Roots of random polynomials under differential flows

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Random Polynomials and their Applications

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Basic question

- How do the roots of a polynomial change as we change the polynomial?
- Main examples in this talk: **heat flow** and **repeated differentiation**
- Will consider both operations in two cases: **real roots** and **complex roots**
- Will find a close connection to **random matrix theory** and **partial differential equations**

POLYNOMIALS WITH ALL REAL ROOTS: HEAT FLOW

Heat flow: definition

- Solve heat equation on real line:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

with polynomial initial condition:

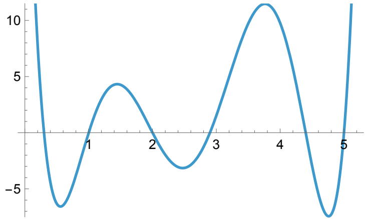
$$u(x,0) = p(x).$$

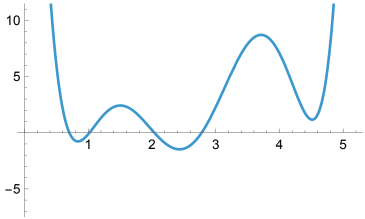
- Can solve as terminating power series in t:

$$u(x,t) = e^{\frac{t}{2}\frac{d^2}{dx^2}} p(x) := \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{2}\right)^k \left(\frac{d^2}{dx^2}\right)^k p(x)$$

Heat flow: definition

- Solution is polynomial in x (same degree as p) for each t
- Roots at time t may be complex, even if roots of p are real
- Extend to complex plane to find roots





Heat flow: definition

- Extend initial condition, solution holomorphically in space variable
- Makes sense with t replaced by **arbitrary complex number** τ
- For high-degree limit, **scale** τ with N
- Define **heat flow operator** as terminating power series:

$$\exp\left\{\frac{\tau}{2N}\frac{d^2}{dz^2}\right\}p(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\tau}{2N}\right)^k\left(\frac{d^2}{dz^2}\right)^k p(z), \quad z \in \mathbb{C}$$

Backward heat flow on polynomials

- Now take $\tau = -t$ and consider **backward heat operator**

$$\exp\left\{-\frac{t}{2N}\frac{d^2}{dz^2}\right\}, \quad t > 0,$$

on polynomials

Theorem (Pólya–Benz 1934)

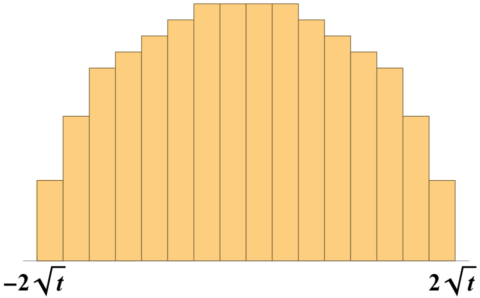
If p has all real roots, so does

$$\exp\left\{-\frac{t}{2N}\frac{d^2}{dz^2}\right\} p(z)$$

for all $t > 0$.

Backward heat operator: first example

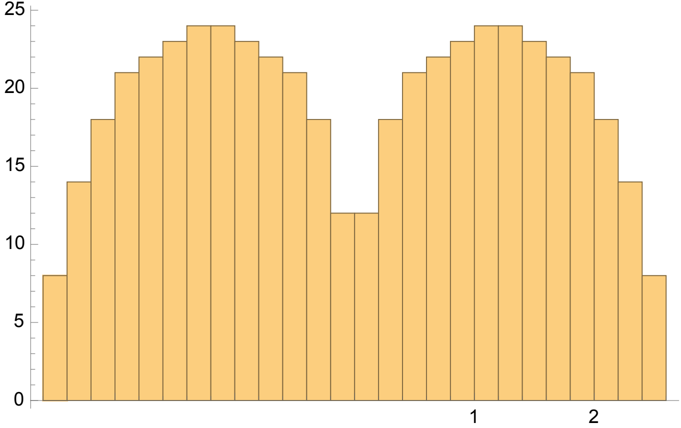
- Apply to z^N, get scaled **Hermite polynomial**
- Histogram of zeros of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}(z^N)$ with $N = 200$



- Zeros have asymptotically **semicircular shape** on $[-2\sqrt{t}, 2\sqrt{t}]$

Backward heat operator: second example

- Take $p(z) = (z-1)^{N/2}(z+1)^{N/2}$
- Half zeros at 1, half at -1
- Histogram of zeros of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}p$ with $N = 500$, $t = 1$



Connection to random matrix theory

- **GUE**: Gaussian unitary ensemble
- Take $N \times N$ Hermitian random matrix X with entries on and above diagonal independent
- Complex Gaussian with mean zero and variance $1/N$ off diagonal
- Real Gaussian with mean zero and variance $1/N$ on diagonal
- Eigenvalues asymptotically have **semicircular distribution** on $[-2, 2]$

Connection to random matrix theory

- Take sequence of real-rooted polynomials p^N of degree N
- Assume root distribution converges to prob. measure μ
- Make **Hermitian matrix** X^N (e.g., diagonal) with eigenvalues equal to roots of p^N
- Take Y^N to be GUE matrix

Claim

Roots of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}(p^N(z))$ resemble eigenvalues of $X^N + \sqrt{t}Y^N$, which can be computed using **free convolution** *of μ with a semicircular distribution.*

- So: backward heat flow is like adding a GUE

Free convolution with semicircular distribution

Theorem (Voit–Woerner 2022, Kabluchko 2025)

If polynomials p^N has real roots and the distribution of roots converges as $N \to \infty$ to μ, then the distribution of roots of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}p^N$ converges to $\mu \boxplus \mathrm{sc}_t$.

- $\mu \boxplus \mathrm{sc}_t$ is **free convolution** $\boxplus$ of μ with semicircular measure of variance t
- Free convolution with sc_t was studied by Biane

A PDE perspective

- Define **Cauchy transform** of measure μ on $\mathbb{R}$ by

$$C_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x), \quad \operatorname{Im} z > 0.$$

- Holomorphic on upper half-plane
- Can recover μ from C_μ by Stieltjes inversion formula

$$d\mu(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \left(\operatorname{Im} C_\mu(x + i\varepsilon)\ dx \right)$$

A PDE perspective

Theorem (Voiculescu)

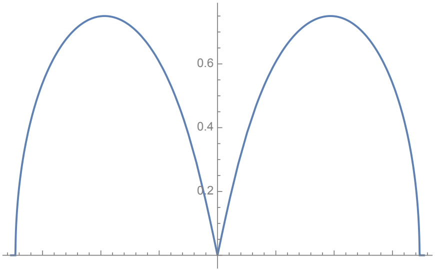
Cauchy transform $C(z,t)$ of $\mu \boxplus \mathrm{sc}_t$ satisfies the "inviscid complex Burger's equation"

$$\frac{\partial C}{\partial t} = -C\frac{\partial C}{\partial z}, \quad \operatorname{Im} z > 0,$$

- Can solve PDE using the **method of characteristics**
- Gives semi-explicit way to compute $\mu \boxplus \mathrm{sc}_t$

Roots at ± 1

- Take μ to have mass $1/2$ at 1 and mass $1/2$ at -1
- Describe polynomial p with zeros at ± 1
- Compute $\mu \boxplus \mathrm{sc}_t$ at, say, $t = 1$



- This gives limiting distribution of zeros of $e^{-\frac{1}{2N}\frac{d^2}{dz^2}} p(z)$

POLYNOMIALS WITH COMPLEX ROOTS: HEAT FLOW

Cauchy transform for measures in plane

- Compactly supported prob. measure μ with bounded density
- Define Cauchy transform as before:

$$C(z) = \int_{\mathbb{C}} \frac{1}{z - w}\, d\mu(w), \quad z \in \mathbb{C}$$

- But C will be **non-holomorphic** inside its support
- Ex: μ uniform on unit disk: $C(z) = \bar{z}$ in disk; $1/z$ outside
- Recover density of measure μ as

$$\frac{1}{\pi} \frac{\partial}{\partial \bar{z}} C(z)$$

General conjecture

Conjecture (Hall–Ho, 2025)

Let μ_t be limiting empirical measure of zeros of

$$\exp\left\{-\frac{t}{2N}\frac{d^2}{dz^2}\right\} p^N(z), \quad t \in \mathbb{R}.$$

Then Cauchy transform $C(z,t)$ satisfies PDE

$$\frac{\partial C}{\partial t} = -C\frac{\partial C}{\partial z} - \bar{C}\frac{\partial \bar{C}}{\partial z} \quad (t \text{ small}). \tag{1}$$

- $\partial/\partial z$ means the Cauchy–Riemann operator
- Second term on RHS vanishes in region where C is holomorphic
- Essentially same PDE as in the real-rooted case!

Heuristic argument for conjecture

- Define Cauchy transform of zeros of polynomial

$$C^N(z,t) := \frac{1}{N}\sum_{j=1}^{N}\frac{1}{z - z_j(t)}$$

where $z_j(t)$ are zeros of heat-evolved polynomials

Theorem

The function C^N satisfies the PDE

$$\frac{\partial C^N}{\partial t} = -C^N\frac{\partial C^N}{\partial z} - \bar{C^N}\frac{\partial \bar{C}^N}{\partial z} - \frac{1}{2N}\left(\frac{\partial^2 C^N}{\partial z^2} + \frac{\partial^2 \bar{C}^N}{\partial z \partial \bar{z}}\right),$$

which **formally** *converges to the PDE in the conjecture as $N \to \infty$.*

Connection to “arbitrary plus elliptic” RM model

- Let X and Y be independent GUEs and t with $-1 < t < 1$
- Take
$$Z_t = \frac{1}{\sqrt{2}}\left(\sqrt{1+t}\,X + i\sqrt{1-t}\,Y\right)$$
- Eigenvalues uniform on ellipse with semi-axes $1 \pm t$
- $t = 0$ gives circular law
- **Model**: $X_0 + Z_t$ where X_0 is indep. of Z_t

Theorem (Hall–Ho)

Cauchy transform $C(z,t)$ of limiting e.v. distribution of $X_0 + Z_t$ satisfies PDE in conjecture.

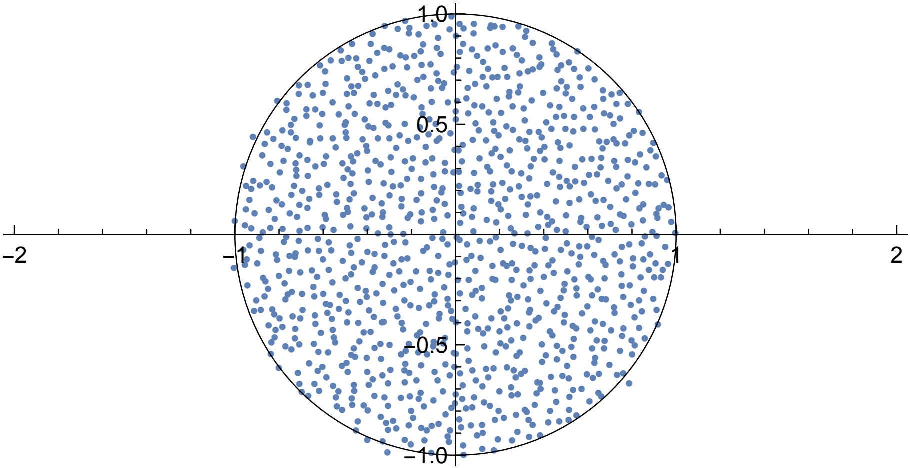
Example: Circular to elliptic

- Theorem provides natural examples for conjecture
- Start from char. poly. p^N of model with parameter t_0
- **Conjecture says**: roots of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}p^N$ should resemble e.v. of model with parameter $t + t_0$
- Running heat flow should be "same" as changing value of t

Example: Characteristic polynomial of Ginibre matrix

- **Example**: Start from Z_0: model with $X_0 = 0$ and $t_0 = 0$
- Z_0 is Ginibre matrix, eigenvalues uniform on disk
- Heat-evolved char. poly. of Z_0 should resemble char. poly. of Z_t
- Roots of heat-evolved char. poly. of Z_0 should be uniform on ellipse

Example: Characteristic polynomial of Ginibre matrix



Rigorous results for random polynomials

- **Kabluchko–Zaporozhets**: large class of random polynomials with independent coefficients

$$p^N(z) = \sum_{j=0}^{N} \xi_j c_j^N z^j$$

- ξ_j: indep. and identically distributed random var.
- c_j^N are deterministic constants (with nice behavior as $N \to \infty$)
- Limiting distribution of zeros is **rotationally invariant** on a disk
- Essentially **any** rot. invariant measure on disk occurs for some c_j^N

Example: Weyl polynomials

- Take
$$W_N(z) = \sum_{j=0}^{N} \xi_j \frac{N^{j/2}}{\sqrt{j!}} z^j$$
- Limiting distribution of zeros **uniform on unit disk**
- **Circular law** for random polynomials!

Rigorous results for random polynomials

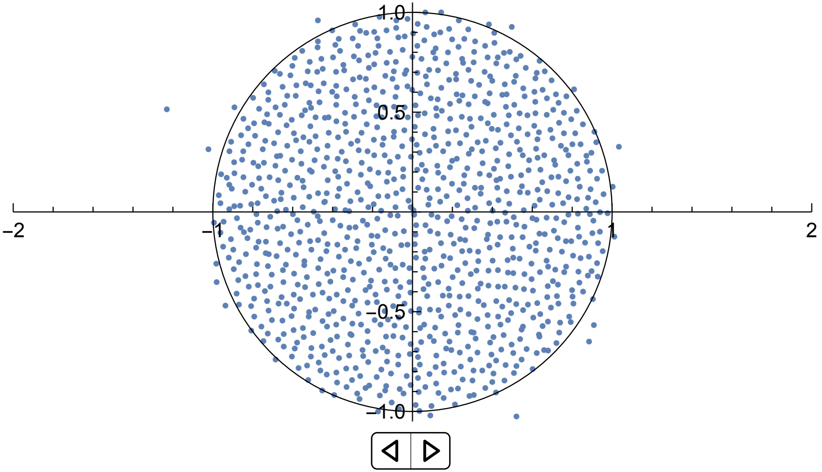
Theorem (Hall–Ho–Jalowy–Kabluchko, 2023a)

The heat-evolved Kabluchko–Zaporozhets polynomials satisfy the Hall–Ho conjecture.
That is, the Cauchy transform of the limiting root distribution satisfies the claimed PDE, for sufficiently small t.

Example: Weyl case

- For $-1 < t < 1$, limiting root distribution of heat-evolved Weyl polynomial is **uniform on ellipse** with semi-axes $1 + t$ and $1 - t$
- For $t \geq 1$, limiting root dist. is **semicircular on $\mathbb{R}$** with variance t
- Case $t = 1$ is "random orthogonal polynomial" with Gaussian weight, matches result of Pritsker–Xie [2015]

Evolution of zeros of Weyl polynomials, $0 \leq t \leq 1$



Transport behavior

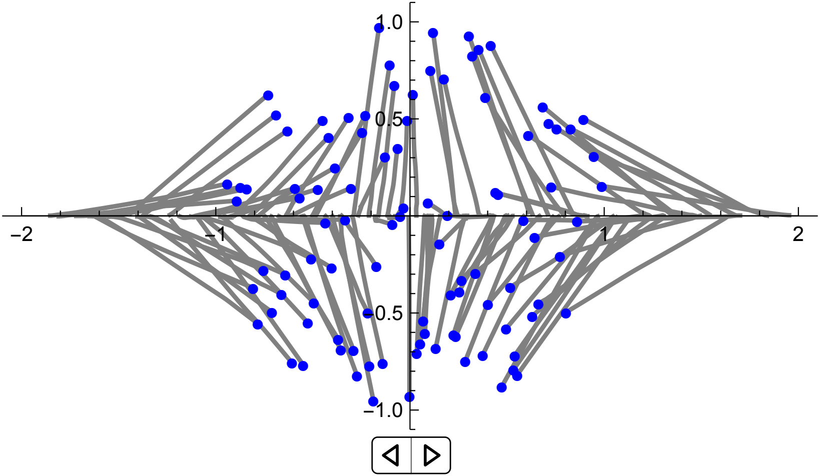
- **Next question**: How do zeros move with t?
- Zeros should evolve approximately in **straight lines** with constant velocity
- Velocity given by the value of Cauchy transform at time 0
- These are **characteristic curves** of the relevant PDE

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023a)

This behavior holds "at the bulk level" for heat-evolved KZ polynomials. That is, for sufficiently small t, the measure μ_t is the push-forward of μ_0 by map obtained by evolving along straight lines.

Straight-line motion

- Motion of sample of zeros of Weyl polynomial under heat flow
- Plotted against predicted straight-line motion



POLYNOMIALS WITH ALL REAL ROOTS: REPEATED DIFFERENTIATION

Repeated differentiation of polynomials with real roots

- Start with polynomial P^N of degree N with real roots
- Then differentiate $\lfloor Nt \rfloor$ times, $0 \leq t < 1$
- Number of deriv. proportional to N
- Roots remain real!
- Assume root dist. of P^N converges to μ_0
- Try to find limiting root dist. μ_t of $\lfloor Nt \rfloor$-th derivative

Connection to random matrix theory

- Assume (at first) that $t = 1 - 1/k$ with $k \in \mathbb{N}$
- Then $\mu_t = \mu_0^{\boxplus k} := \mu_0 \boxplus \cdots \boxplus \mu_0$, rescaled by a factor of $1 - t$
- $\mu_0^{\boxplus k}$ is like adding k indep.Hermitian matrices with e.v. distribution μ
- Then extend definition to arbitrary t (i.e., fractional k)
- "Fractional free convolution" introduced by Bercovici–Voiculescu ($k \gg 1$) and Nica–Speicher ($k \geq 1$)
- Equivalently: take **corner** of size $\lfloor (1-t)N \rfloor$ of $N \times N$ matrix with e.v. distribution μ

Connection to random matrix theory

Theorem (Hoskins–Kabluchko, '21; Arizmendi–Garza-Vargas–Perales, '23)

If polynomials P^N have limiting root distribution μ_0 then $\lfloor Nt \rfloor$-th derivative of P^N has limiting root distribution equal to

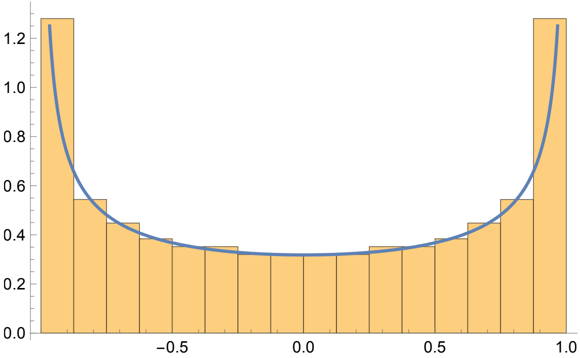
$$\mu_0^{\boxplus k}, \quad k = \frac{1}{1-t},$$

rescaled by a factor of $1-t$, for $0 \leq t < 1$.

- Results motivated by work of Steinerberger, 2019

Example: Roots at ± 1

- Take $P^N(z) = (z-1)^{N/2}(z+1)^{N/2}$; i.e. $\mu_0 = \frac{1}{2}(\delta_1 + \delta_{-1})$
- Take $t = 1/2$—i.e., take $N/2$ derivatives—so $k = 2$
- Then $\mu_0^{\boxplus k} = \mu_0 \boxplus \mu_0$ can be computed explicitly
- After rescaling, get "arcsin" distribution $d\mu_t(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\,dx$



PDE for the Cauchy transform

- Use **rescaled** measure $(1-t)\mu_t$ of mass $1-t$
- Let $C(z,t)$ be Cauchy transform of $(1-t)\mu_t$
- Use PDE for Cauchy transform of $\mu^{\boxplus k}$ by Shlyakhtenko–Tao

Theorem

The Cauchy transform $C(z,t)$ of $(1-t)\mu_t$ satisfies the PDE

$$\frac{\partial C}{\partial t} = \frac{1}{C}\frac{\partial C}{\partial z}.$$

- Compare to $\frac{\partial C}{\partial t} = -C\frac{\partial C}{\partial z}$ for backward heat flow

POLYNOMIALS WITH COMPLEX ROOTS: REPEATED DIFFERENTIATION

Repeated differentiation of random polynomials

- First observation: derivative of polynomial with independent (not necessarily i.i.d.) coefficients still has independent coefficients
- Feng and Yao showed that repeated differentiation of Kabluchko–Zaporozhets polynomial gives another KZ polynomial, with computable change in the deterministic coefficients

Transport behavior for random polynomials

- Next question: How do the roots evolve with t?
- Answer must recognize that differentiation kills roots!

Idea

Let μ_0 be the (radial) limiting root distribution of the initial polynomials and let $m_0(z)$ be its Cauchy transform. Then under repeated differentiation, roots evolve approximately **radially with constant speed** *according to*

$$z(t) \approx z_0 - \frac{t}{m_0(z_0)}$$

until they reach the origin, at which point they die.

Rigorous result at bulk level

- We verify idea at the level of the bulk distribution
- Let μ_t be the limiting root distribution of P_t^N.

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023b)

1. *Restrict μ_0 to the outer annulus with mass $1 - t$. I.e., remove inner disk with mass t.*
2. *After normalization, μ_t is the push-forward of μ_0 restricted to this annulus by the map*

$$T_t(z) = z\left(1 - \frac{t}{\alpha_0(|z|)}\right)$$

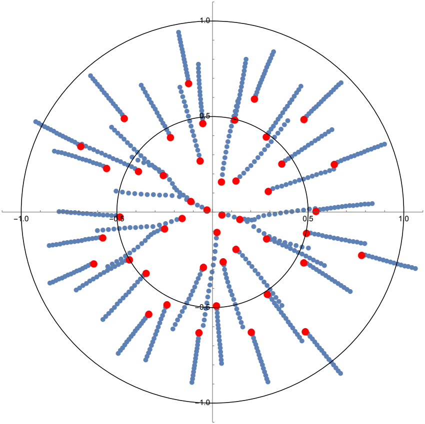
where $\alpha_0(|z|) = \mu_0(B(0, |z|))$.

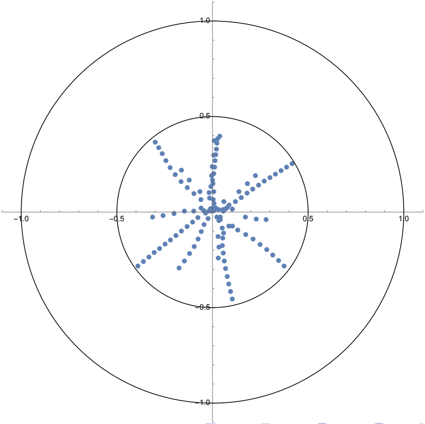
Example for Weyl case

- Example: Repeated differentiation of Weyl polynomial
- $T_t(z) = z\left(1 - \frac{t}{|z|^2}\right)$
- $N = 60$, $t = 1/4$. Showing roots of all derivatives up to the 15th derivative

Example for Weyl case

- Red dots: roots of 15th derivative
- Blue dots: roots of all previous derivatives
- Left: roots in annulus survive to time t
- Right: roots in disk die before time t





Further results

- PDE for Cauchy transform
- Random matrix interpretation (Campbell–O'Rourke–Renfrew) in terms of fractional convolution of R-diagonal operators
- Both similar to the case of real roots

Conclusion

THANK YOU FOR YOUR ATTENTION