Complex Roots of Random Sums

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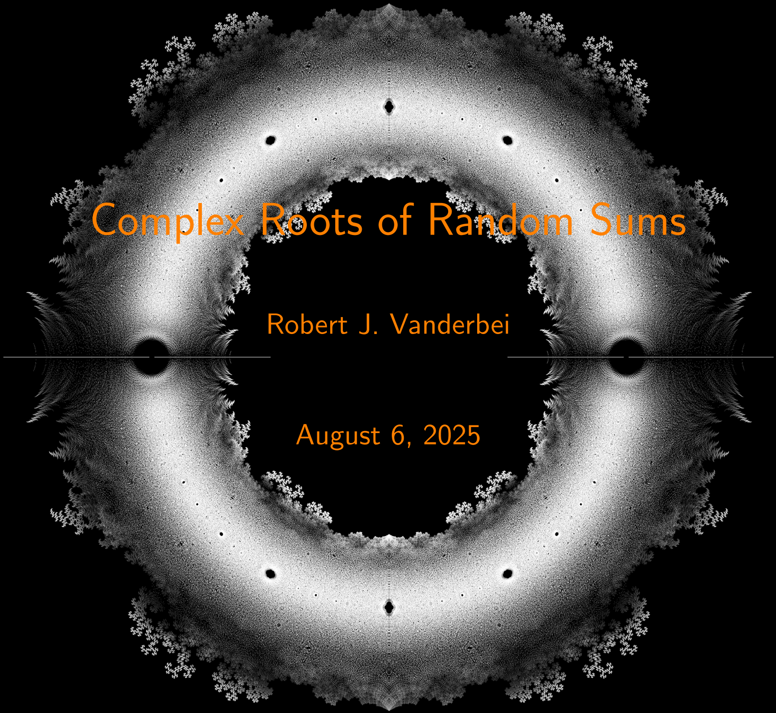
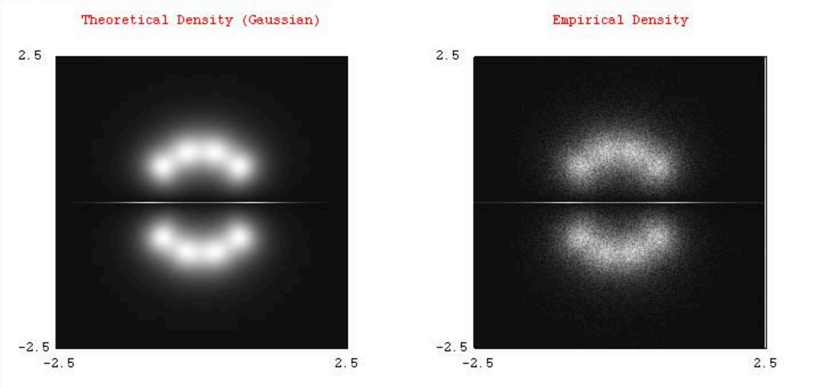


Zeros of Random Polynomials (work with Larry Shepp)

Consider a random polynomial:

$$P_n(z) = \sum_{j=0}^{n} \eta_j z^j, \quad z \in \mathbb{C},$$

where $\eta_0, \ldots, \eta_n$ are independent standard normal random variables.
The distribution of the zeros for, say, $n = 5$ looks like this...



Explicit Formula for any n:

Let $\nu_n(\Omega)$ denote the number of zeros in a set Ω in the complex plane. For each measurable set $\Omega \subset \mathbb{C}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x+iy)dxdy + \int_{\Omega\cap\mathbf{R}} g_n(x)dx,$$

where

$$h_n = \frac{B_2 D_0^2 - B_0(B_1^2 + |A_1|^2) + B_1(A_0\overline{A_1} + \overline{A_0}A_1)}{\pi|z|^2 D_0^3},$$

and

$$g_n = \frac{(B_0 B_2 - B_1^2)^{1/2}}{\pi|z|B_0}$$

and where

$$B_k(z) = \sum_{j=0}^n j^k |z|^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1, 2,$$

$$A_k(z) = \sum_{j=0}^n j^k\, z^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1,$$

and

$$D_0(z) = \sqrt{B_0^2(z) - |A_0|^2(z)}.$$

Explicit Formula for Limiting Density

Limiting density in the complex plane:

$$h = \lim_{n \to \infty} h_n = \frac{|B|D}{\pi}.$$

Limiting density on the real axis:

$$g = \lim_{n \to \infty} g_n = \frac{1}{\pi}|B|.$$

Where

$$B(z) = \frac{1}{1-|z|^2}, \quad z \in \mathbb{C},$$

$$A(z) = \frac{1}{1-z^2}, \quad z \in \mathbb{C},$$

and

$$D(z) = \sqrt{B^2(z) - |A|^2(z)}.$$

Large n and the Limit

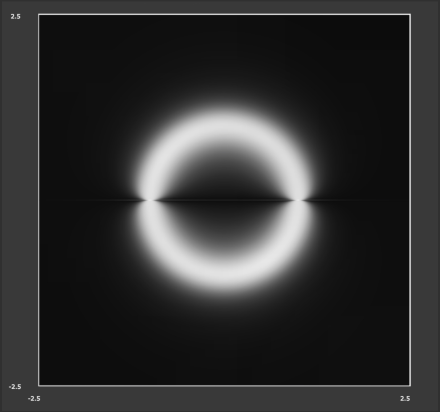
$n = 36$

$n = \infty$

Nonlinear Stretches $n = \infty$

asinh

atan



New Work (10 Years Ago): Zeros of Random Sums

Consider a random sum of basis functions $f_j(z)$:

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z), \quad z \in \mathbb{C}$$

We assume that the basis functions are real on the real line.
Some interesting choices include:

Polynomials:	$f_j(z) = z^j$	(okay, not so interesting)
Taylor Polynomials:	$f_j(z) = \dfrac{z^j}{j!}$	
Weyl Polynomials:	$f_j(z) = \dfrac{z^j}{\sqrt{j!}}$	
Kostlan Polynomials:	$f_j(z) = \sqrt{\binom{n}{j}}\, z^j$	
Fourier Cosine Series:	$f_j(z) = \cos(jz)$	
Fourier Sine/Cosine Series:	$f_j(z) = \begin{cases} \cos(\frac{j}{2}z) & j \text{ even} \\ \sin(\frac{j+1}{2}z) & j \text{ odd} \end{cases}$	

Theorem 1

For each measurable set $\Omega \subset \{z = x + iy \in \mathbb{C} \mid D_0(z) \neq 0\}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x+iy)dxdy,$$

where

$$h_n = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3},$$

and where

$$\begin{aligned}
A_0(z) &= \sum_{j=0}^{n} f_j(z)^2, & B_0(z) &= \sum_{j=0}^{n} |f_j(z)|^2, \\
A_1(z) &= \sum_{j=0}^{n} f_j(z) f_j'(z), & B_1(z) &= \sum_{j=0}^{n} \overline{f_j(z)} f_j'(z), \\
A_2(z) &= \sum_{j=0}^{n} f_j'(z)^2, & B_2(z) &= \sum_{j=0}^{n} |f_j'(z)|^2,
\end{aligned}$$

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}.$$

Comparison to Polynomial Result

For each measurable set $\Omega \subset \{z = x + iy \in \mathbb{C} \mid y \neq 0\}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x+iy)\,dxdy,$$

where

$$h_n = \frac{B_2 D_0^2 - B_0(B_1{}^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0} B_1 A_1)}{\pi D_0^3},$$

and where

$$\begin{aligned} A_0(z) &= \sum_{j=0}^{n} z^{2j}, & B_0(z) &= \sum_{j=0}^{n} |z|^{2j}, \\ A_1(z) &= \sum_{j=0}^{n} j z^{2j-1}, & B_1(z) &= \sum_{j=0}^{n} j|z|^{2j-1}, \\ A_2(z) &= \sum_{j=0}^{n} j^2 z^{2j-2}, & B_2(z) &= \sum_{j=0}^{n} j^2|z|^{2j-2}, \end{aligned}$$

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}.$$

Theorem 2

On the real line, the function D_0 vanishes. For each Borel measurable set $\Omega \subset \mathbb{R}$,

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega g_n(x)dx,$$

where

$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}.}{\pi B_0(x)}$$

Main Idea

From the *argument principle*, we get

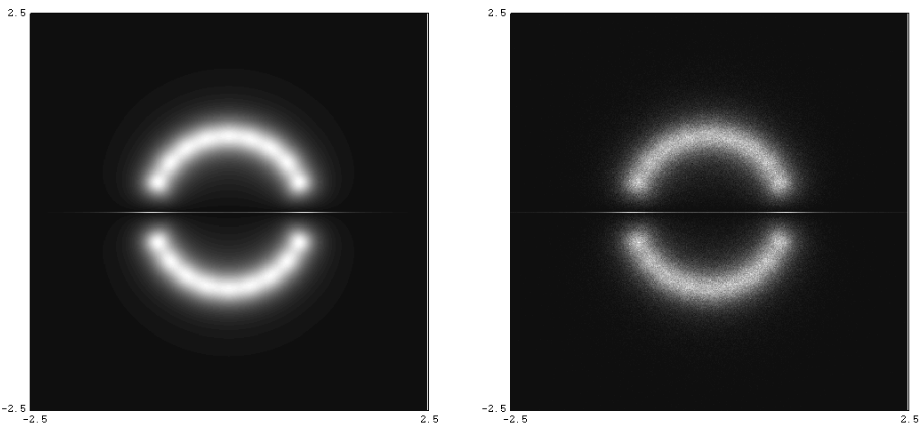
$$\nu_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)} dz.$$

Taking the expectation, interchanging expectation and integration and then applying *Stoke's Theorem*, we get

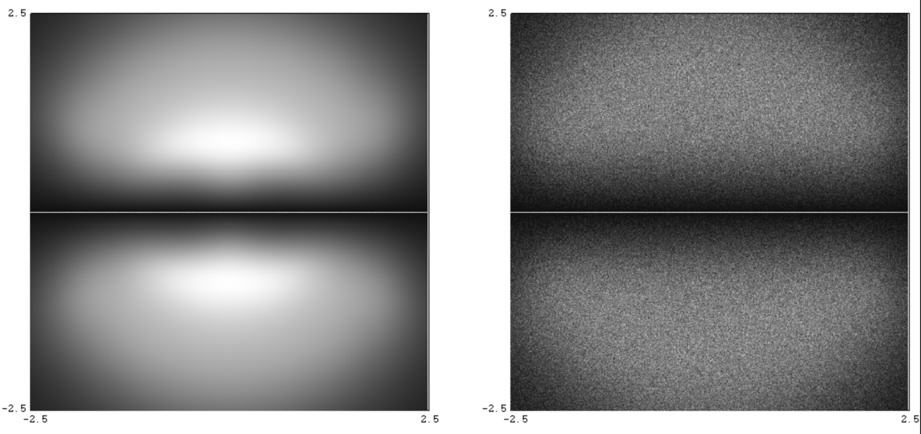
$$\begin{aligned} \mathbf{E}\nu_n(\Omega) &= \frac{1}{2\pi i} \int_{\partial\Omega} \mathbf{E} \frac{P_n'(z)}{P_n(z)} dz \\ &= \frac{1}{\pi} \int_{\Omega} \frac{\partial}{\partial \overline{z}} \mathbf{E} \frac{P_n'(z)}{P_n(z)} dxdy \end{aligned}$$

The rest is tedious (and perhaps nontrivial) algebra/calculus.

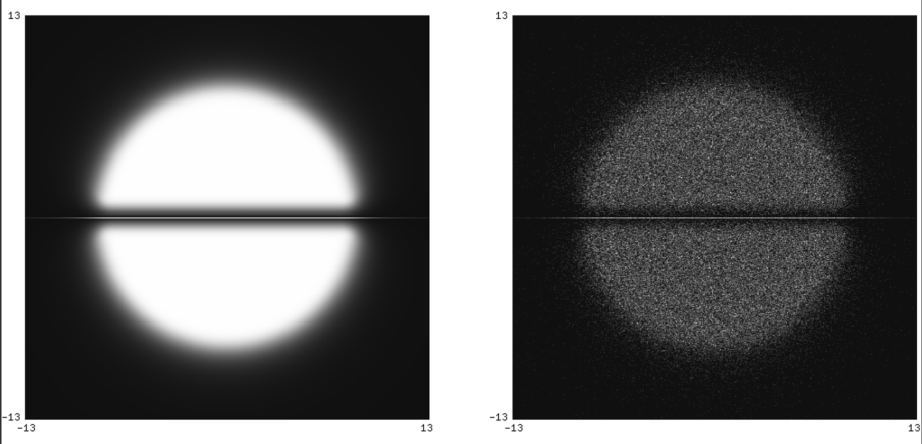
Polynomials: $f_j(z) = z^j, \quad n = 10$



Weyl Polynomials: $f_j(z) = z^j/\sqrt{j!}$, $n = 10$



Weyl Polynomials: $f_j(z) = z^j/\sqrt{j!}$, $n = 80$



Limiting Case for $f_j(z) = z^j/\sqrt{j!}$ (Weyl Polynomials)

$$A_0(z) = \sum_{j=0}^{\infty} f_j(z)^2 = e^{z^2} \qquad B_0(z) = \sum_{j=0}^{\infty} |f_j(z)|^2 = e^{|z|^2}$$

$$A_1(z) = \sum_{j=0}^{\infty} f_j(z) f_j'(z) = z e^{z^2} \qquad B_1(z) = \sum_{j=0}^{\infty} \overline{f_j(z)} f_j'(z) = \bar{z} e^{|z|^2}$$

$$A_2(z) = \sum_{j=0}^{\infty} f_j'(z)^2 = (z^2+1) e^{z^2} \qquad B_2(z) = \sum_{j=0}^{\infty} |f_j'(z)|^2 = (|z|^2+1) e^{|z|^2}$$

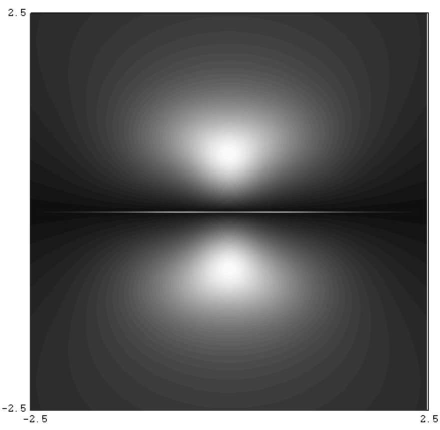
$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2} = \sqrt{e^{2|z|^2} - e^{z^2+\bar{z}^2}}$$

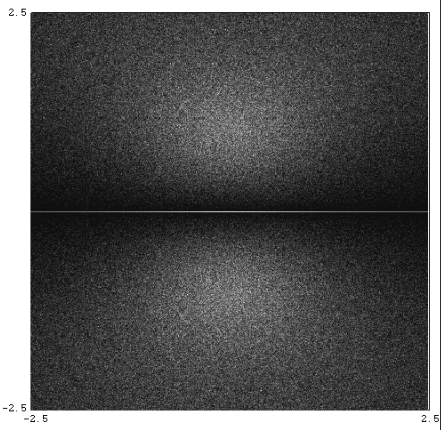
$$h(z) = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0}\,\overline{B_1} A_1)}{\pi D_0^3} = \frac{\left(1 + ((z-\bar{z})^2 - 1) e^{(z-\bar{z})^2}\right)}{\pi \left(1 - e^{(z-\bar{z})^2}\right)^{3/2}}$$

$$= \frac{\left(1 + (-4y^2 - 1) e^{-4y^2}\right)}{\pi \left(1 - e^{-4y^2}\right)^{3/2}}$$

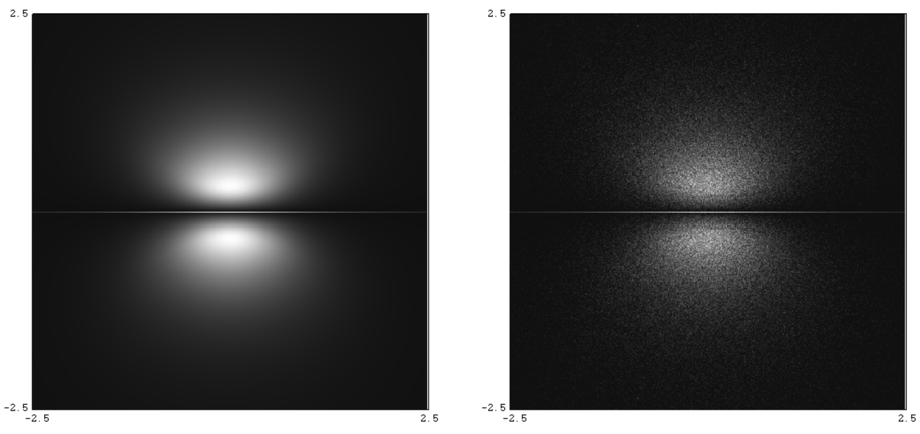
$$g(x) = \frac{\sqrt{A_2(x) A_0(x) - A_1(x)^2}}{\pi B_0(x)} = \frac{1}{\pi}$$

Taylor Polynomials: $f_j(z) = z^j/j!$, $n = 10$





Kostlan Polynomials: $f_j(z) = \sqrt{\binom{n}{j}}\, z^j, \quad n = 10$



Kostlan Polynomials: $f_j(z) = \sqrt{\binom{n}{j}} z^j$

$$A_0(z) = (1+z^2)^n \qquad B_0(z) = (1+|z|^2)^n$$

$$A_1(z) = nz(1+z^2)^{n-1} \qquad B_1(z) = n\bar{z}(1+|z|^2)^{n-1}$$

$$A_2(z) = n(1+nz^2)(1+z^2)^{n-2} \qquad B_2(z) = n(1+n|z|^2)(1+|z|^2)^{n-2}$$

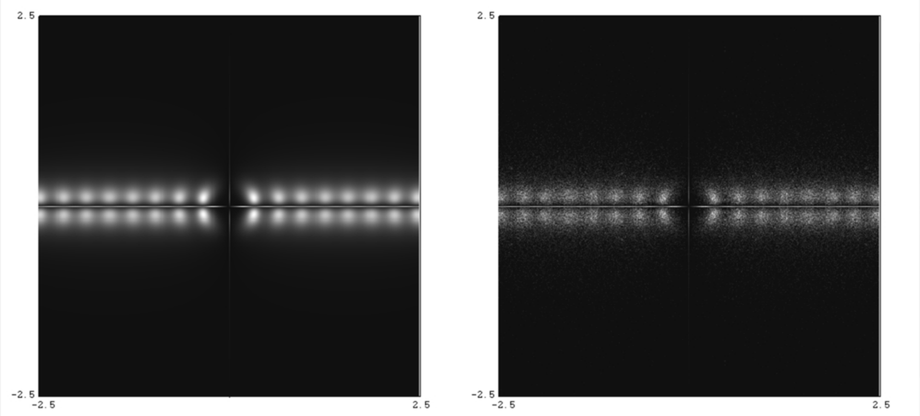
$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2} = \sqrt{(1+|z|^2)^{2n} - |1+z^2|^{2n}}$$

$$h_n = \frac{B_2 D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0 B_1 \overline{A_1} + \overline{A_0 B_1} A_1)}{\pi D_0^3} = ???$$

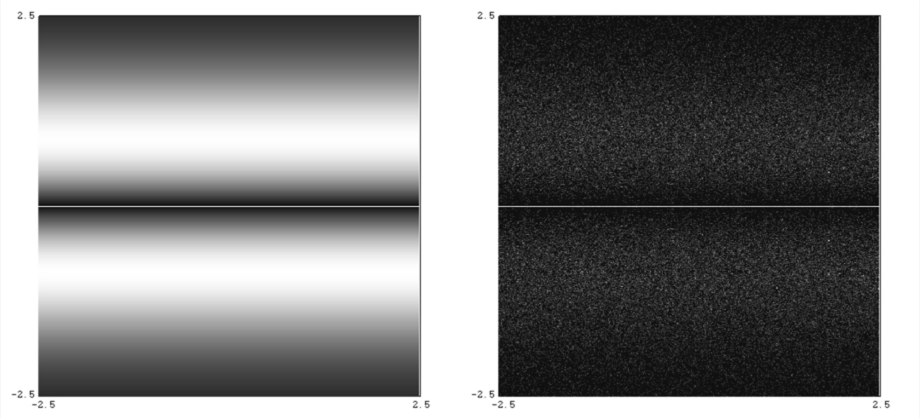
$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}}{\pi B_0(x)} = \frac{\sqrt{n}}{\pi}\frac{1}{1+x^2} \quad \longleftarrow \quad \text{Cauchy distribution}$$

$$\int_{-\infty}^{\infty} g_n(x)dx = \sqrt{n} \quad \longleftarrow \quad \text{Number of real roots}$$

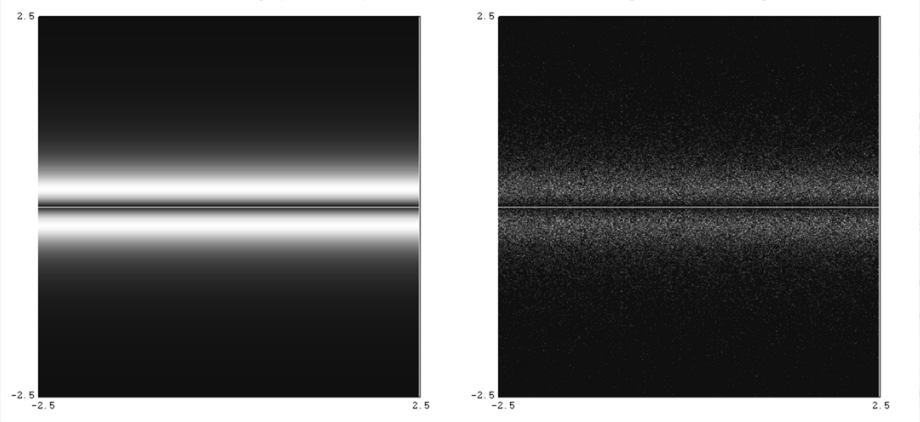
Fourier: $\alpha_0 + \alpha_1 \cos(z) + \alpha_2 \cos(2z) + \cdots + \alpha_9 \cos(9z) + \alpha_{10} \cos(10z)$



Sine/Cosine Fourier: $\alpha_0 + \alpha_1 \sin(z) + \alpha_2 \cos(z)$



$\alpha_0 + \alpha_1 \sin(z) + \alpha_2 \cos(z) + \cdots + \alpha_9 \sin(5z) + \alpha_{10} \cos(5z)$



Fourier Sine/Cosine Series $(n = 2m)$

$$A_0(z) = m + 1 \qquad B_0(z) = m + 1 + 2\sum_{j=1}^{m} \sinh^2(jy)$$

$$A_1(z) = 0 \qquad B_1(z) = -i\sum_{j=1}^{m} j\cosh(jy)\sinh(jy)$$

$$A_2(z) = m(m+1)(2m+1)/6 \qquad B_2(z) = 2\sum_{j=1}^{m} j^2\sinh^2(jy)$$

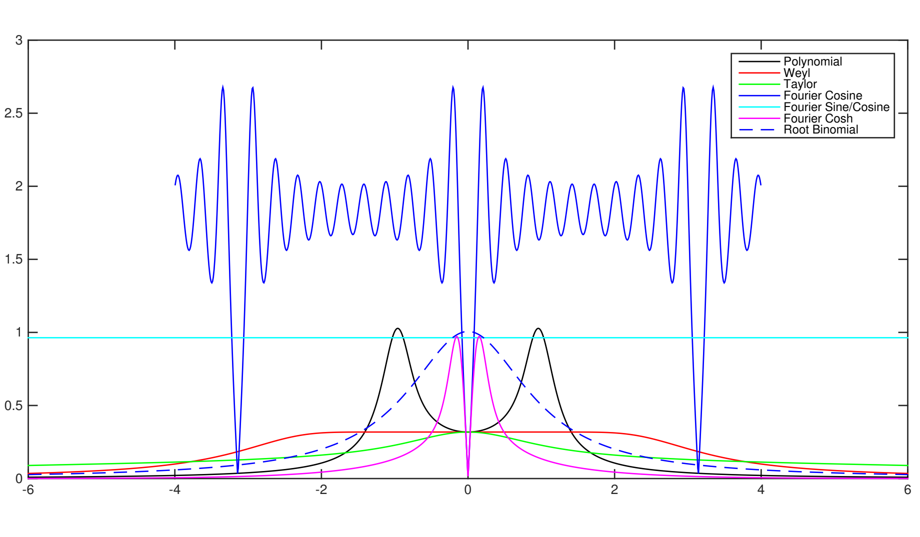
$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}$$

$$h_n = \frac{B_2D_0^2 - B_0(|B_1|^2 + |A_1|^2) + (A_0B_1\overline{A_1} + \overline{A_0B_1}A_1)}{\pi D_0^3} = \frac{B_2D_0^2 - B_0|B_1|^2}{\pi D_0^3}$$

$$g_n(x) = \frac{\sqrt{A_2(x)A_0(x) - A_1(x)^2}}{\pi B_0(x)}$$

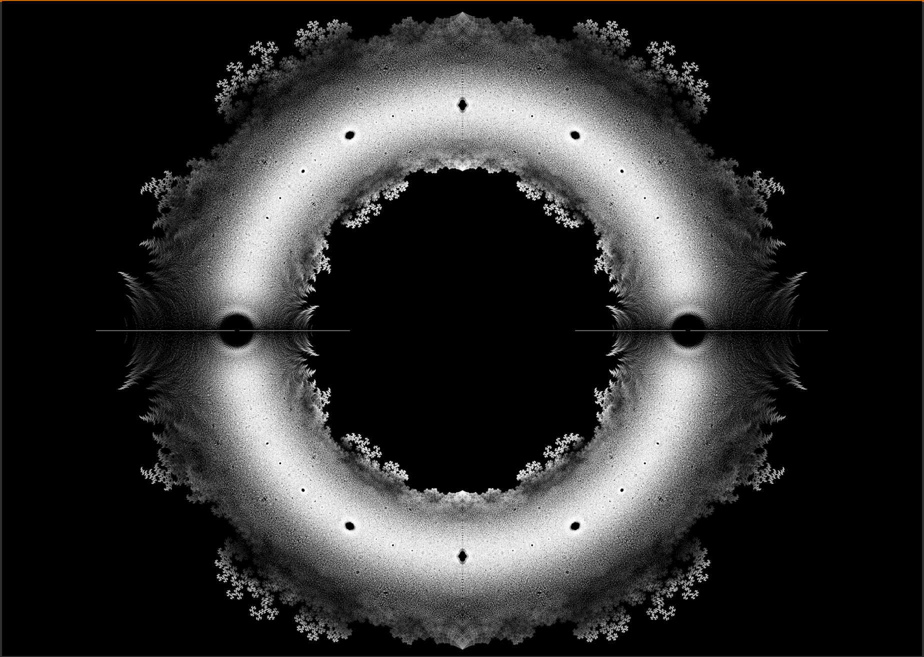
Densities of the Real Roots $n = 10$

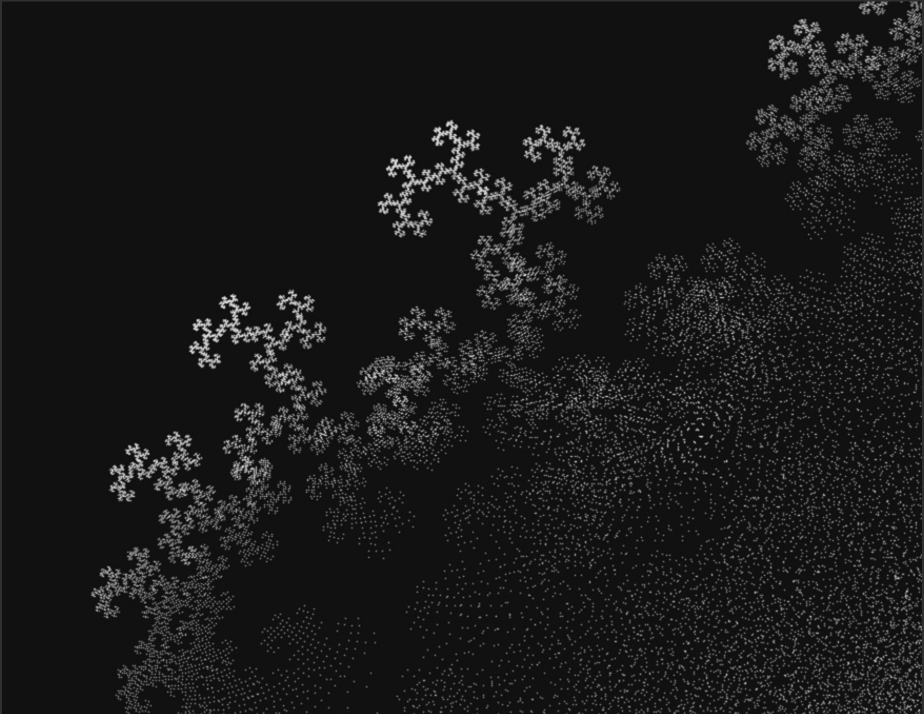
The function g_n for $n = 10$ for several choices of the f_j's:



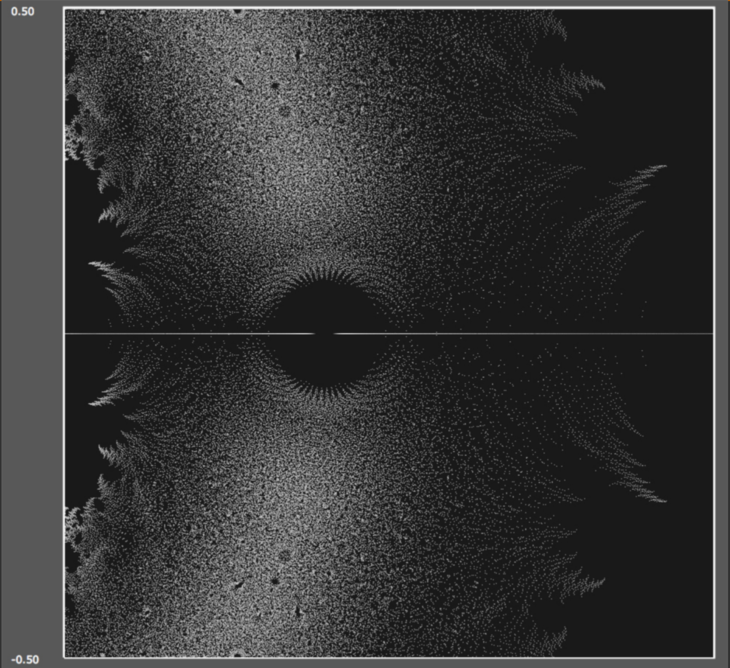
Let's Go Bernoulli!

Polynomial Roots $\eta_j = \pm 1$ $n = 18$

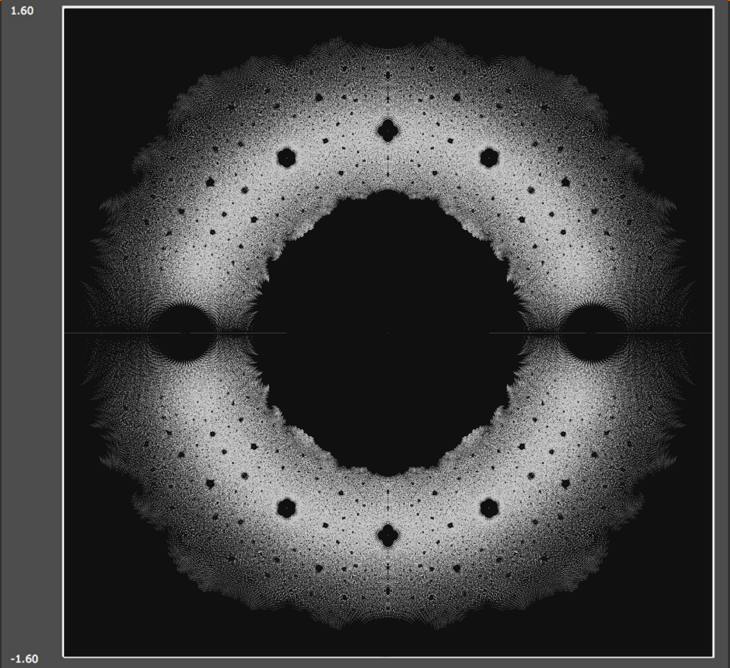




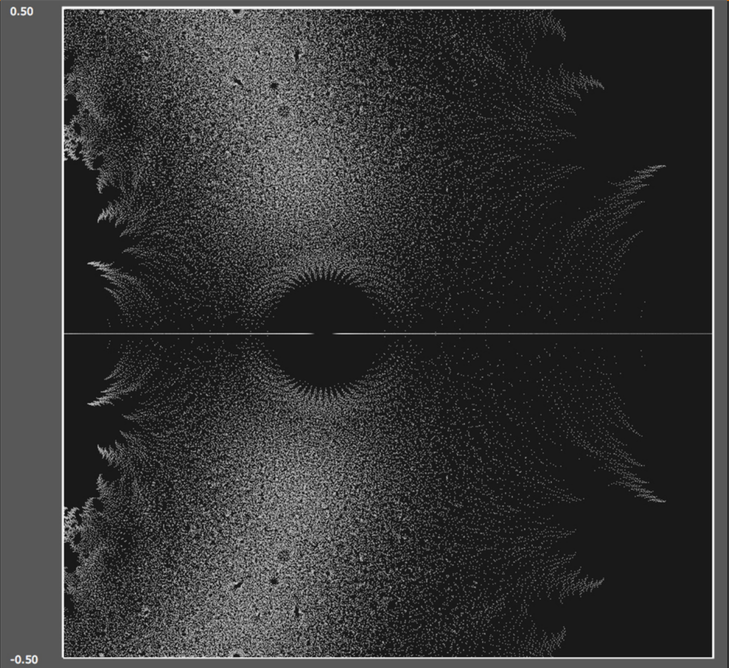
Close-Up View $\eta_j = \pm 1$ $n = 16$



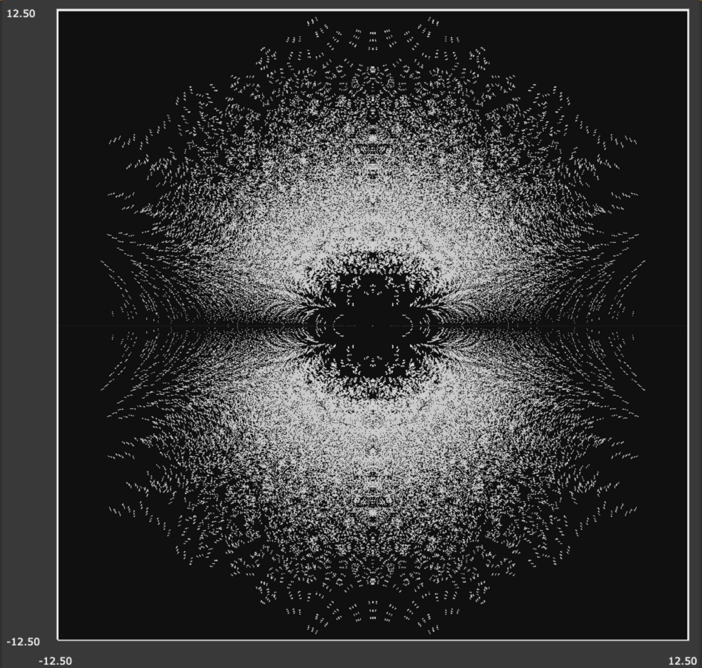
$\eta_j = -1, 0, +1 \qquad n = 12$



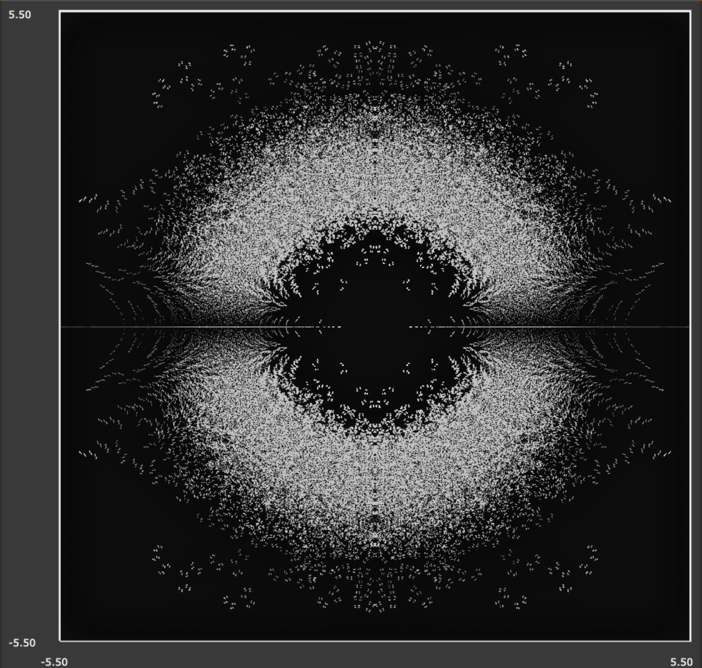
Close-Up View $\eta_j = -1, 0, +1$ $n = 12$



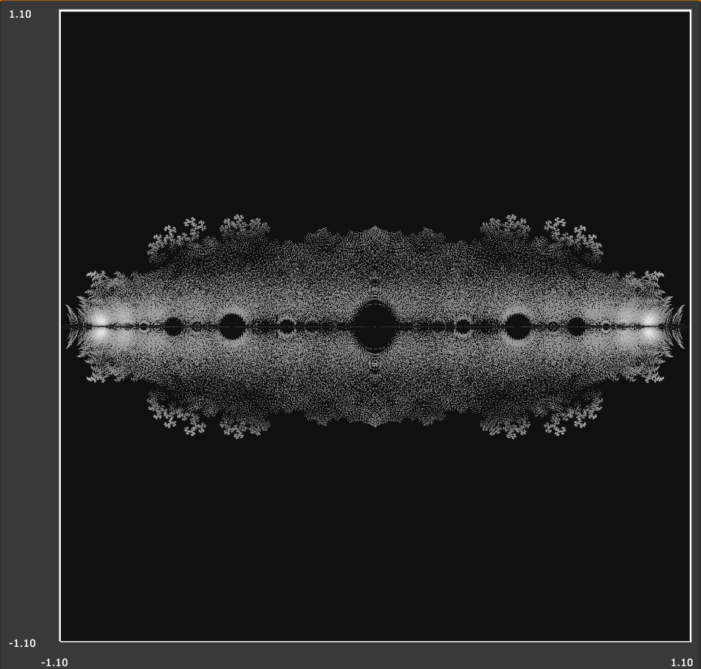
10-degree Taylor polynomial case with +1,0,-1 coefficients



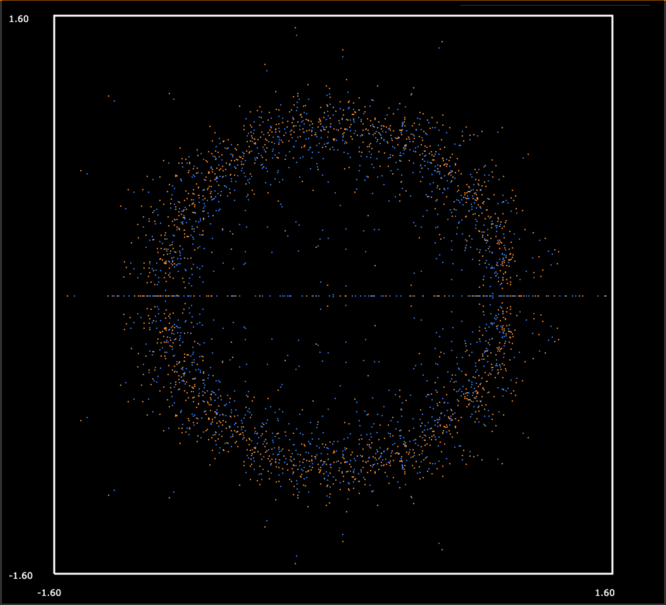
15-degree Sqrt-Taylor polynomial case with +1,0,-1 coefficients



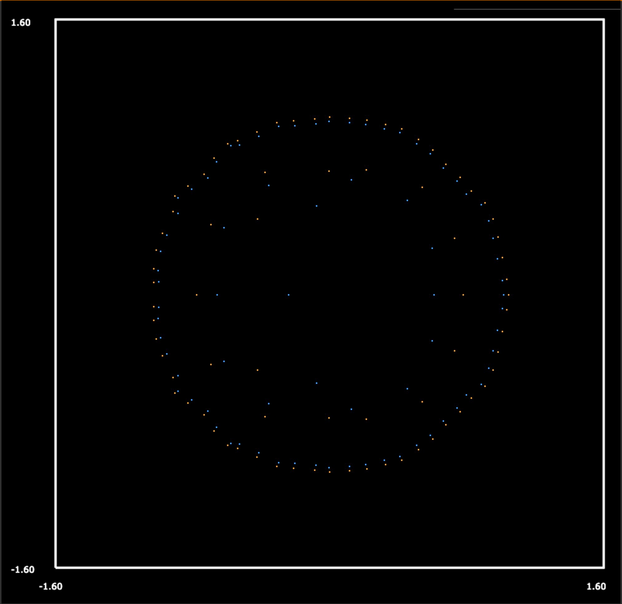
15-degree Chebyshev 2nd kind polynomial case with $\eta_j = \pm 1$ coefficients



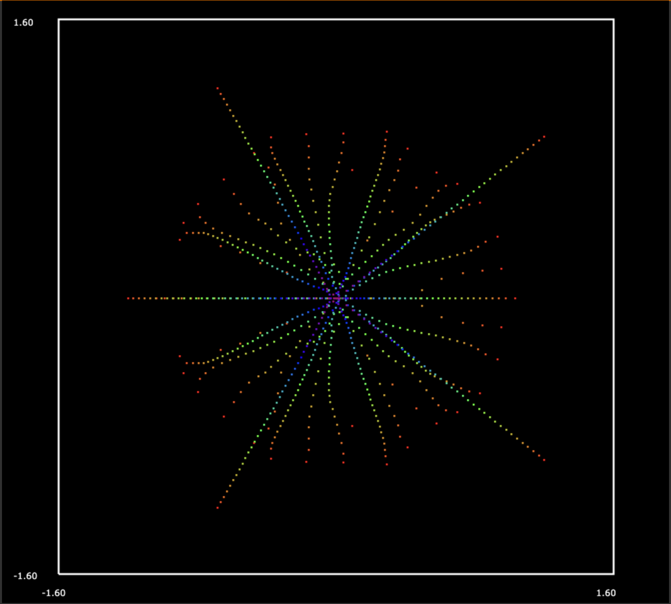
Zeros and Critical Points, $f(z) = \sum_{j=0}^{40} \alpha_j z^j$, 40 polynomials



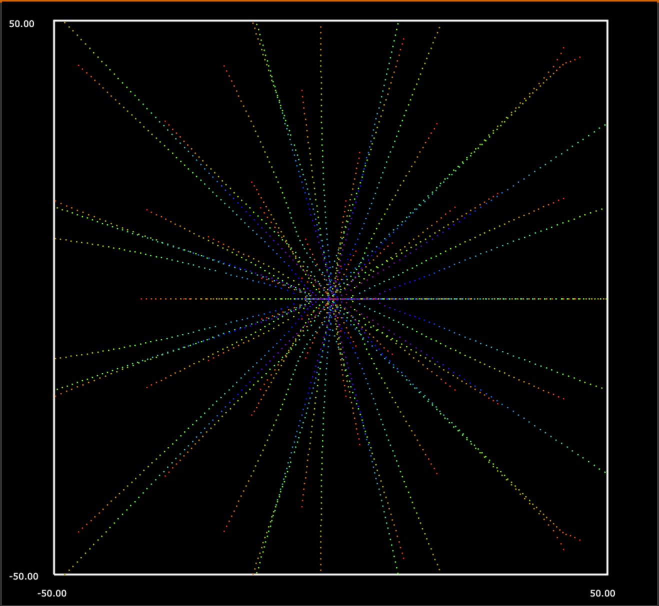
Zeros and Critical Points, $f(z) = \sum_{j=0}^{78} \alpha_j z^j$ (ref. Boris Hanin)



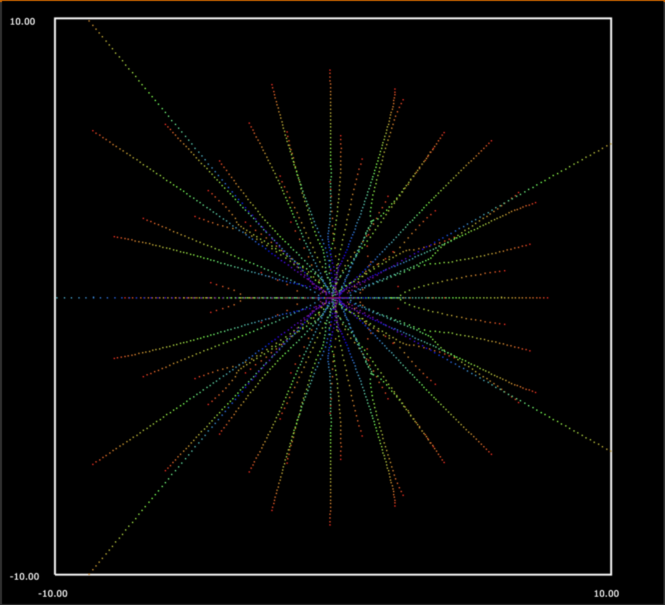
Zeros of all Derivatives, $f(z) = \sum_{j=0}^{40} \alpha_j z^j$



Zeros of all Derivatives, $f(z) = \sum_{j=0}^{j=80} \alpha_j \frac{z^j}{j!}$



Zeros of all Derivatives, $\quad f(z) = \sum_{j=0}^{j=80} \alpha_j \frac{z^j}{\sqrt{j!}}$



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Thank You!