Persistence of random polynomials

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Outline

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- Question: What is the chance that the polynomial $Q_n(\cdot)$ has no real roots?

- More precisely, setting

$$p_n := \mathbb{P}(Q_n(x) \neq 0, \text{ for all } x \in \mathbb{R}),$$

we want to study asymptotics of p_n.

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$$p_n = \mathbb{P}(\sup_{x\in\mathbb{R}} Q_n(x) < 0) + \mathbb{P}(\inf_{x\in\mathbb{R}} Q_n(x) > 0).$$

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- q_n is the probability that $Q_n(\cdot)$ persists below the origin.

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- More precisely,

$$b_0 = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}(\sup_{t\in[0,T[} Y^{(0)}(t) < 0),$$

where $\{Y^{(0)}(t), t \geq 0\}$ is a centered GSP with the above covariance function.

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- This value agrees with the above rigorous bounds, and all existing simulation results.

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- For $\alpha \leq -1$ set $b_\alpha = 0$.

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- Here $\hat{b}_\infty$ is the persistence exponent of a centered GSP with covariance $\exp\Big(-(t-s)^2/8 \Big)$.

Main question

- It was conjectured in Poonen-Stoll, AoM-99 that universality of the asymptotics of q_n should hold whenver $\{a_i\}_{i \geq 0}$ is in the domain of attraction of $N(0,1)$.

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- To answer both these questions in a unified framework, in Ghosal-M., Arxiv-24 we set $a_i = \sqrt{R(i)}\xi_i$ for $i \geq 0$.

- Here $\{\xi_i\}_{i\geq 0}$ is a sequence of IID random variables with finite second moment, and $R(i)$ is a regularly varying function of order $\alpha > -1$.

Connection to previous examples

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- Thus the leading order asymptotics does not depend on the distribution of $\{\xi_i\}_{i \geq 0}$ under the second moment assumption.

- Without the second moment assumption, it is expected that the exponent will change.

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$$\limsup_{n\to\infty} \frac{1}{n} \log q_n \leq -2(b_0 + b_\alpha).$$

- This works for all n (as expected).

Outline

Lower bound

- Partition the domain $\mathbb{R}$ into 5 disjoint parts

$$\begin{aligned}
A_0 =& \Big[-\frac{K}{n}, \frac{K}{n}\Big], \\
A_{+1} =& \Big(\frac{K}{n}, \frac{h}{\log n}\Big], \\
A_{+2} =& \Big(\frac{h}{\log n}, \infty), \\
A_{-1} =& \Big[-\frac{h}{\log n}, -\frac{K}{n}\Big), \\
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- Note that $A_{-1} = -A_1$, and $A_{-2} = -A_{+2}$.

- Also,

$$\cup_{r \in \mathcal{R}} A_r = \mathbb{R},$$

where $\mathcal{R} = \{0, \pm 1, \pm 2\}$.

Lower bound

- Similarly, partition the set of indices $[n] := \{0, 1, 2, \cdots, n\}$ into 5 parts,

$$\begin{aligned}
B_0 =& [n] \cap \left[\frac{n}{D}, n - \frac{n}{D}\right], \\
B_{+1} =& [n] \cap \left(n - \frac{n}{D}, n - L \log n\right], \\
B_{+2} =& [n] \cap \left(n - L \log n, n\right], \\
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- Again note that

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- Also we can write

$$Q_n(x) = \sum_{r \in \mathcal{R}} Q_n^{(r)}(x),$$

where the polynomials $Q_n^{(r)}(x) = \sum_{i \in B_r} a_i x^i$ are independent.

Lower bound

- Using the above decomposition, and changing variables gives (with $\sigma_n^2(u) = Var(Q_n(\pm e^u))$),

$$\left\{\sup_{x\in\mathbb{R}} Q_n(x) < 0\right\} = \left\{\sup_{u\in\mathbb{R}} \sum_{r\in\mathcal{R}} \frac{Q_n^{(r)}(\pm e^u)}{\sigma_n(u)} < 0\right\}.$$

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- Using the above decomposition, one way to guarantee persistence is

$$\Big\{\sup_{u\in A_r} \frac{Q_n^{(r)}(\pm e^u)}{\sigma_n(u)} < -\delta, \max_{s\neq r} \sup_{u\in A_s} \frac{Q_n^{(r)}}{\sigma_n(u)} < \frac{\delta}{4}\Big\}.$$

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- Also these events are independent for different values of r, giving the lower bound

$$\prod_{r\in\mathcal{R}} \mathbb{P}\left(\sup_{u\in A_r} \frac{Q_n^{(r)}(\pm e^u)}{\sigma_n(u)} < -\delta, \max_{s\neq r} \sup_{u\in A_s} \frac{Q_n^{(r)}}{\sigma_n(u)} < \frac{\delta}{4}\right).$$

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- Thus we need to lower bound

$$\mathbb{P}\left(\sup_{u \in A_{-1}} \frac{Q_n^{(-1)}(\pm e^u)}{\sigma_n(u)} < -\delta, \sup_{u \notin A_{-1}} \frac{Q_n^{(-1)}(\pm e^u)}{\sigma_n(u)} < \frac{\delta}{4}\right).$$

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- Recall that

$$-A_{-1} = \left(\frac{K}{n}, \frac{h}{\log n} \right], \quad B_{-1} = [n] \cap \left[L \log n, \frac{n}{D} \right).$$

$r = -1$

- Fix $M > 0$, and partition the set B_{-1} as

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are mutually independent.

- Using this, one way to guarantee persistence is to demand

$$\Big\{ \sup_{u \in A_{-1,p}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} \leq -2\delta,$$

$$\max_{|q-p|>1} \sup_{u \in A^{-1,q}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} < \delta\rho(|p-q|),$$

$$\max_{r \in \{0,1,2\}}, \sup_{u \in A_r} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} < \delta\rho(n+1-p),$$

$$\sup_{u \in A_{-2}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} < \delta\rho(p)\Big\}.$$

Fix p

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- A crude lower bound using $\mathbb{P}(E_1 \cap E_2) \geq P(E_1) - \mathbb{P}(E_2^c)$ gives the lower bound

$$\mathbb{P}\Big(\sup_{u\in A_{-1,p}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} \leq -2\delta,$$

$$\max_{1\leq|q-p|\leq\Gamma}\sup_{u\in A^{-1,q}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} < \delta\rho(|p-q|)\Big)$$

$$-\mathbb{P}\Big(\max_{|q-p|>\Gamma}\sup_{u\in A^{-1,q}} \frac{Q_n^{(-1,p)}(\pm e^u)}{\sigma_n(u)} > \delta\rho(|p-q|)\Big)$$

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- For the main term, we have the process convergence, after parametrizing $u = -\frac{h}{M^p \log n} t$:

$$\left\{ \frac{Q_n^{(-1,p)}(\zeta e^u)}{\sigma_n(u)} \right\}_{u \in \cup_{q:|q-p| \le \Gamma} A^{-1,q}, \zeta \in \{\pm\}} \stackrel{d}{\to} \{Z_M(t,\zeta)\}_{t \in [M^{-\Gamma}, M^{\Gamma+1}], \zeta \in \{\pm\}}$$

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- Here $Z_M(t,\zeta)$ is a centered (non-stationary) Gaussian process with correlation

$$C_M\Big((t_1,\zeta_1),(t_2,\zeta_2)\Big) = \frac{g_M(t_1+t_2)}{\sqrt{g_M(2t_1)}\sqrt{g_M(2t_2)}}$$

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- Finally

$$g_M(t) = \int_{\frac{\lambda}{M}}^{\lambda} x^\alpha e^{-xt} dx, \text{ where } \lambda = hL.$$

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- By weak convergence of stochastic processes, the main term is approximately

$$\mathbb{P}\Big(\sup_{t\in[1,M]} Z_M(t) < -2\delta, \sup_{t\in[M^\ell,M^{\ell+1}]} Z_M(t) < \rho(|\ell|)\delta, \text{ for } \ell = \pm 1, \cdots, \pm\Gamma\Big)^2.$$

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- By Slepian's inequality, a further lower bound is

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[1,M]} Z_M(t) < -2\delta\Big)^2 \\
&\times \mathbb{P}(\sup_{t\in[M^\ell,M^{\ell+1}]} Z_M(t) < \rho(|\ell|)\delta, \text{ for } \ell = \pm 1,\cdots,\pm\Gamma\Big)^2 \\
\geq &\mathbb{P}\Big(\sup_{t\in[M^\delta,M^{1-\delta}]} Z_M(t) < -2\delta\Big)^2 \mathbb{P}(\sup_{t\in[1,M^\delta]\cup[M^{1-\delta},M]} Z_M(t) < -2\delta\Big)^2 \\
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Analyze Gaussian processes

- In the domain $t \in [M^\delta, M^{1-\delta}]$, the correlations of the process $Y_M(t)$ are well approximated by that of $Z_\infty(t)$, whose correlation is
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- Using the continuity of persistence exponents (Dembo-M., AoP-15), one has
$$\lim_{M\to\infty}\frac{1}{M}\log\mathbb{P}\Big(\sup_{t\in[1,M]} Z_M(t) < -2\delta\Big)$$
$$=\lim_{M\to\infty}\frac{1}{\log M}\log\mathbb{P}\Big(\sup_{s\in[0,(1-2\delta)]\log M]} Y^{(\alpha)}(s) \le -2\delta\Big).$$

Analyze Gaussian processes

- Finally one takes $\delta \to 0$, which on using continuity of levels of persistence gives the answer

$$\lim_{M \to \infty} \frac{1}{\log M} \log \mathbb{P}\Big(\sup_{s \in [0, \log M]} Y^{(\alpha)}(s) < 0 \Big).$$

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- The other products in the main term, i.e.

$$\mathbb{P}(\sup_{t\in[1,M^\delta]\cup[M^{1-\delta},M]} Z_M(t) < -2\delta \Big)^2$$

$$\times \mathbb{P}(\sup_{t\in[M^\ell,M^{\ell+1}]} Z_M(t) < \rho(|\ell|)\delta, \text{ for } \ell = \pm 1, \cdots, \pm\Gamma \Big)^2.$$

can be lower bounded by first using Slepian to decouple the terms, then bounding each term using Borel-TIS inequality.

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- The only other case where we need the second moment bound is the process convergence, which utilizes the Lindeberg-Feller CLT.

Outline

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- Since we cannot through away coefficients (unlike the intervals), set

$$B_0 = [n]/(B_+ \cup B_-).$$

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- An upper bound to q_n is thus given by ignoring part of the domain, to get the bound

$$\mathbb{P}\left(\max_{\zeta\in\{-1,1\}} \sup_{u\in\bigcup_{q\in[\varepsilon T_n,(1-\varepsilon)T_n]} A_{\zeta,q}} \frac{Q_n(\pm_2 e^{\zeta_1 u})}{\sigma_n(u)}\sigma_n(u) < 0\right).$$

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$$Q_n^{(-1)}(x) = \sum_{p=1}^{T_n} Q_n^{(-1,p)}(x), \text{ and } Q_n^{(+1)}(x) = \sum_{p=1}^{T_n} Q_n^{(+1,p)}(x).$$

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- Fixing $q \in [\varepsilon T_n, (1-\varepsilon)T_n]$ and $\zeta = -1$, if $\sup_{u \in A^{-1,q}} Q_n\left(\frac{\pm e^u}{\sigma_n(u)}\right) < 0$, then one of the following must hold:

$$E_q = \left\{ \sup_{u \in A^{(-1,q)}} \frac{Q_n^{(-1,q)}(\pm e^u)}{\sigma_n(u)} < 2\delta \right\},$$

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- If the first event E_q happens for most q's, then we are happy.

Upper bound

- We need to show that $F_q = \cup_{p \neq q} F_{p,q}$ doesn't happen for many q's, and $F_{0,q}$ doesn't happen for many q's.

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- The challenge here comes from the fact that the events

$$\{F_q\}_{\varepsilon T_n \leq q \leq (1-\varepsilon) T_n}$$

have dependence across themselves, and also with

$$\{E_q\}_{\varepsilon T_n \leq q \leq (1-\varepsilon) T_n}.$$

Main technical challenge

- To control this, we need a quantitive bound on events of the form

$$\mathbb{P}\Big(\cap_{q\in S} F_q \bigcap \cap_{r\in[\varepsilon T_n,(1-\varepsilon)T_n]/S} E_r \Big),$$

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- By a union bound, focus on one such event of the form

$$\Big\{ \cap_{q\in S} F_{q,s_q} \bigcap \cap_{r\in[\varepsilon T_n,(1-\varepsilon)T_n]/S} E_r \Big\}.$$

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- These events are finally independent, and so decouple.

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- For $F_{q,s}$ with $|q-s|$ large, we use our Kolmogorov-continuity lemma.
- Combining and summing over, we get the desired quantitative bound, allowing us to ignore sets of the form

$$\Big\{\cap_{q\in S}F_q\bigcap\cap_{r\in[\varepsilon T_n,(1-\varepsilon)T_n]/S}E_r\Big\}$$

whenever $|S|$ is not too small.

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- Changing to stationary scale $s = \log t$ and taking limits as $M \to \infty$, we again get the limiting Gaussian process with covariance $\operatorname{sech}\left(\frac{s_1 - s_2}{2}\right)^{\alpha+1}$.
- The fact that this convergence allows the exponents to converge again follows form Dembo-M., AoP-15, as in the lower bound.

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- If one wants to only consider no roots in the domain $[0, 1]$ or $[1, \infty)$ or $[0, \infty)$, n even is not needed.

Outline

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- Generalize to higher dimensions?


The End