Persistence Probabilities: in Polynomials and Auto-Regressive Sequences

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[1]Based on joint works with Jian Ding, Fuchang Gao, Sumit Mukherjee, Bjorn Poonen, Qi-Man Shao, Jun Yan and Ofer Zeitouni.

Persistence probabilities

$(X_t, t \geq 0)$ is $\mathbb{R}$-valued $(X_0 = 0)$. First passage time

$$\tau_z = \inf\{t > 0 : X_t > z\}\,.$$

- Typically $z = 0$.
- Persistence (power law) exponent b if $\mathbb{P}(\tau_z \geq T) = T^{-b+o(1)}$.
- Large deviations problem (hitting probab. of Markov processes).
- In many examples, little gained from general theories (?).
- Focus on few examples of much interest.

Real zeros of algebraic polynomials

$x \in \mathbb{R}$, n integer, $J \subseteq \mathbb{R}$ an interval, ξ_i i.i.d. of symmetric law,

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- Up to factor 2 same as $\mathbb{P}(Q_n(\cdot)$ has no zeros in $J)$.

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- $\mathbb{E}(N_n) \sim c_\alpha \log n$ if ξ_i attracted to $\alpha \in (0, 2]$ stable [Ibragimov-Maslova/Logan-Shepp '68-'71; following Kac '43]. Recent work on SLLN & CLT for N_n, expansions for $\mathbb{E}(N_n)$ (yesterday talks - Yattselev & Nhan Nguyen)

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- Much work on complex zeros (e.g. [Edelman-Kostlan '95; Ibragimov-Zeitouni '97, Sodin-Tsirelson '04]; Yakir & Michelen talks).

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- $O(e^{-Cn}) \leq p_{\mathbb{R}}(2n) \leq O(1/\log n)$ if $p > 2$ finite moments [Maslova '74; after Littlewood-Offord '38-'43].

Persistence for real zeros

$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary, zero-mean, Gaussian process $t \mapsto Y_t^{(\kappa)}$ of covariance $\mathbb{E}[Y_s^{(\kappa)} Y_t^{(\kappa)}] = (\mathrm{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

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I. Open: α-stable ξ_i ($\alpha = 1$, exponent $0.86 > 4b_0$ - numerics).

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  - Auto-correlation of $Q_n(e^{-e^{-s}})$ explicit & close to that of $Y_s^{(0)}$.

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Independent $\xi_i \sim N(0, L(i))$, $i \mapsto L(i)$ regularly varying, order κ $(L(\lambda i)/L(i) \to \lambda^\kappa$ as $i \to \infty)$.

- $p_{\pm[0,1]}(n) \approx n^{-b_\kappa} \quad (b_\kappa = 0, \kappa \leq -1) \quad \& \quad p_{\pm[1,\infty)}(n) \approx n^{-b_0}$.
[Dembo-Mukherjee '15] (predicted in [Schehr-Majumdar '07-'08]).
  - For general $L(\cdot)$ arbitrary convergence rate ε_n for auto-correlations of $Q_n(e^{-e^{-s}})$ to those of $Y_s^{(\kappa)}$.

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  - Key to [Dembo-Mukherjee '15] is continuity result for persistence exponents in Gaussian processes of non-negative covariance.
  - Utilized for Elliptic [DM15] & Weyl polynomials [Can-Pham '19].

Persistence exponents: Markov, Gaussian, Continuity

Stationary, zero-mean, normalized ($\mathbf{E}[Z_k^2(t)] = 1$), Gaussian processes $Z_k(t)$, $1 \leq k \leq \infty$, of non-negative covariance.

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- [Aurzada-Mukherjee-Zeitouni '21]: $\bar{b}$ & Markov kernel of $i \mapsto Z(i)$.

b_κ & Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x}, t)}{\partial t} = \Delta \phi_d(\mathbf{x}, t)$$

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    & In no-real eigenvalue for certain random matrices
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- **II. Open:** Prove $b_0 = 3/16$; Explain these connections!

Gaussian interface models (slow decay of correlation)

- $\nabla\phi$-interface model in $(d+1)$ dimensions solves SDS:

$$d\phi_t(\mathbf{x}) = -\frac{\partial H}{\partial \phi(\mathbf{x})}\big(\phi_t(\mathbf{x})\big) + \sqrt{2}dB_t(\mathbf{x})$$

with $\phi_0(\mathbf{x}) = 0$, $B_t(\mathbf{x})$ independent Brownian motions,

$$H(\phi) = \frac{1}{4}\sum_{\mathbf{x}}\sum_{\mathbf{y}} q(\mathbf{x}-\mathbf{y})(\phi(\mathbf{x}) - \phi(\mathbf{y}))^2$$

- q symmetric law of bounded support $\Longleftrightarrow$ jumps for $\boldsymbol{Z}^d$-irreducible RW S_t of expected local time $L(\cdot)$ at $\mathbf{0}$.
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  - $p_3(T) \approx O(\sqrt{T}\log T)$, $p_4(T) \approx O(T\log\log T/\log T)$.

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- **III. Open:** decay rate for $p_2(T)$ (the slowly-var case).

ℓ-iterated sums & integrals of Brownian motion

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- $r_\ell \downarrow b_0$ for $\ell \uparrow \infty$ [Li-Shao '05].

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IV. Open: Find r_ℓ, $\ell \geq 3$; Uniform bounds above and below?

Auto-regressive sequences: stability

$\underline{a} := (a_1, a_2, \dots, a_L) \in \mathbb{R}^L$ non-random.

For $\xi_i \sim N(0,1)$ i.i.d.

$$X_k = 0, \forall k \le 0, \quad \& \quad X_k = \sum_{i=1}^{L} a_i X_{k-i} + \xi_k, \qquad \forall k \ge 1$$

is a Gaussian auto-regressive sequence
($\underline{a} = (1)$ for S_k; $\underline{a} = (2,-1)$ for $S_k^{(2)}$; $\underline{a} = (3,-3,1)$ for $S_k^{(3)}$).

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$k \mapsto X_k$ stable if and only if $\rho < 1$:

$$\rho := \max\{|z| : z \in \Lambda\}, \qquad \Lambda := \{z \in \mathbb{C} : z^L - \sum_{i=1}^{L} a_i z^{L-i} = 0\}.$$

Persistence for AR sequences

- [Dembo-Ding-Yan '19] study asymptotics of

$$f_{\underline{a}}(n) := -\log \mathbb{P}(X_k < 0, \quad \forall k \in [1, n]) \, .$$

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V. Open: Prove $\exists$ exponents; Find $r_{\underline{a}}$; Related to persistence for d-dimensional BM staying in cones? ([Banuelos-Smits '97]).

Thank you!