# New Chaos Decomposition of Gaussian Nodal Volumes

*Based on joint work with Michele Stecconi*

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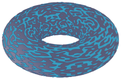
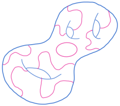
# Setting

- $(M, g)$ Riemannian manifold of $dimM = n$, compact, possibly with boundary.
- $f : M \to \mathbb{R}$ Gaussian field, smooth, centered.
- The nodal volume $\mathcal{L}_f(M) = Vol^{n-1}(f^{-1}(0))$

Credit: Alex Barnett



Credit: webpage https://unirandom.univ-rennes1.fr/



Credit: Michele Stecconi

# Wiener-Itô chaos decomposition

- Hermite polynomials:

$$\sum_{q\in\mathbb{N}} H_q(x)\frac{t^q}{q!} = e^{tx-\frac{t^2}{2}}.$$

$H_0 \equiv 1,\ H_1(t) = t,\ H_2(t) = t^2 - 1,\ H_3(t) = t^3 - 3t,$
$H_4(t) = t^4 - 6t^2 + 3.$

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- Given two Gaussian variables $\gamma_1, \gamma_2 \sim N(0,1)$,

$$\mathbb{E}[H_q(\gamma_1)H_{q'}(\gamma_2)] = \begin{cases} q!(\mathbb{E}[\gamma_1\gamma_2])^q & \text{if } q = q' \\ 0 & \text{if } q \neq q' \end{cases}$$

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- $F \in L^2 \Rightarrow F = \sum_{q\in\mathbb{N}} F[q]$
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- $$\mathbb{V}\mathrm{ar}(F) = \sum_{q=1}^{\infty} \mathbb{V}\mathrm{ar}(F[q])$$

# Assumptions on the field and chaos decomposition of nodal volume

$\| \cdot \|_{g^f}$ is the norm induced by $f$ on $T_x M$

- $f \in \mathcal{C}^2$ Gaussian random field:

$$\mathbb{E}\left\{|f(x)|^2\right\} = 1, \quad \text{and} \quad \|u\|^2_{g^f} := \mathbb{E}\left\{|d_x f(u)|^2\right\} > 0,$$

for every $x \in M$ and $u \in T_x M \setminus \{0\}$;

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- Gass-Stecconi 2024, Ancona-Letendre 2025, Angst-Poly 2020:

$$\mathcal{L}_f(M) \in L^2 \Rightarrow \mathcal{L}_f(M) = \sum_{q\in\mathbb{N}} \mathcal{L}_f(M)[q]$$

$$\Rightarrow \mathbb{V}\mathrm{ar}(\mathcal{L}_f(M)) = \sum_{q=1}^{\infty} \mathbb{E}[(\mathcal{L}_f(M)[q])^2]$$

# Existing chaos decomposition

$$\mathcal{L}_f(M)[2q+1] = 0, \text{ for}$$

$$\mathcal{L}_f(M)[2q] = \sum_{p=0}^{q} \frac{\beta_{2q-2p}}{(2q-2p)!}$$

$$\times \sum_{\substack{s=(s_1,\dots,s_n)\in\mathbb{N}^n \\ s_1+\dots+s_n=p}} \frac{\alpha_{2s_1,\dots,2s_n}}{(2s_1)!\dots(2s_n)!} \int_M H_{2q-2p}(f(x)) \prod_{j=1}^{n} H_{2s_j}(\partial_j f(x))\, dx,$$

where $\beta_{2q-2p} = \frac{1}{\sqrt{2\pi}} H_{2q-2p}(0)$ and, for $s = (s_1, \dots, s_n) \in \mathbb{N}^n$,

$$\alpha_{2s_1,\dots,2s_n} = \sum_{i=0}^{\infty} \frac{1}{i!2^i} \frac{\sqrt{2}\Gamma(\frac{n}{2}+i+\frac{1}{2})}{\Gamma(\frac{n}{2}+i)}$$

$$\times \sum_{\substack{j_1+\dots+j_n=i \\ j_1\le s_1,\dots,j_n\le s_n}} \binom{i}{j_1,\dots,j_n} \frac{(-1)^{s_1-j_1+\dots+s_n-j_n}(2s_1)!\dots(2s_n)!}{(s_1-j_1)!\dots(s_n-j_n)!2^{s_1-j_1+\dots+s_n-j_n}}.$$

Cammarota19, Marinucci-Peccati-Rossi-Wigman16, Rossi18, Marinucci-Rossi-Wigman20, Kratz-León 01

# Problems:

- Computation of the variance:

$$\mathbb{E}[H_{2q-2p}(f(x))\prod_{j=1}^{n} H_{2s_j}(\partial_j f(x))H_{2q-2p'}(f(y))\prod_{j=1}^{n} H_{2s'_j}(\partial_j f(y))]$$

- depends on a basis $\partial_1(x), \dots, \partial_n(x)$ of $T_x M$
- assumes that $\partial_1 f(x), \dots, \partial_n f(x)$ are i.i.d. $\sim N(0,1)$

$\partial_{u_1}(x), \dots, \partial_{u_n}(x) \sim N(0,1)$ iid $\Leftrightarrow u_1, \dots, u_n$ is $g_x^f$-orthonormal

In general $g \neq g^f$ (Nash' Isometric Embedding theorem)

# Main result: new chaos decomposition

Assumptions above on $f$.
Let $\|u\|_{g^f}^2 = \mathbb{E}[|d_x f(u)|^2]$ be the norm induced by $f$ on $T_x M$

## Theorem (Stecconi-T.2025)

*For all $q$ even*

$$\mathcal{L}_f(M)[q]$$

$$= \sum_{a,b\in\mathbb{N}, a+b=\frac{q}{2}} \frac{\Theta(a,b)}{s_n} \int_M \int_{S(T_x M)} H_{2a}(f(x)) H_{2b}\left(\frac{\langle d_x f, u\rangle}{\|u\|_{g^f}}\right) \|u\|_{g^f} du dx,$$

*where*

$$\Theta(a,b) = \frac{(-1)^{a+b-1}}{2^{a+b}(2b-1)a!b!}, \text{ for all } a,b \in \mathbb{N}$$

*and* $s_n := \mathrm{Vol}^n(S^n) = \dfrac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$

# Diagram Formula

### Lemma

*Let* $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \sim N(0,1)$ *with correlations* $\mathbb{E}\gamma_i\gamma_j = C_{ij}$ *and assume that* $C_{12} = C_{34} = 0$. *Then,*

$$\mathbb{E}\left\{H_a(\gamma_1)H_b(\gamma_2)H_{a'}(\gamma_3)H_{b'}(\gamma_4)\right\}$$

$$= \begin{cases} a!b!a'!b'! \displaystyle\sum_{k=\max\{0,a'-b\}}^{\min\{a,a'\}} \frac{C_{13}^{k} C_{14}^{a-k} C_{23}^{a'-k} C_{24}^{b-a'+k}}{k!(a-k)!(a'-k)!(b-a'+k)!} \\ 0 \qquad \text{if } a+b \neq a'+b'. \end{cases}$$

(Caramellino-Giorgio-Rossi 2024)

## Semplified Variance

$C(x,y) := \mathbb{E}[f(x)f(y)]$, $C'_{x,y}(\cdot) := \mathbb{E}[f(x)d_y f(\cdot)]$, $C'_{y,x} := \mathbb{E}[d_x f(\cdot) f(y)]$,

$C''_{x,y} := \mathbb{E}[d_x f(\cdot) d_y f(\cdot)]$, $\quad (C''_{x,x} = g^f_x)$

### Corollary

$$\mathbb{V}\mathrm{ar}\left(\mathcal{L}_f(M)\right) = \sum_{q\in\mathbb{N}} \mathbb{V}\mathrm{ar}\left(\mathcal{L}_f(M)[q]\right)$$

$$= \sum_{q\in\mathbb{N}} \frac{1}{ns_n^2} \int_M \int_M \int_{S^{n-1}(T_xM)} \int_{S^{n-1}(T_yM)} \sum_{\substack{a,b\in\mathbb{N}\\ 2a+2b=q}} \sum_{\substack{a',b'\in\mathbb{N}\\ 2a'+2b'=q}}$$

$$\sum_{k=\max\{0,2a'-2b\}}^{\min\{2a,2a'\}} \frac{\Theta(a,b)\Theta(a',b')(2a)!(2b)!(2a')!(2b')!}{k!(2a-k)!(2a'-k)!(2b-2a'+k)!}$$

$$\times \left(C_{xy}\right)^k \left(\frac{C'_{xy}(v)}{\|v\|_{g^f}}\right)^{2a-k} \left(\frac{C'_{yx}(u)}{\|u\|_{g^f}}\right)^{2a'-k} \left(\frac{C''_{xy}(u,v)}{\|v\|_{g^f}\|u\|_{g^f}}\right)^{2b-2a'+k} \|v\|_{g^f}\|u\|_{g^f}\, dvdudydx$$

# Remarks

- Marinucci-Rossi-T. 2023: *Hermite-Laguerre expansion*

$$\mathcal{L}_f(M)[2q] = \sum_{p=0}^{q} \frac{\beta_{2q-2p}}{(2q-2p)!}\sqrt{2}C(n,p) \times \int_M H_{2q-2p}(f(x)) L_p^{(n/2-1)}\left(\frac{||\nabla f(x)||^2}{2}\right) dx.$$

  the number of terms involved grows linearly with $q$ and is constant in the dimension $n$.

- Notarnicola 2023: Matrix-Hermite expansion of $f_0(XX^T)$ for Gaussian matrix $X$ (with independent columns).

(Homothetic case)

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# Comparison with laguerre

- $F(\xi) \in L^2$, $\Rightarrow F(\xi)[q]$ is its $L^2$ projection onto the closed subspace of $L^2$ generated by the random variables $\left\{H_q(\langle \xi, v \rangle) : v \in S^{n-1}\right\}$, i.e., the space

$$W_q[\xi] := \mathrm{span}\left\{H_q(\langle \xi, v \rangle) : v \in S^{n-1}\right\}.$$

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## Definition

Let $S_{n,q} : \mathbb{R}^n \to \mathbb{R}$, be the function

$$S_{n,q}(x) := \int_{S^{n-1}} H_q\left(\langle x, v\rangle\right) \mathcal{H}^{n-1}(dv).$$

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- Any $L^2$ random variable of the form $F(\|\xi\|)$ has a chaotic decomposition of the form

$$F(\|\xi\|) = \sum_{q \in \mathbb{N}} c_q S_{n,q}(\xi).$$

# Comparison with Laguerre

- In particular, since for all even $q$, the Laguerre polynomial $L_{\frac{q}{2}}^{(\frac{n}{2}-1)}\left(\frac{\|x\|^2}{2}\right)$ is in the $q^{th}$ chaos space $W_q[\xi]$ (Marinucci-Rossi-T. 2023+):

$$\frac{(-1)^{q/2}}{2^{q/2}} \sum_{s_1+\cdots+s_n=q} \frac{1}{s_1!\cdots s_n!} \prod_{j=1}^{n} H_{s_j}(x_j) = L_{q/2}^{n/2-1}\left(\frac{||x||^2}{2}\right)$$

$\Rightarrow$ there exist constants $c_{n,q} \in \mathbb{R}$ such that

$$L_{\frac{q}{2}}^{(\frac{n}{2}-1)}\left(\frac{\|\xi\|^2}{2}\right) = c_{n,q} \cdot S_{n,q}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

# $||u||_{g^f} \approx const$

$$g^f_x = \frac{\lambda^2}{n}(g_x + O(\varepsilon))$$

## Definition

- *frequency of f:*

$$\begin{aligned} \lambda(f)^2 &:= \frac{1}{Vol^n(M)} \int_M \mathbb{E}[||\nabla_x f||^2]\, dx \\ &= \frac{1}{Vol^n(M)} \int_M \mathbb{E}[-f(x)\Delta f(x)] = -\text{"}eigenvalue\text{"} \end{aligned}$$

- *maximal eccentricity of f:* $\varepsilon = \varepsilon(f) := \max_{x \in M} ||\frac{n}{\lambda^2} g^f_x - g_x||$

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- *maximal eccentricity of f:* $\varepsilon = \varepsilon(f) := \max_{x \in M} ||\frac{n}{\lambda^2} g_x^f - g_x||$
- $f$ is *homothetic* if $\varepsilon(f) = 0$ *(Pistolato-Stecconi 2024)*

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- $f$ is *homothetic* if $\varepsilon(f) = 0$ *(Pistolato-Stecconi 2024)*

For random waves $\sum_{\lambda_i \in [\lambda, \lambda + o(\lambda)]} \gamma_i \varphi_i(x)$ : $g^f = \dfrac{\lambda^2}{n}(g + O(\varepsilon))$

## Homothetic formula: $||u||_{g^f} = const$

$$g^f = \frac{\lambda^2}{n} g, \qquad \lambda^2 = \lambda(f)^2 := \frac{1}{Vol^n(M)} \int_M \mathbb{E}[||d_x f||^2] dx$$

$$\mathcal{L}_f[q] = \lambda \sum_{a,b \in \mathbb{N},\ a+b=\frac{q}{2}} \frac{\Theta(a,b)}{s_n \sqrt{n}} \int_M \int_{S(T_x M)} H_{2a}(f(x)) H_{2b}\left( \langle \nabla_x f, v \rangle \frac{\sqrt{n}}{\lambda} \right) dv dx.$$

$$\frac{\mathbb{E}\left\{ \mathcal{L}_f(M) \right\}}{\lambda} = \frac{s_{n-1}}{s_n \sqrt{n}} \mathrm{Vol}^n(M)$$

## HOMOTHETIC CASE: FIRST CHAOSES

$$\frac{\mathcal{L}_f(M)[2]}{\lambda} = -\frac{s_{n-1}}{2s_n\sqrt{n}}\left(\|f\|^2_{L^2(M)} - \|\lambda^{-1}\nabla f\|^2_{L^2(M)}\right)$$

$$= -\frac{s_{n-1}}{2s_n\sqrt{n}}\cdot\left(\int_M f\left(f+\frac{\Delta f}{\lambda^2}\right) - \int_{\partial M} f\langle\lambda^{-1}\nabla f,\nu\rangle\right);$$

$$\frac{\mathcal{L}_f(M)[4]}{\lambda} = \frac{s_{n-1}}{24 s_n\sqrt{n}}\Bigg[\int_M H_4(f) - \int_{S(T_xM)} H_4\left(\langle\lambda^{-1}\nabla f, v\rangle\sqrt{n}\right) dv$$

$$+2H_3(f)\left(f+\frac{\Delta f}{\lambda^2}\right)dx - \frac{2}{\lambda}\int_{\partial M} H_3(f)\langle\lambda^{-1}\nabla f,\nu\rangle dx\Bigg].$$

$$\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{L}_f(M)}{\lambda}\right) \geq \mathbb{V}\mathrm{ar}\left(\frac{\mathcal{L}_f(M)[2]}{\lambda}\right) + \mathbb{V}\mathrm{ar}\left(\frac{\mathcal{L}_f(M)[4]}{\lambda}\right)$$

# Upper bound

## Definition

- the *pointwise frequency of $f$* as the function $x \mapsto \lambda(f,x) > 0$ such that
$$\lambda(f,x)^2 := \mathbb{E}[\|d_x f\|^2] = n \int_{S(T_xM)} \|u\|^2_{g^f} du,$$
for all $x \in M$.
-
$$\|j''_{x,y} C\|_{g^f} := \max_{\substack{u \in T_xM \setminus \{0\} \\ v \in T_yM \setminus \{0\}}} \left\{ C(x,y), \frac{|C'_{y,x}(u)|}{\|u\|_{g^f}}, \frac{|C'_{x,y}(v)|}{\|v\|_{g^f}}, \frac{|C''_{x,y}(u,v)|}{\|u\|_{g^f}\|v\|_{g^f}} \right\}$$

## Theorem

*Let $f : M \to \mathbb{R}$ as above.*
$$\mathbb{V}\mathrm{ar}\left(\mathcal{L}_f(M)[q]\right) \leq 2^q \cdot \int_{M\times M} \frac{\lambda(f,x)\lambda(f,y)}{n} \cdot \|j''_{x,y} C\|^q_{g^f} dxdy.$$

# Semplified Variance

$C(x,y) := \mathbb{E}[f(x)f(y)]$, $C'_{x,y}(\cdot) := \mathbb{E}[f(x)d_y f(\cdot)]$, $C'_{y,x} := \mathbb{E}[d_x f(\cdot) f(y)]$,

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## Corollary

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# Reduction to the homothetic case

$$g_x^f = \frac{\lambda^2}{n}(g_x + O(\varepsilon))$$

$$\frac{\tilde{\mathcal{L}}_f(M)\{q\}}{\lambda} := \sum_{a,b\in\mathbb{N}, a+b=\frac{q}{2}} \frac{\Theta(a,b)}{s_n\sqrt{n}} \int_M \int_{S^{n-1}(T_xM)} H_{2a}(f(x)) H_{2b}\left(\frac{\partial_u f(x)\sqrt{n}}{\lambda}\right) dudx$$

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$$\frac{\mathcal{L}_f(M)[q]}{\lambda} = \frac{\tilde{\mathcal{L}}_f(M)\{q\}}{\lambda} + O(\varepsilon)$$

$$\mathbb{E}\left[\frac{\mathcal{L}_f(M)}{\lambda}\right] = \frac{1}{s_n}\int_M \int_{S(T_xM)} ||u||_{g^f}\, dudx = \frac{s_{n-1}}{s_n\sqrt{n}} Vol^n(M)(1+O(\varepsilon))$$

# Proof Main result: new chaos decomposition

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# Proof of the main result: chaotic expansion

- The nodal volume can be expressed

$$\mathcal{L}_f(M) := \int_M \delta_0(f(x)) \|\nabla_x f\| dx$$

**Lemma**

*For $\gamma \sim N(0,1)$, the chaos expansion of the distribution $\delta_0$ makes sense and is equal to:*

$$\delta_0(\gamma) = \sum_{a \in \mathbb{N}} H_{2a}(\gamma) \frac{(-1)^a}{2^a a! \sqrt{2\pi}}.$$

# Proof of the main result: chaotic decomposition of $||\xi||$

**Theorem (Stecconi-T.2025)**

*Let $\xi \sim \mathcal{N}(0, G)$ be a Gaussian random vector in $\mathbb{R}^n$, with non-degenerate covariance matrix $G$. Then, the $q^{th}$ chaotic component of $\|\xi\|$ vanishes if $q$ is odd and for $q$ even is*

$$(\|\xi\|)\,[q] = A(n,q) \int_{S^{n-1}} H_q\left(\frac{\langle \xi, v\rangle}{\sqrt{v^T G v}}\right) \sqrt{v^T G v}\; dv,$$

*where*

$$A(n, 2b) = \frac{\pi}{s_n} \frac{(-1)^{b-1}}{2^{b-1}\sqrt{2\pi}(2b-1)b!},$$

*for all $b \in \mathbb{N}$. In particular, when $G = \mathbb{1}_n$ the above formula yields the chaotic components of a chi random variable of parameter $n$.*

# Proof of the main result: chaotic expansion

Observe that in an orthonormal basis of $T_x M$, the covariance matrix $G$ of $\nabla_x f$ is such that

$$v^T G v = \mathbb{E}\left\{ |\langle \nabla_x f, v\rangle|^2 \right\} = g_x^f(v,v),$$

for any $v \in S(T_x M)$.

$$\mathrm{Vol}^{n-1}\left(f^{-1}(0)\right) = \int_M \delta_0(f(x)) \|\nabla_x f\| dx$$

$$= \int_M \left( \sum_{a\in\mathbb{N}} H_{2a}(f(x)) \frac{(-1)^a}{2^a a! \sqrt{2\pi}} \right) \left( \sum_{b\in\mathbb{N}} A(n,2b) \int_{S(T_x M)} H_{2b}\left( \frac{\langle \nabla_x f, v\rangle}{\sqrt{g^f(v,v)}} \right) \sqrt{g^f(v,v)} dv \right) dx$$

$$= \sum_{q\in\mathbb{N}} \sum_{2(a+b)=q} \tilde{\Theta}(n,a,b) \int_M \int_{S(T_x M)} H_{2a}(f(x)) H_{2b}\left( \frac{\langle \nabla_x f, v\rangle}{\sqrt{g^f(v,v)}} \right) \sqrt{g^f(v,v)} dv dx$$

so that

$$\tilde{\Theta}(n,a,b) = \frac{(-1)^a}{2^a a! \sqrt{2\pi}} \cdot A(n,2b) = \frac{(-1)^a}{2^a a! \sqrt{2\pi}} \cdot \frac{\pi}{s_n} \frac{(-1)^{b-1}}{2^{b-1}\sqrt{2\pi}(2b-1)b!},$$

thus we conclude.

# Proof of the chaotic expansion of $||\xi||$

## Lemma

*Let $\xi \in \mathbb{R}^n$ be any vector, then*

$$\|\xi\| = \frac{\pi}{s_n} \int_{S^{n-1}} |\langle \xi, v \rangle| dv.$$

## Proof.

The integral depends only on the norm of $\xi$, by rotational invariance,

$$\int_{S^{n-1}} |\langle \xi, v \rangle| dv = \|\xi\| \int_{S^{n-1}} |v_1| dv = \|\xi\| \beta(n, 1);$$

where $\beta(n, 1) = \frac{s_n}{\pi}$. $\square$

# Proof of the chaotic expansion of $||\xi||$

- We start by expressing $\|\xi\|$ as a circle average

$$\|\xi\| = \frac{\pi}{s_n}\int_{S^{n-1}} |\langle \xi, v\rangle| dv = \frac{\pi}{s_n}\int_{S^{n-1}} \frac{|\langle \xi, v\rangle|}{\sqrt{v^T G v}}\sqrt{v^T G v}\cdot dv.$$

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- $\frac{\langle \xi, v\rangle}{\sqrt{v^T G v}}$ is a unit variance Gaussian variable.

# PROOF OF THE CHAOTIC EXPANSION OF $||\xi||$

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- $\frac{\langle \xi, v\rangle}{\sqrt{v^T G v}}$ is a unit variance Gaussian variable.
- $\gamma \sim N(0,1)$,

$$|\gamma| = \sum_{b\in\mathbb{N}} c_\chi(2b) H_{2b}(\gamma), \quad \text{with} \quad c_\chi(2b) = \frac{(-1)^{b-1}}{2^{b-1}\sqrt{2\pi}(2b-1)b!}.$$

(Marinucci-Peccati-Rossi-Wigman 16, Cammarota 19, Stecconi-T.2025)

## Proof of the chaotic expansion of $||\xi||$

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  (Marinucci-Peccati-Rossi-Wigman 16, Cammarota 19, Stecconi-T.2025)

- setting $A(n,q) := \frac{\pi}{s_n} c_\chi(q)$ concludes the proof.

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