Limit law for root separation in random polynomials

Oren Yakir
Stanford University

ICERM workshop, August 2025

Joint work with Marcus Michelen (UIC)

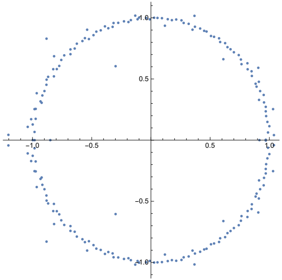
(arXiv 2505.02723)

Kac polynomials

- Let $\xi_0, \ldots, \xi_n$ be i.i.d. random variables, think Gaussians or random signs $\{-1, +1\}$.
- The Kac model for random polynomials:

$$f_n(z) = \sum_{k=0}^{n} \xi_k z^k$$

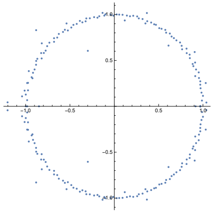
- What can we say about spatial distribution of roots, as $n \to \infty$?



Kac polynomials

- Let $\xi_0, \ldots, \xi_n$ be i.i.d. random variables, think Gaussians or random signs $\{-1, +1\}$.
- The Kac model for random polynomials:

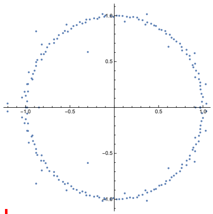
$$f_n(z) = \sum_{k=0}^{n} \xi_k z^k$$



- What can we say about spatial distribution of roots, as $n \to \infty$?
- Ibragimov-Zaporozhets (2011): Picture holds $\iff \mathbb{E}\log(1 + |\xi_0|) < \infty$.
- Universality phenomena:

Distribution of roots does not depend on particular distribution of random coefficients

Roots around the unit circle



Finer results are known:

- Famously, $\mathbb{E}[\text{real roots}] \asymp \log n$ (Kac 40s, Erdős-Offord 50s, ...)
- Shepp-Vanderbei, Ibragimov-Zeitouni 90s:

  most roots are at distance $1/n$ from the unit circle

- Michelen-Sahasrabudhe, Cook-Nguyen-Y.-Zeitouni 2020s:

$$\left\{ n^2 (1 - |z|) \ : \ f_n(z) = 0 \right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\left(\frac{1}{12} \mathrm{d}t\right)$$

- Qualitative repulsion for Gaussian roots (Shiffman-Zelditch 2003)
- Local universality for k-point functions (Tao-Vu, O. Nguyen-Vu, ... 2000s)

Main result

- Let $\alpha_1, \ldots, \alpha_n$ denote the roots of f_n. To study repulsion **quantitatively**, set

$$m_n = \min\left\{|\alpha_i - \alpha_j| \,:\, i \neq j\right\}.$$

- Clearly $m_n = 0 \iff f_n$ has a double root.

Main result

- Let $\alpha_1, \ldots, \alpha_n$ denote the roots of f_n. To study repulsion **quantitatively**, set

$$m_n = \min\left\{|\alpha_i - \alpha_j| \,:\, i \neq j\right\}.$$

- Clearly $m_n = 0 \iff f_n$ has a double root.

Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ with i.i.d. sub-Gaussian coefficients and $\mathbb{P}(\xi_0 = 0) = 0$. Then

$$\left\{n^{5/4}\,|\alpha_i - \alpha_j| \,:\, i \neq j\right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\left(\mathrm{c}_* t^3 \,\mathrm{d}t\right)$$

on $\mathbb{R}_{\geq 0}$ the non-negative reals, for some $\mathrm{c}_* > 0$. In particular, for all $t \geq 0$

$$\lim_{n\to\infty} \mathbb{P}\left(n^{5/4} m_n \geq t\right) = \exp\left(-\frac{\mathrm{c}_*}{4} t^4\right).$$

Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ *with i.i.d. sub-Gaussian coefficients and* $\mathbb{P}(\xi_0 = 0) = 0$. *Then*

$$\left\{ n^{5/4} \left|\alpha_i - \alpha_j\right| \,:\, i \neq j \right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\left(\mathrm{c}_* t^3 \, \mathrm{d}t\right)$$

on $\mathbb{R}_{\geq 0}$ *the non-negative reals, for some* $\mathrm{c}_* > 0$. *In particular, for all* $t \geq 0$

$$\lim_{n\to\infty} \mathbb{P}\left(n^{5/4} m_n \geq t\right) = \exp\left(- \frac{\mathrm{c}_*}{4} t^4 \right).$$

- The exponent $5/4$ reflects on the repulsion, more on that in the next slide.
- The assumption $\mathbb{P}(\xi_0 = 0) = 0$ is not essential.
- Corollary: $\mathbb{P}\big(f_n$ has a double root$\big) \to 0$ as $n \to \infty$.
- Corollary (and more) was previously proved in Peled-Sen-Zeitouni (2016) Feldheim-Sen (2017) for ξ_0 random bounded integer with no atom bigger that $1/\sqrt{3}$.

Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ *with i.i.d. sub-Gaussian coefficients and* $\mathbb{P}(\xi_0 = 0) = 0$. *Then*

$$\left\{ n^{5/4}\, |\alpha_i - \alpha_j| \,:\, i \neq j \right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\big(c_* t^3 \,\mathrm{d}t\big) \qquad \textit{on} \;\; \mathbb{R}_{\geq 0}\,.$$

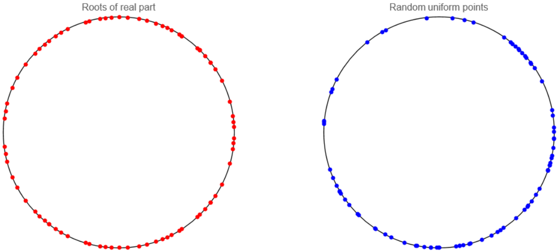
- The normalization $n^{-5/4}$ reflects the repulsion between random roots.
- Let $x_1, \ldots, x_n$ be i.i.d. uniform points in $\left\{1 - \frac{1}{n} \leq |z| \leq 1 + \frac{1}{n}\right\}$.
- This is a simple toy model where we pretend the roots do not interact.

Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ *with i.i.d. sub-Gaussian coefficients and* $\mathbb{P}(\xi_0 = 0) = 0$. *Then*

$$\left\{ n^{5/4} \, |\alpha_i - \alpha_j| \, : \, i \neq j \right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\left(c_* t^3 \, \mathrm{d}t\right) \qquad \text{on } \mathbb{R}_{\geq 0} \, .$$

- The normalization $n^{-5/4}$ reflects the repulsion between random roots.
- Let $x_1, \ldots, x_n$ be i.i.d. uniform points in $\left\{1 - \frac{1}{n} \leq |z| \leq 1 + \frac{1}{n}\right\}$.
- This is a simple toy model where we pretend the roots do not interact.



Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ *with i.i.d. sub-Gaussian coefficients and* $\mathbb{P}(\xi_0 = 0) = 0$*. Then*

$$\left\{ n^{5/4} |\alpha_i - \alpha_j| \, : \, i \neq j \right\} \xrightarrow[n \to \infty]{d} \mathrm{PoissonPP}\left(\mathrm{c}_* t^3 \, \mathrm{d}t \right) \qquad \textit{on} \;\; \mathbb{R}_{\geq 0} \, .$$

- Let $x_1, \ldots, x_n$ be i.i.d. uniform points in $A = \{1 - n^{-1} \leq |z| \leq 1 + n^{-1}\}$.
- For this toy model, the minimal separation scale like $n^{3/2}$
- Heuristically, for "most" $z \in A$ we have

$$\mathbb{P}\Big(\exists \text{ two distinct roots of } f_n \text{ in } \mathbb{D}(z, \delta) \Big) \approx n^2 \times (n\delta^2)^2 \times (n\delta)^2 \qquad (*)$$

- We can cover the annulus with $n^{-1}\delta^{-2}$ such disks, so $\delta = n^{-5/4}$ is meaningful.
- To show $(*)$, we need to:
  - Use randomness to reduce quadratic approximation to two linear approximations.
  - Develop accurate local CLT á la Konyagin-Schlag (1999), Cook-Nguyen (2021).

Local Central Limit Theorem

For $f_n(z) = \sum_{k=0}^{n} \xi_k z^k$ with $\xi_k \sim \text{Uniform}\{\pm 1\}$, difficulties of Diophantine nature appear:

- $\mathbb{P}\big(f_n(1) = 0\big) = \mathbb{P}\big(\sum_k \xi_k = 0\big) \asymp n^{-1/2}$ for n odd.
- $\mathbb{P}\big(f_n(e^{i\frac{\pi}{4}}) = 0\big) \asymp n^{-1}$ over multiples of 4.

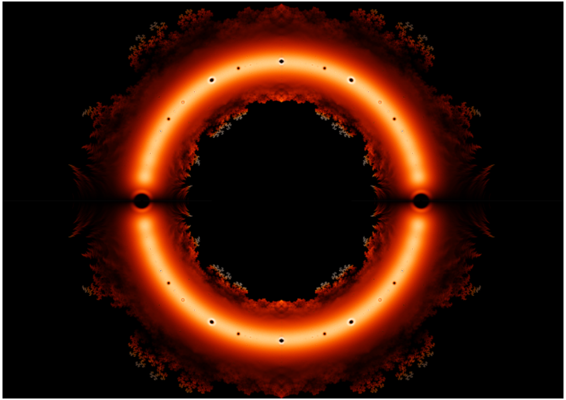
Theorem (Michelen - Y. 2025)

For $z \in \overline{\mathbb{D}}$ we let $d_n(z) = \min\{n, (1-|z|)^{-1}\}$. For any "smooth" z and any rectangle $Q \subset \mathbb{C}$ with side lengths at least $d_n(z)^{-K}$,

$$\mathbb{P}_{\pm 1}\big(f_n(z) \in Q\big) = \big(1 + o(1)\big) \cdot \mathbb{P}_G\big(f_n(z) \in Q\big) \qquad \text{as } \; d_n(z) \to \infty.$$

- In fact, we need a local CLT for $\big(f_n(z), f_n'(z), f_n''(z)\big)$
- We also need the local CLT for tuples of well-separated, smooth points.

What is a “smooth” point?



Picture taken from “The Beauty of Roots”, John Baez (2011).

Bird's eye view of the proof

Theorem (Michelen - Y. 2025)

Let $f_n = \sum_{k=0}^{n} \xi_k z^k$ with i.i.d. sub-Gaussian coefficients and $\mathbb{P}(\xi_0 = 0) = 0$. Then

$$\left\{ n^{5/4} |\alpha_i - \alpha_j| \,:\, i \neq j \right\} \xrightarrow[n\to\infty]{d} \mathrm{PoissonPP}\left(\mathrm{c}_* t^3 \,\mathrm{d}t\right) \qquad \text{on } \mathbb{R}_{\geq 0}\,.$$

(A) Poisson scaling in $\left\{1 - \frac{K}{n} \leq |z| \leq 1 + \frac{K}{n}\right\}$ as $n \to \infty$, followed by $K \to \infty$.

(B) Roots in $\left\{r \leq |z| \leq 1 - \frac{K}{n}\right\}$ do not contribute as $n \to \infty$, $K \to \infty$, $r \to 1$.

(C) Roots in $\left\{|z| \leq r\right\}$ do not contribute for all $r < 1$ as $n \to \infty$.

- (A) and (B) follows from previous slides, noting that $f_n(z)$ behaves effectively like a Kac polynomial of degree $\min\{n, (1 - |z|)^{-1}\}$ and summing dyadically.
- How to handle (C)? There is no "Gaussian" universality in the disk.

Random Taylor series in the disk

Let $\{\zeta_k\}$ be i.i.d. (complex) random variables such that $\mathbb{E}\log(1+|\zeta_0|) < \infty$. Set

$$F(z) = \sum_{k=0}^{\infty} \zeta_k z^k$$

- F is a random analytic function in the unit disk $\mathbb{D}$.
- Obtained as the weak distributional limit of Kac polynomial in compact subsets of $\mathbb{D}$.

Random Taylor series in the disk

Let $\{\zeta_k\}$ be i.i.d. (complex) random variables such that $\mathbb{E}\log(1+|\zeta_0|) < \infty$. Set

$$F(z) = \sum_{k=0}^{\infty} \zeta_k z^k$$

- F is a random analytic function in the unit disk $\mathbb{D}$.
- Obtained as the weak distributional limit of Kac polynomial in compact subsets of $\mathbb{D}$.

Theorem (Michelen - Y. 2025)

We have $\mathbb{P}\big(F \text{ has a double zero}\big) = \big(\mathbb{P}(\zeta_0 = 0)\big)^2$. In particular, if $\mathbb{P}(\zeta_0 = 0) = 0$, then almost surely F has no double zero.

- Previously proved only for Gaussian coefficients (Peres-Virág 2005). Even for Gaussians the proof is not trivial, as basic density arguments are not applicable.
- Proved assuming only that $\mathbb{E}\log(1+|\zeta_0|) < \infty$. Based on a perturbative argument.

Thank you!