Limit cycles in random polynomial systems

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joint work w/ Manju Krishnapur and Oanh Nguyen (Annals of Probability, to appear)

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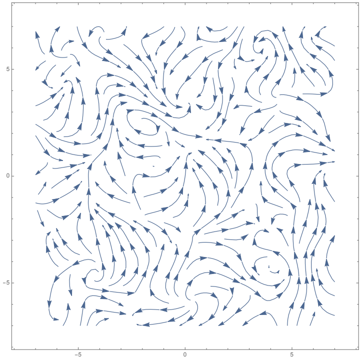
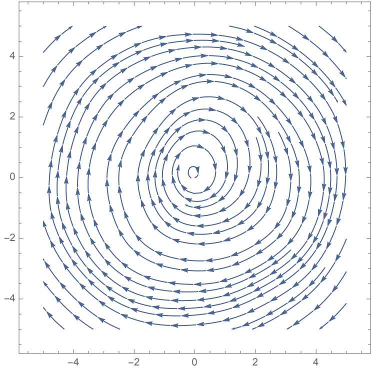
How many limit cycles do we expect to see?



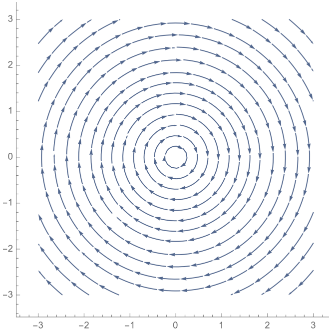


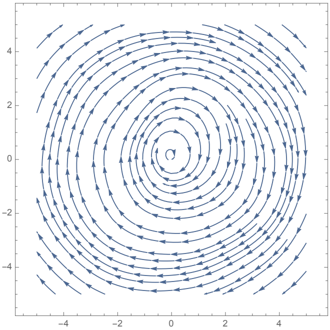
$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

$$\begin{cases} \dot{x} = \partial_y H(x, y) + \varepsilon p(x, y) \\ \dot{y} = -\partial_x H(x, y) + \varepsilon q(x, y) \end{cases}$$

For $H(x, y) = (x^2 + y^2)/2$ and $\varepsilon \to 0^+$ the question becomes:
"How many zeros of a random polynomial are real?"

Interesting simple case (perturbed linear center)





$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$\begin{cases} \dot{x} = y + \varepsilon p(x,y) \\ \dot{y} = -x + \varepsilon q(x,y) \end{cases}$$

With $p, q \sim$ Kac, $\mathbb{E} \lim_{\varepsilon \to 0^+} \#$ LCs $\rightsquigarrow$ expected number of zeros of random (univariate) generalized Kac with parameter $\tau = -1/2$.

Hilbert's sixteenth problem

First Part: Study the number and relative positions of the connected components of a real algebraic curve.

Second Part: Study the number and relative positions of the limit cycles of a planar polynomial ODE system.

(LCs important for planar systems $\leftarrow$ Poincaré-Bendixson Thm.)

"Except for the Riemann hypothesis, this seems to be the most elusive of Hilbert's problems." -Stephen Smale

Simpler problem promoted by V.I. Arnold: Study bifurcating LCs in perturbed Hamiltonian systems.

$$\begin{cases} \dot{x} = \partial_y H(x,y) + \varepsilon p(x,y) \\ \dot{y} = -\partial_x H(x,y) + \varepsilon q(x,y) \end{cases}$$

Hilbert's sixteenth problem (second part)

Several survey papers (and books) on the subject (Y. Ilyashenko, 2001 and J. Li, 2003).

- (Y. Ilyashenko, 1991, J. Écalle 1992) finiteness
- (C. J. Christopher and N. G. Lloyd, 1995 and M. Han and J. Li 2012) lower bounds of order $d^2 \log d$
- (A. Brudnyi, 2001, 2003) upper bound $O(\log d)$ for $\mathbb{E}\ \#$ limit cycles near a randomly perturbed center ($d =$ deg of perturbation)

Arnold's "infinitesimal" version (deg-d perturbation of deg-n Hamiltonian):

- (A.N. Varchenko, 1984 and A.G. Khovanskii, 1984) upper bound depending on degree (but non-explicit)
- (G.S. Petrov, 1997) linear in d, non-explicit dependence on n
- (G. Binyamini, D. Novikov, S. Yakovenko 2010) fully explicit upper bound depending on d and n
- (G. Binyamini, G. Dor 2012) fully explicit upper bound with linear dependence on d (still tower exponential dependence on n)

Randomly perturbed linear center

$$\begin{cases} \dot{x} = y + \varepsilon p(x,y) \\ \dot{y} = -x + \varepsilon q(x,y), \end{cases}$$

$$p(x,y) = \sum_{1 \le j+k \le d} a_{j,k} x^j y^k, \quad q(x,y) = \sum_{1 \le j+k \le d} b_{j,k} x^j y^k.$$

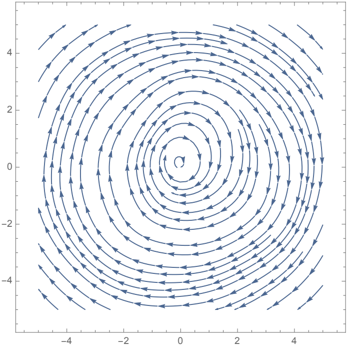
(Brudnyi, 2001)

$$\mathbb{E}\#LC \text{ in } \mathbb{D}_{1/2} = O(\log d)$$

as $d \to \infty$ and $\varepsilon = d^{-1/2} \to 0$.

(E.L., 2023)

$$\#LC \text{ in } \mathbb{D}_{1/2} \xrightarrow{a.s.} \underbrace{RV}_{\mathbb{E}(\cdot)<\infty}$$



Q. What about $N_d := \lim_{\varepsilon \to 0^+} \#LC$?

Number of bifurcating LC

$$\begin{cases} \dot{x} = y + \varepsilon p(x,y) \\ \dot{y} = -x + \varepsilon q(x,y), \end{cases}$$

$$p(x,y) = \sum_{1 \le j+k \le d} a_{j,k} x^j y^k, \quad q(x,y) = \sum_{1 \le j+k \le d} b_{j,k} x^j y^k.$$

Theorem (M. Krishnapur, E.L., O. Nguyen)

For p, q w/ i.i.d. Gaussian coefficients, the number N_d of bifurcating LCs satisfies

$$\mathbb{E}N_d \sim \frac{1}{2\pi} \log d, \quad \text{as } d \to \infty.$$

Moreover, the expected number of limit cycles in the unit disk satisfies

$$\mathbb{E}N_d(\mathbb{D}) \sim \frac{1}{\pi} \sqrt{\log d}, \quad \text{as } d \to \infty.$$

(We obtained same asymptotic for $\mathbb{E}N_d$ in non-Gaussian setting.)

How many zeros of a random polynomial are real?

The result on bifurcating limit cycles is a corollary of:

Theorem (M. Krishnapur, E.L., O. Nguyen)

Let

$$f_n(x) = \sum_{m=0}^{n} \xi_m x^m,$$

be a random (univariate) polynomial with independent Gaussian coefficients ξ_m having variance $\sigma_m^2 \sim m^{-1}$. Then as $n \to \infty$

$$\mathbb{E} N_{f_n}(1, \infty) = \frac{1 + o(1)}{2\pi} \log n,$$

and

$$\mathbb{E} N_{f_n}(0, 1) = \frac{1 + o(1)}{\pi} \sqrt{\log n}.$$

Interesting: The power -1 in $\sigma_m^2 \sim m^{-1}$ turns out to be "critical".

Generalized Kac models (power-law variance)

Theorem

Assume that ξ_m are independent mean zero Gaussians with variance $\sigma_m^2 \sim m^\lambda$. Then for

$$f_n(x) = \sum_{m=0}^{n} \xi_m x^m,$$

we have

$$\mathbb{E}N_{f_n}(0,1) = \begin{cases} \frac{\sqrt{\lambda+1}+o(1)}{2\pi} \log n & \text{if } \lambda > -1 \\ \frac{1+o(1)}{\pi}\sqrt{\log n} & \text{if } \lambda = -1 \\ \Theta(1) & \text{if } \lambda < -1. \end{cases}$$

The Melnikov function

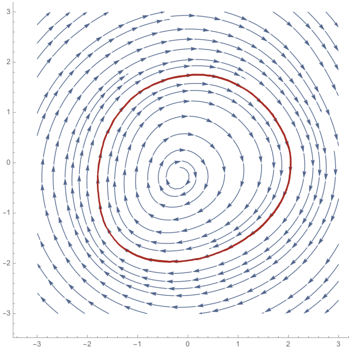
As $\varepsilon \to 0^+$, limit cycles of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} + \varepsilon \begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix}$$

are determined by zeros of

$$\mathscr{M}(r) = \int_{C_r} p\,dy - q\,dx$$

=net effect of perturbation to first-order.



Poincaré first-return map:
$\mathscr{P}(r) = r + \varepsilon \mathscr{M}(r) + O(\varepsilon^2)$.

LCs $\leftrightarrow$ fixed points of $\mathscr{P}$ $\leftrightarrow$ zeros of $\mathscr{M}(r)$

Integrating in polar coordinates

$$p(x,y) = \sum_{1 \le j+k \le d} a_{j,k} x^j y^k, \quad q(x,y) = \sum_{1 \le j+k \le d} b_{j,k} x^j y^k.$$

$$\begin{aligned}\mathscr{M}(r) &= \int_{x^2+y^2=r^2} p\,dy - q\,dx \\ &= \int_0^{2\pi} p(r\cos(\theta), r\sin(\theta)) r\cos(\theta)d\theta + q(r\cos(\theta), r\sin(\theta)) r\sin(\theta)d\theta \\ &= \sum_{m=0}^{\lfloor (d-1)/2 \rfloor} \xi_m r^{2m+2} = r^2 f_n(r^2),\end{aligned}$$

with ξ_m being linear combin of $a_{j,k}, b_{j,k}$ w/ coeff:

$$\int_0^{2\pi} (\cos(\theta))^{j+1} (\sin(\theta))^k d\theta$$

Perturbations with $p, q \sim$ Kostlan

$$\begin{cases} \dot{x} = y + \varepsilon p(x,y) \\ \dot{y} = -x + \varepsilon q(x,y) \end{cases}, \quad p, q \sim \text{Kostlan}(d).$$

Theorem (EL, 2023)

For $\rho > 0$ let $N_{d,\varepsilon}(\rho)$ denote # LCs in the disk $\mathbb{D}_\rho$.
As $\varepsilon \to 0$, $N_{d,\varepsilon}(\rho)$ converges almost surely to a random variable $N_d(\rho)$ that satisfies the following asymptotic in d.

$$\mathbb{E}N_d(\rho) \sim \frac{\arctan \rho}{\pi}\sqrt{d}, \quad \text{as } d \to \infty.$$

Kac-Rice formula applied to Melnikov function

$$\mathscr{M}(r) = \int_{C_r} p\,dy - q\,dx$$

The Kac-Rice formula gives

$$\begin{aligned}\mathbb{E}|\{r \in [a,b] : \mathscr{M}(r) = 0\}| &= \int_a^b \mathbb{E}|\mathscr{M}'(\tau)|\delta(\mathscr{M}(\tau))d\tau \\ &= \int_a^b \sqrt{\frac{\partial^2}{\partial r \partial t} \log \mathscr{K}(r,t)|_{r=t=\tau}}\, d\tau,\end{aligned}$$

$$\mathscr{K}(r,t) := \mathbb{E}\mathscr{M}(r)\mathscr{M}(t) \quad \text{(covariance kernel)}$$

For $p, q \sim$ Kostlan(d)

$$\mathscr{K}(r,t) = \int_{C_r}\int_{C_t} (1 + x_1x_2 + y_1y_2)^d \left[dy_1dy_2 + dx_1dx_2\right].$$

Future direction: Zeros of random Abelian integrals

Fix a generic polynomial Hamiltonian H of degree n, and consider

$$\begin{cases} \dot{x} = \partial_y H(x,y) + \varepsilon p(x,y) \\ \dot{y} = -\partial_x H(x,y) + \varepsilon q(x,y) \end{cases}, \quad p, q \sim \text{Kostlan}(d).$$

Limit cycle births are determined by the Melnikov function.

$$\mathscr{M}(t) = \int_{C_t} p\,dy - q\,dx, \quad C_t := \{(x,y) : H(x,y) = t\}.$$

The Kac-Rice formula applies, but asymptotics appear delicate.

Conjecture: There exists $C = C(H)$ such that

$$\mathbb{E} N_d \sim C \cdot \sqrt{d}, \quad \text{as } d \to \infty,$$

and C satisfies $C(H) \leq C_0 n^2$ for some absolute constant $C_0 > 0$.

Nonautonomous perturbations and chaotic dynamics

(current project with my student Ali Ittayem)

Random trigonometric polynomials (a_k, b_k i.i.d. Gaussians)

$$\mathscr{M}(t) = a_0 + \sum_{k=1}^{N} \omega k \operatorname{sech}\left(\frac{k\pi\omega}{2}\right)(a_k \cos(k\omega t) + b_k \sin(k\omega t)),$$

appear as Melnikov integrals in a different setting:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases} \xrightarrow{\text{add forcing/damping}} \begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \varepsilon(f(t) - \delta y) \end{cases}$$

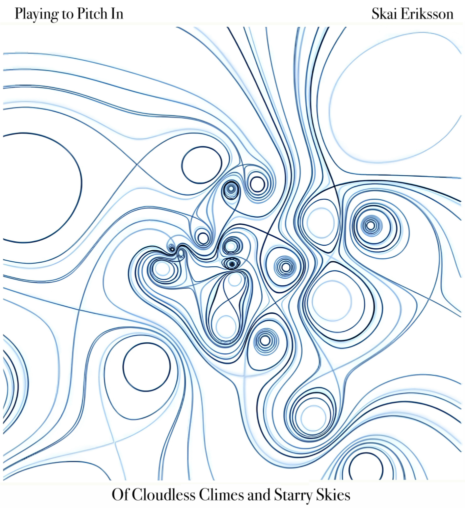
A zero of $\mathscr{M}(t)$ corresponds to a transverse intersection of stable and unstable manifolds $\rightsquigarrow$ homoclinic tangle (and chaos in the sense of Smale).

Kac-Rice gives expected number of zeros in a period interval $\to C_\omega$ as $N \to \infty$. Also, $C_\omega \to 0$ as $\omega \to \infty$, and $C_\omega \sim \sqrt{\frac{14}{5}\frac{2}{\omega}}$ as $\omega \to 0$.

Future directions / open problems

- Open problems (several) about the critical case $\lambda = -1$ of generalized Kac: variance? CLT? precise asymptotic for $\mathbb{E}N_n(0,1)$ in the non-Gaussian case?
- A future direction: perturbed Hamiltonian systems $\rightarrow$ zeros of random Abelian integrals (randomized problem of V.I. Arnold)
- Predator-prey: Study limit cycles in randomly perturbed Lotka-Volterra models.
- Chaotic dynamics: Use another form of Melnikov's method to study occurrences of homoclinic tangles in non-autonomous perturbations of Hamiltonian systems (e.g., nonlinear oscillator with small forcing). Reduces to random trigonometric polynomials.
- Study random Liénard systems (a non-perturbative problem) motivated by Smale's 13th problem.
- Study limit cycles on a cylinder, motivated by Lins-Neto's revised version of Pugh's problem (a non-perturbative problem).

Thanks for listening!



Neoclassical piano album released August 2025 (all royalties to charity)