The giant component of excursion sets of spherical Gaussian ensembles

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Random polynomials and their applications
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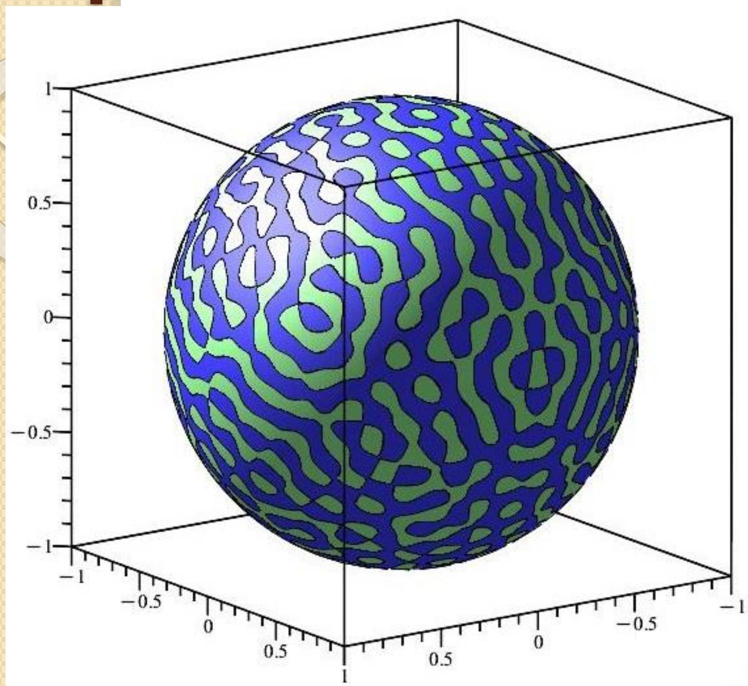
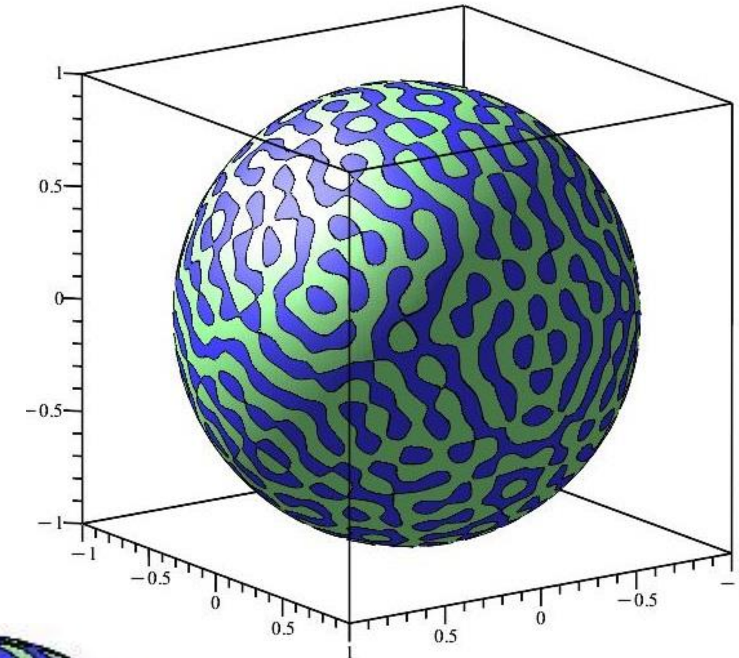
1. Motivation

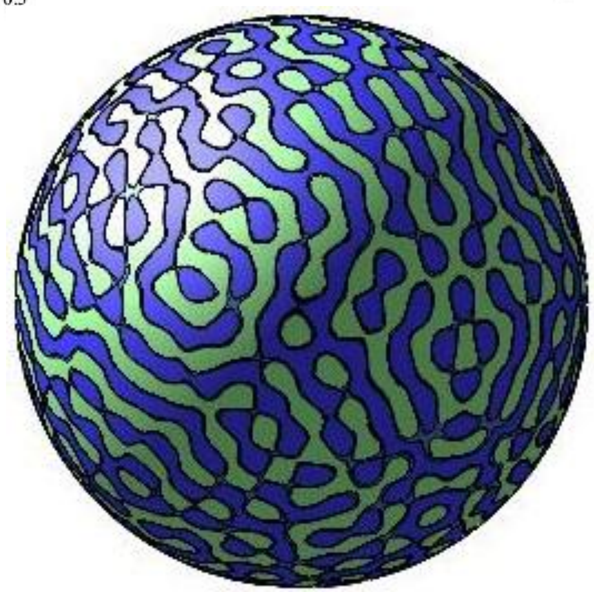
Spherical harmonics degree 40



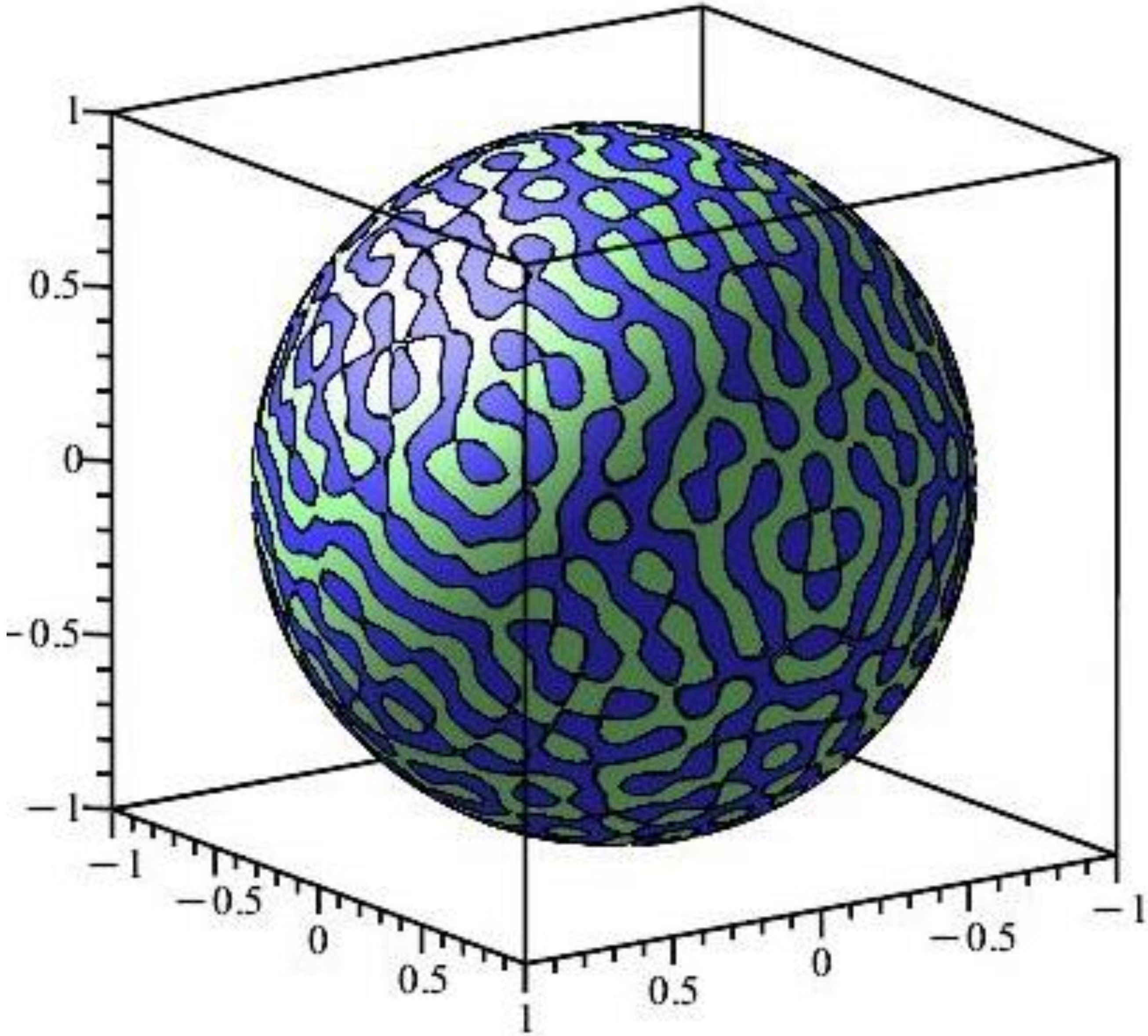
$f < -0.1$



$f < +0.1$



From level -0.05 to 0.05



Giant component

- T_l random spherical harmonic of degree $l \to \infty$
- Excursion set $t \in \mathbb{R}$, $\mathcal{V}_u = T_l^{-1}(-\infty, u)$
- 1. **Supercritical** level $\mathrm{u} > 0$
- With high probability $\mathcal{V}_u$ contains *unique macroscopic* connected component $\mathcal{W}_u$. Positive proportion of sphere.
- $Area(\mathcal{W}_t) \sim c_u \cdot 4\pi \qquad c_u = \varphi(1, u) > 0$
- $\varphi(\alpha, u) > 0$ "limit percolation density" defined $\alpha \in [0,1]$ "band", $\mathrm{u} > 0$ level.
- No other components of $\mathcal{V}_u$ of area $> \varepsilon$

Giant component (cont.)

- T_l random spherical harmonic of degree $l \to \infty$
- Excursion set $\mathrm{u} \in \mathbb{R}$, $\mathcal{V}_u = T_l^{-1}(-\infty, u)$
- 1. **Supercritical** level $\mathrm{u} > 0$
- $\mathcal{V}_t$ contains unique macroscopic connected component.
- 2. **Subcritical** level $\mathrm{u} < 0$
- With high probability no component of $\mathcal{V}_u$ of area $> \varepsilon$
- 3. Critical $\mathrm{u} = 0$. Open, conjecturally no macroscopic component ("no percolation").

II. Nodal structures of band-limited functions

Band-limited functions

- (M, g) Riemannian d-manifold (e.g. closed), e.g. surface
- $\triangle$ Laplacian (Laplace-Beltrami).
- Boundary conditions if non-empty.
- Eigenfunctions & eigenvalues $\triangle \varphi_j + \lambda_j^2 \varphi_j = 0$
- $\alpha \epsilon [0,1]$ band, $\lambda \to \infty$ spectral parameter
- Band-limited functions (unit variance) $a_j \sim N(0,1)$ i.i.d.

$$f_\lambda(x) = \frac{1}{\lambda^{d/2}} \sum_{\alpha\lambda < \lambda_j < \lambda} a_j \cdot \varphi_j(x)$$

- Monochromatic $\alpha = 1$ $\eta(\lambda) = o(\lambda)$ $(\eta(\lambda) \to \infty)$

$$\frac{1}{\sqrt{\lambda \cdot \eta(\lambda)}} \sum_{\lambda - \eta(\lambda) < \lambda_j < \lambda} a_j \cdot \varphi_j(x)$$

Band-limited functions (sphere)

- Band-limited functions

$f_\lambda(x) = \frac{1}{\lambda^{d/2}} \sum_{\alpha\lambda<\lambda_j<\lambda} a_j \cdot \varphi_j(x)$ $\quad a_j \sim N(0,1)$ i.i.d.

- Important case: $M = \mathcal{S}^2$.
- Eigenvalues: $l \geq 1$ integer $\rightsquigarrow \lambda_l^2 = l \cdot (l+1)$ multiplicity $2l+1$
- Eigenfunctions – spherical harmonics basis $Y_{lm}, \quad m = -l, \ldots, l.$
- Random spherical harmonics

$$T_l(x) = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} a_m \cdot Y_{lm}(x) \qquad a_m \text{ Gaussian i.i.d.}$$

- Appears in "Fourier" expansion of "any" rotation invariant field
- Band-limited on $\mathcal{S}^2 \leftrightsquigarrow$ superposition of $T_l(x)$ energy window

Band-limited functions (sphere)

- Band-limited functions

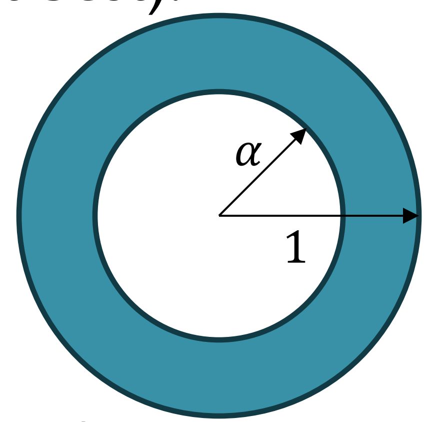
$f_\lambda(x) = \frac{1}{\lambda^{d/2}} \sum_{\alpha\lambda<\lambda_j<\lambda} a_j \cdot \varphi_j(x)$ $\quad a_j \sim N(0,1)$ i.i.d.

- Around every reference $x \epsilon M$ can rescale by λ "Planck scale"

$f_\lambda\left(x + \frac{y}{\lambda}\right) \longrightarrow F_\alpha(y) \qquad x + \frac{y}{\lambda} \leftrightsquigarrow exp_x(y/\lambda)$

$y \epsilon B_0(R)$ of fixed size (very slowly increasing at best).

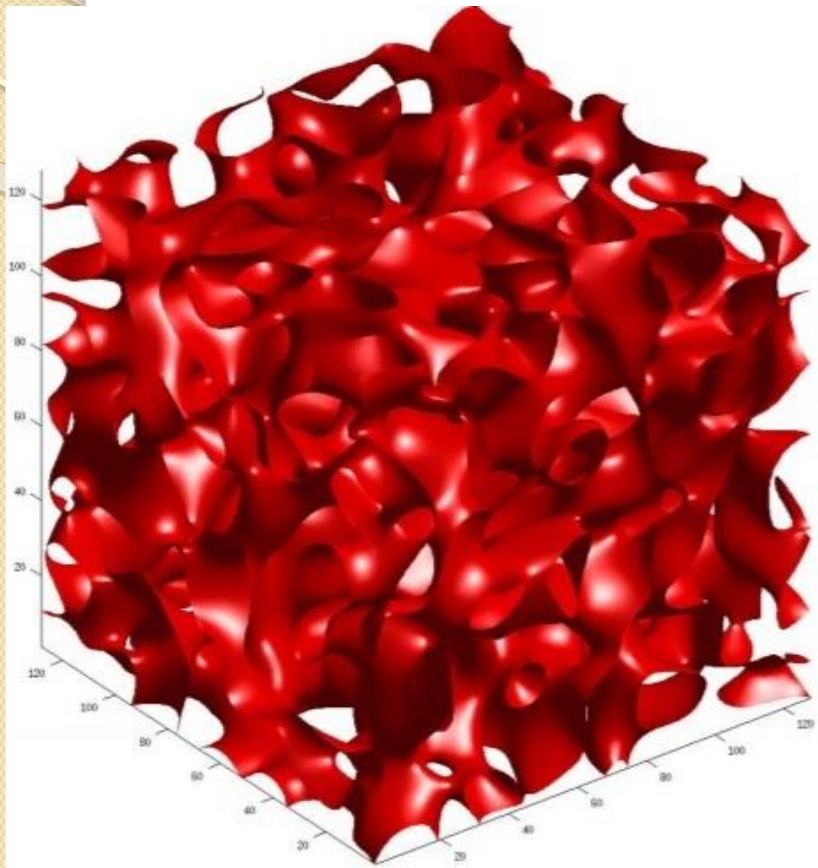
- $F_\alpha: \mathbb{R}^d \to \mathbb{R}$ stationary isotropic Gaussian
- Only depends on d, α (not on M).
- $\mathrm{d} = 2, \alpha = 1$ Berry's random waves
- Covariance $\mathbb{E}[F_\alpha(x) \cdot F_\alpha(y)] = J_0(|x-y|)$
- E.g. random spherical harmonics (special functions)

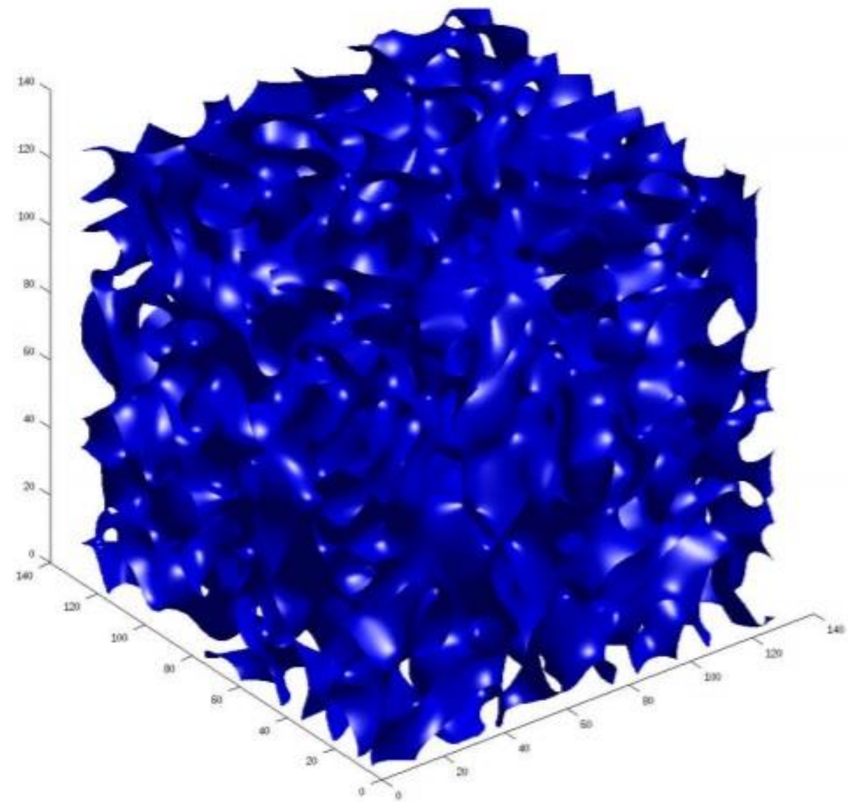


Nodal domains (components)

- $F: \mathbb{R}^d \to \mathbb{R}$ "nice" stationary Gaussian random field
- $N_F(R)$ - counts nodal domains (components) of F in B_R. Connected components of $F^{-1}(0)$ fully in B_R (of $F^{-1}(0)$)
- Nazarov-Sodin: $N_F(R) = c_{NS}(F) \cdot R^d + o_{R\to\infty}\left(R^d\right)$ mean / a.s.
- Nazarov-Sodin constant $c_{NS}(F) > 0$
- Band-limited $f_\lambda(x) = \frac{1}{\lambda^{d/2}} \sum_{\alpha\lambda<\lambda_j<\lambda} a_j \cdot \varphi_j(x)$
- Scales on Planck scale patches $f_T\left(x + \frac{y}{\lambda}\right) \longrightarrow F_\alpha(y)$
- Nazarov-Sodin: $N(f_\lambda) = c_{NS}(F_\alpha) \cdot \lambda^d + o_{T\to\infty}\left(\lambda^d\right)$ in mean
- Sarnak-W limit law topologies of domains/components.
- Works nicely in simulations (2d).

Structures 3d





"Kostlan's ensemble"

Monochromatic waves / 3d spherical harmonics

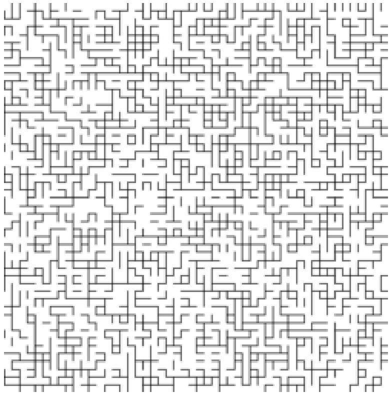
Pictures by A. Barnett

III. Crash course in percolation, and Sarnak's ansatz

Bernoulli bond percolation

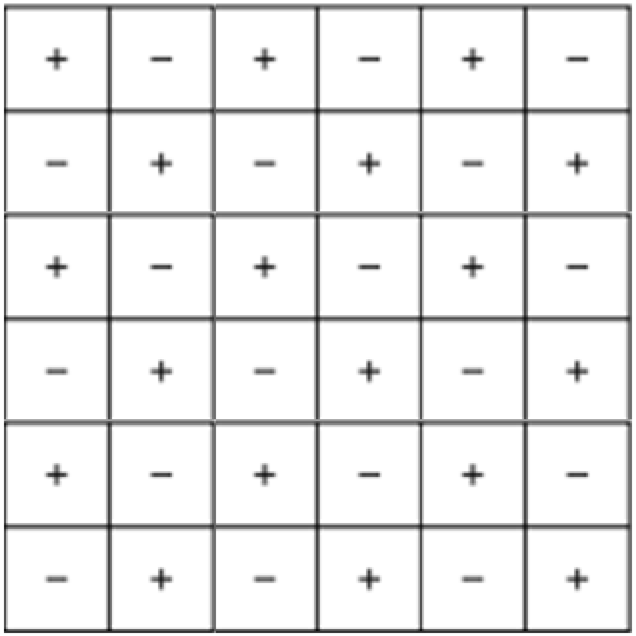
- $d \geq 2$ dimension, $p \in [0,1]$
- For every d, there exists $p_c = p_c(\mathrm{d}) \epsilon (0,1)$ so that
- Supercritical $p > p_c$ a.s. exists **unique** *unbounded* **percolation cluster** magnificent properties: ubiquity, positive density…
- For $p < p_c$ a.s. no unbounded percolation clusters.
- Percolation at criticality?
- $d = 2 \leadsto p_c = 1/2$ (Russo-Seymour-Welsh + H. Kesten).
- Corresponds to level 0 Gaussian field.
- No percolation at criticality.
- $d \geq 3 \leadsto p_c < 1/2$ value not rigorously known.

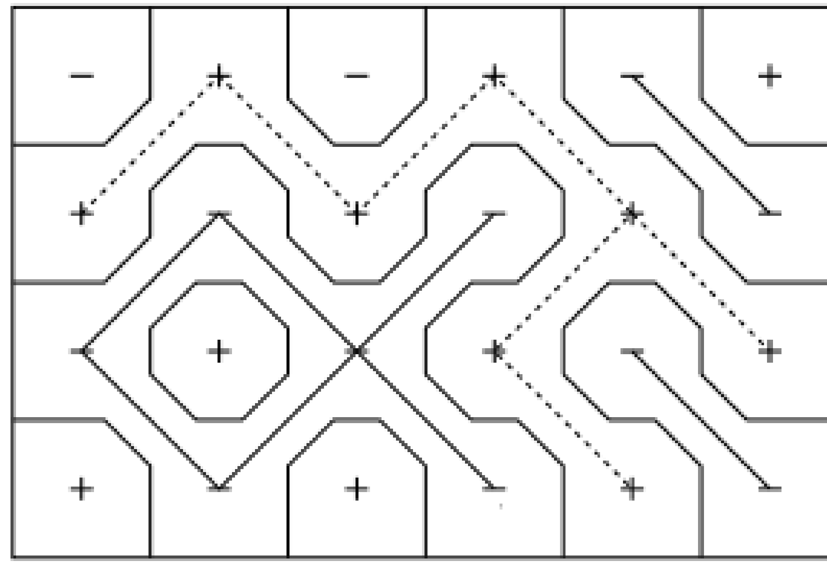
  $\Rightarrow$ percolation at $p = 1/2$.



Percolation 2d (Bogomolny-Schmit `02)

- Two mutually dual lattices (maxima & minima)
- Bond with (critical) probability ½ independent





- Percolation clusters $\leftrightsquigarrow$ Nodal domains
- Square side λ $\leftrightsquigarrow$ Eigenvalue λ

Percolation of random fields 2d

- $F: \mathbb{R}^d \to \mathbb{R}$ "nice" stationary (e.g. Gaussian) random field
- Molchanov-Stepanov 1980's (general): **Critical level** $u_c \epsilon [-\infty, +\infty]$
  $u > u_c$ a.s. $F^{-1}(-\infty, u)$ has unbounded "**percolating**" component
  $u < u_c$ no percolating component.
- F percolates above u_c, does not percolate below u_c
- Beffara-Gayet (2016): F_{BF} Bargmann-Fock 2d $r(x) = e^{-\|x\|^2/2}$ positive & rapidly decaying. $u_c \geq 0$, no percolation at criticality.
- Rivera-Vanneuville: $u_c \leq 0$ same assumptions $\to u_c = 0$.
- Muirhead-Rivera-Vanneuville: Wide class 2d random fields
  $u_c = 0$ including Berry's RWM. No determination <u>at</u> criticality.
- Akin to $p_c(2) = 1/2$ Bernoulli percolation.

Percolation random fields (cont.)

- Muirhead-Rivera-Vanneuville: $F: \mathbb{R}^2 \to \mathbb{R}$ wide class $u_c = 0$
- **Density.** $u > 0$: percolating component $\mathcal{W}_u$ of $F^{-1}(-\infty, u)$.

$$\frac{Area(\mathcal{W}_u \cap B_R)}{Area(B_R)} \to \varphi_F(u) = \mathcal{P}(0 \epsilon \mathcal{W}_u) > 0 \quad \text{ergodicity}$$

- Sarnak's ansatz:

Giant 3d $\leftrightsquigarrow$ $u_c < 0$ $\leftrightsquigarrow$ $p_c < 1/2$ $\leftrightsquigarrow$ percolation at level 0?

- Duminil-Copin-Rivera-Rodriguez-Vanneuville:

$F: \mathbb{R}^d \to \mathbb{R}$ Bargmann-Fock $d \geq 3.\, u_c < 0$ for $u < u_c$ a.s. $F^{-1}(-\infty, u)$ unbounded ubiquitous component. Same components.

- F. Severo: uniqueness of percolating component.
- Uses positivity of correlations substantially. Uses rapid decay.
- Rivera: Same monochromatic, high dimension (not effective).

IV. Principal results

Local density of giant domains

- Recall $\alpha\epsilon[0,1]$ band, $\lambda \to \infty$ spectral parameter
- Band-limited functions (unit variance)

$$f_\lambda(x) = \frac{1}{\lambda^{d/2}} \sum_{\alpha\lambda<\lambda_j<\lambda} a_j \cdot \varphi_j(x) \qquad a_j \sim N(0,1) \text{ i.i.d.}$$

$f_T\left(x + \frac{y}{\lambda}\right) \longrightarrow F_\alpha(y)$ local scaling

- For $\alpha, \mathrm{u} > 0$ given, let $\varphi(\alpha, u)$ be density of percolating domain $\mathcal{W}_u = \mathcal{W}_u(F_\alpha)$ level u.

$$\frac{Area(\mathcal{W}_u \cap B_R)}{Area(B_R)} \to \varphi(\alpha, u) = \mathcal{P}(0\epsilon\mathcal{W}_u) > 0$$

- Fact: $\varphi(\alpha, u)$ continuous 2 variables.
- For every $\alpha, u \mapsto \varphi(\alpha, u)$ **strictly increasing** to 1
- $\varphi(\alpha, u) \to 0$ as $u \to 0$??? Conjecture.

Giant domains on $\mathcal{S}^2$

- $\varphi(1,u)$ density percolating domains of local limit F_1 (Berry)
- Random spherical harmonics $d=2, \alpha=1$ most difficult

$T_l(x) = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} a_m \cdot Y_{lm}(x) \qquad a_m$ Gaussian i.i.d.

For $u\epsilon\mathbb{R} \rightsquigarrow \mathcal{V}_u = T_l^{-1}(-\infty, u), \mathcal{W}_u$ component largest diameter on $\mathcal{S}^2$

- Theorem 1 (S. Muirhead-IW):

1. Supercritical: a. **Existence** $\forall u > 0\ \forall \varepsilon > 0\ \exists c > 0$

$$\mathcal{P}(|Area(\mathcal{W}_u) - 4\pi \cdot \varphi(1,u)| > \varepsilon) < \exp\left(-c\sqrt{\log(l)}\right)$$

b. **Uniqueness** $\mathcal{P}(\mathcal{V}_u \backslash \mathcal{W}_u \ni domain\ Area > \varepsilon) < c\,\frac{\log(l)^{2+\delta}}{l^{1/2}}$

2. Subcritical. $\forall u < 0\ \forall \varepsilon > 0\ \exists c > 0 \quad \mathcal{P}(Area(\mathcal{W}_u) > \varepsilon) < c\,\frac{\log(l)^{2+\delta}}{l^{1/2}}$

- Critical $u = 0$. Open.
- By coupling $\rightsquigarrow$ local giants Planck scale. Can't infer.
- Do local giants merge to a single macroscopic giant?

Band-limited functions on $\mathcal{S}^2$

- $\varphi(\alpha, u)$ density percolating domains of local limit F_α (0 for $u < 0$)
- Band-limited $\alpha < 1 \rightsquigarrow$ $g_l(x) = C_l \sum_{l'=\lfloor \alpha l \rfloor}^{l} \sqrt{2l'+1} T_l(x)$
- $\alpha = 1$ $g_l(x) = C_l \sum_{l'=l-\lfloor \eta(l) \rfloor}^{l} \sqrt{2l'+1} T_l(x)$ $\eta(l) = l^\beta$ $\beta < 1$

  $u \epsilon \mathbb{R} \rightsquigarrow \mathcal{V}_u = {g_l}^{-1}(-\infty, u), \mathcal{W}_u$ component largest diameter
- Theorem 2 (S. Muirhead-IW):

  1. Supercritical: a. **Existence** $\forall u > 0$ $\forall \varepsilon > 0$ $\exists c > 0$

$\mathcal{P}(|Area(\mathcal{W}_u) - 4\pi \cdot \varphi(\alpha, u)| > \varepsilon) < \exp(-c \cdot l)$ optimal

b. **Uniqueness.** $\mathcal{P}(\mathcal{V}_u \backslash \mathcal{W}_u \ni domain\ Area > \varepsilon) < \exp(-c \cdot l)$

c. **Ubiquity and local uniqueness.** With probability $1 - \exp(-l^c)$

Every cap $\mathcal{D} \subseteq \mathcal{S}^2$ of radius $r > l^{-1+\delta}$ (i.) $\mathcal{V}_u \cap \mathcal{D}$ contains unique component of diameter $> \frac{r}{100}$ (ii.) contained in $\mathcal{W}_u$.

2. Subcritical. $\forall u < 0$ $\forall \varepsilon > 0$ $\exists c > 0$

$$\mathcal{P}(Area(\mathcal{W}_u) > \varepsilon) < \exp\left(-c \cdot l^{4/3}\right)$$

V. Kostlan's ensemble

Kostlan's ensemble random polynomials

- $f_n(x) = \sum_{|J|=n} \sqrt{\binom{n}{J}} a_J x^J \quad |J| = j_0 + j_1 + j_2$

  $x = (x_0 : x_1 : x_2) \epsilon \mathbb{RP}^2 \quad x^J = x_0^{j_0} x_1^{j_1} x_2^{j_2}$

  $\binom{n}{J} = \frac{n!}{j_0! \cdot j_1! \cdot j_2!} \quad a_J$ iid standard Gaussian
- Unique Gaussian ensemble homogeneous polynomials invariant unitary transformations of $\mathbb{CP}^2$
- Naturally defined $f_n : \mathcal{S}^2 \to \mathbb{R}$
- Local scaling: $f_n\left(x + \frac{y}{\sqrt{n}}\right) \longrightarrow F_{BF}(y)$
- $F_{BF} : \mathbb{R}^2 \to \mathbb{R}$ Bargmann-Fock isotropic $r(x) = e^{-\|x\|^2/2}$
- $u > 0 \rightsquigarrow \vartheta(u)$ density of percolating component ${F_{BF}}^{-1}(-\infty, u)$

Kostlan's ensemble random polynomials

- Naturally defined $f_n: \mathcal{S}^2 \to \mathbb{R}$ Kostlan
- Local scaling: $f_n\left(x + \frac{y}{\sqrt{n}}\right) \longrightarrow F_{BF}(y)$ $u > 0 \rightsquigarrow \vartheta(u)$ density
- Theorem 3 (S. Muirhead-IW):

1. Supercritical: a. **Existence** $\forall u > 0\ \exists \vartheta(u)\ \forall \varepsilon > 0\ \exists c > 0$
   $\mathcal{P}(|Area(\mathcal{W}_u) - 4\pi \cdot \vartheta(u)| > \varepsilon) < \exp(-c \cdot \sqrt{n})$ optimal

   b. **Uniqueness.** $\mathcal{P}(\mathcal{V}_u \backslash \mathcal{W}_u \ni domain\ Area > \varepsilon) < \exp(-c \cdot \sqrt{n})$

   c. **Ubiquity and local uniqueness.** With probability $1 - n^{-c}$

Every cap $\mathcal{D} \subseteq \mathcal{S}^2$ of radius $r > \frac{\log(n)}{\sqrt{n}}$ (i.) $\mathcal{V}_u \cap \mathcal{D}$ contains unique component of diameter $> \frac{r}{100}$ (ii.) contained in $\mathcal{W}_u$.

- $1 - n^{-c}$ Optimal for this scale. $\frac{1}{100}$ - arbitrary.

Kostlan's ensemble (cont.)

- Naturally defined $f_n : \mathcal{S}^2 \to \mathbb{R}$ Kostlan
- Theorem 3 (S. Muirhead-IW):

2. Subcritical & critical. $\forall u \leq 0 \; \forall \varepsilon > 0 \; \exists c > 0$

$$\mathcal{P}(Area(\mathcal{W}_u) > \varepsilon) < \exp(-c \cdot n)$$

3. $\Rightarrow$No percolation at criticality $u = 0$.

Mind $\forall \varepsilon > 0 \;\; \exists c > 0, \mathcal{P}(diam(\mathcal{W}_u) > c) > 1 - \varepsilon$

(D. Beliaev-S. Muirhead-IW, 2021)

- $\vartheta(u)$ density of percolating domain $\mathcal{W}_u, u > 0$.
- Fact: 1: $\vartheta(u)$ continuous, strictly increasing on $\mathbb{R}_{>0}$

2. $\vartheta(u) \to 0$ as $u \to 0$ (no percolation)

3. $\vartheta(u) \to 1$ as $u \to +\infty$