \text{char. by}$$

$$F_{\Phi(\mu)}(T_\mu(t)) = t \qquad \text{for all } t \in (0,1).$$

[Fujie and Ueda '23]
Given $p \in \mathcal{P}_d(\mathbb{R}_{\geq 0})$, there exists

$$\Phi_d(p) := \lim_{m\to\infty} (p^{\boxtimes_d m})^{\langle 1/m \rangle},$$

with roots $\quad \lambda_k(\Phi_d(p)) = \dfrac{e_k(p)}{e_{k-1}(p)}.$



Case: compact support away from 0

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Assume the roots of the polynomials are contained in $C = [\alpha, \beta] \subset (0, \infty)$. *Then*

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Idea of the proof:

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Since $\mu\left[\!\left[p^{(d-k)}\right]\!\right] \to \mu_t := \mathrm{Dil}_t(\mu^{\boxplus 1/t})$, then

$$\longrightarrow \quad \int_0^\infty x^{-1}\mu_t(dx) = -G_{\mu_t}(0) = S_\mu(-t)$$

Future directions

Ongoing project

Open problems

Repeated polar differentiation (ongoing project with Zhiyuan Yang)

For $a \in \mathbb{C}$, the polar derivative of a polynomial is

$$D_a p(x) := dp(x) - (x - a)p'(x).$$

D_∞ is differentiation.

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Short ans: $$\nu = F_0^s := T^{-1}(\mathrm{Dil}_{\frac{1}{t}}(T\mu)^{\boxplus t}),$$

where $T(z) := \frac{1}{z-a}$ and $T\mu$ denotes the push-forward measure $T\mu(E) := \mu(T^{-1}(E))$.

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Question. Fix $t \in (0,1)$ and $a \in \mathbb{R}$. Let $(p_d)_{d=1}^\infty$ with $p_d \in \mathcal{P}_d(\mathbb{R})$, and $\mu [\![p_d]\!] \longrightarrow \mu$. Define

$$q_d := D_a^{(\lfloor (1-t)d \rfloor)} p_d.$$

What is the limiting measure ν of the sequence $(q_d)_{d=1}^\infty$?

Short ans: $$\nu = F_0^s := T^{-1}(\mathrm{Dil}_{\frac{1}{t}}(T\mu)^{\boxplus t}),$$

where $T(z) := \frac{1}{z-a}$ and $T\mu$ denotes the push-forward measure $T\mu(E) := \mu(T^{-1}(E))$.

Connection $\boxtimes_d$ when $a = 0$:

$$D_0 p = p \boxtimes_d d(x-1)^{d-1}.$$

Repeated polar differentiation (ongoing project with Zhiyuan Yang)

For $a \in \mathbb{C}$, the polar derivative of a polynomial is

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Repeat this $d-k$ times, and let $d \to \infty$ while $\frac{d}{k} \to s$, then

$$\frac{1}{s} F_0^s \mu + \frac{s-1}{s}\delta_\infty = \mu \boxtimes \left(\frac{1}{s}\delta_1 + \frac{s-1}{s}\delta_\infty \right).$$

Generalized rectangular convolution and additive of even

Definition. Let $d \in \mathbb{N}$ and $\alpha > -1$. The (d, α)-rectangular convolution of $p, q \in \mathcal{P}_d$, is the polynomial $p \boxplus_d^\alpha q \in \mathcal{P}_d$ with

$$\frac{\mathrm{e}_k\left(p \boxplus_d^\alpha q\right)}{(d+\alpha)^{\underline{k}}} = \sum_{i+j=k} \binom{k}{i} \frac{\mathrm{e}_i\left(p\right)}{(d+\alpha)^{\underline{i}}} \frac{\mathrm{e}_j\left(q\right)}{(d+\alpha)^{\underline{j}}},$$

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- Case $\alpha = \frac{1}{2}$ [Talk by Gribinski, and Campbell, Morales, P 25+]
- Other cases remain open.

Finite Free Commutator

Given $p \in \mathcal{P}_d$, define $\mathrm{Sym}(p) := p \boxplus_d (\mathrm{Dil}_{-1} p)$

Definition (J. Campbell '22)

For polynomials $p, q \in \mathcal{P}_d$ define

$$p \,\square_d\, q := \mathrm{Sym}(p) \boxtimes_d \mathrm{Sym}(q) \boxtimes_d z_d, \qquad \text{where}$$

$$z_d(x) := \sum_{k=0}^{\lfloor d/2 \rfloor} x^{d-2k} (-1)^k \binom{d}{2k} (d)_{\underline{k}} \frac{k!}{(2k)!} \frac{d+1-k}{d+1}$$

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Theorem (J. Campbell)

Let A and B be $d \times d$ matrices with char pol $p = \chi(A)$ and $q = \chi(B)$. Then

$$p \,\square_d\, q = \mathbb{E}_Q \chi[i(AQBQ^* - QBQ^* A)] \qquad \text{where } Q \sim \text{Haar unitary}$$

Conjecture

For $p, q \in \mathcal{P}_d(\mathbb{R})$, we have $p \,\square_d\, q \in \mathcal{P}_d(\mathbb{R})$.

Thanks!