The Discrete Logarithm Problem (DLP) and its Generalization to the Semigroup Action Problem (SAP)

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Outline

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The Discrete Logarithm Problem (DLP)

Definition

Let G be an arbitrary group, $\alpha \in G$ an arbitrary element and $H := < \alpha > \subset G$ the cyclic group generated by α. Assume $\beta \in H$ is an arbitrary element. The unique integer n having the property that $1 \leq n < |H|$ and $\alpha^n = \beta$ is called the discrete logarithm of β to the base α.

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One has the usual computations:

$$\alpha^{(\log_\alpha \beta)} = \beta, \quad \log_\alpha(\alpha^n) = n$$

$$\log_\alpha(\beta_1\beta_2) = \log_\alpha(\beta_1) + \log_\alpha(\beta_2) \mod |H|$$

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Remark

Using so called 'consecutive squaring' allows Alice efficiently α^a *even for very large integers a. (polynomial time in the number of input bits). On the other hand the best algorithm known to compute* $\log_\alpha \beta = n$ *has exponential running time in the number of input bits.*

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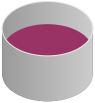
Illustration of Diffie-Hellman-Protocol



base color

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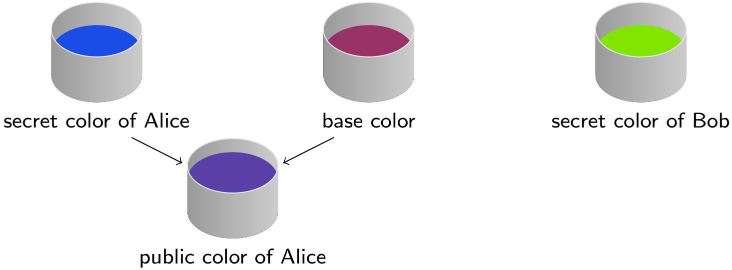

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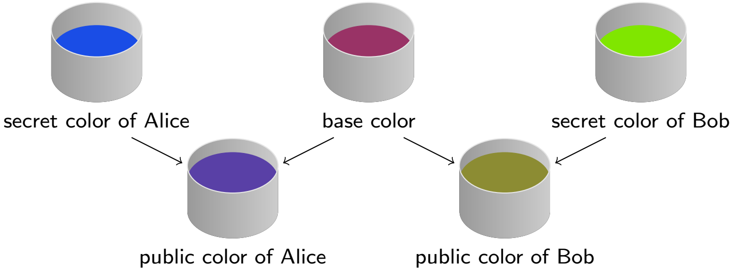

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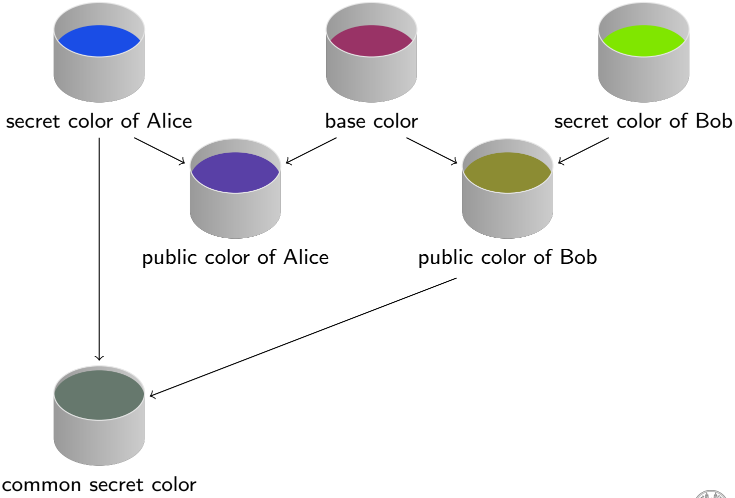

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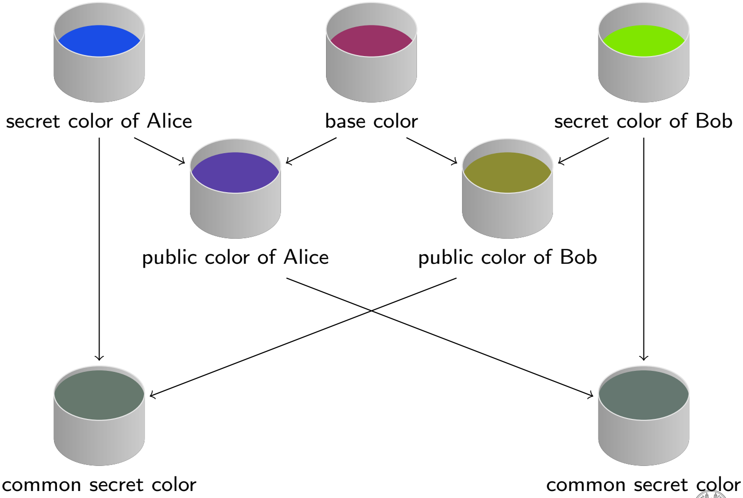

One way trapdoor functions and asymmetric keys [DH76]

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- φ *is injective*
- *With the help of a 'private key' it is possible to compute:*

$$\varphi^{-1} : \varphi(X) \longrightarrow X.$$

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Remark

In practices $x \in X$ represents often the key for some secret key system. The importance of one-way trapdoor functions was recognized by Diffie and Hellman in 1976.

El Gamal one way trapdoor function:

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Bob's Public Key:	(α, β, G)
Bob's Private Key:	$n = \log_\alpha \beta$.
Encryption:	$H \longrightarrow H \times H$
	$x \longmapsto (\alpha^k, x\beta^k) =: (c_1, c_2)$,

where k has been randomly chosen by Alice.

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Bob, with the knowledge of n is able to compute x from the cipher text c_1, c_2:

$$x = c_2 \left((c_1)^n\right)^{-1}.$$

Semigroups and Loops

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Let L be a set with a binary operation $(a, b) \longmapsto ab$. Then L is a loop if:

- *For $a, b, c \in L$, the knowledge of any two elements in the equation $ab = c$ uniquely specifies the third.*
- *There exists a neutral element e such that $ea = ae = a$ for all $a \in L$.*

Moufang Loops

Definition

A loop M is called a Moufang loop if the Moufang identities

$$(ab)(ca) = a((bc)a)$$
$$a(b(ac)) = ((ab)a)c$$
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Remark

One can show that if M is Moufang loop and $\alpha \in M$ *then the subloop* $< \alpha > \subset M$ *forms a group. In particular the discrete logarithm problem* $\log_\alpha \beta$ *is well defined and efficient algorithms such as square and multiply are possible.*

Ruth Moufang, 1905–1977

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For general Moufang loops and general semigroups it seems to be unknown if a quantum-polynomial algorithm exists.

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Semigroups and actions on sets

Another natural generalization to the DLP are semigroup actions first introduced by G.Maze, C.Monico and the speaker in 2002 [MMR02, MMRC02, Mon02, Maz03].

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Let G be a semigroup, let X be a set. A *semigroup action* of G on X is a map

$$\begin{array}{rrcl} \varphi: & G \times X & \longrightarrow & X \\ & (a,x) & \longmapsto & ax \end{array}$$

having the property, that

$$(a \cdot b)x = a(bx) \text{ for all } a,b \in G \text{ and } x \in X.$$

Semigroup Action Problem (SAP)

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Given a semigroup action G on X and elements $a \in G$ and $x \in X$. Given the elements x and $y := ax$. The semigroup action problem *asks for the computation of an element $\tilde{a} \in G$ such that $y = \tilde{a}x$.*

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Remark

Given a semigroup action. It has been shown in [MMR07] that

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\}$$

is a sub-semigroup and for cryptographic purposes what matters is the size of

$$\frac{\#G}{\#\mathrm{Stab}(x)}$$

Generalization of the DLP

Remark

Integers $(\mathbb{Z}, \cdot)$ *act on a group* G *through* $(a, g) \mapsto g^a$. *This leads to the usual discrete logarithm problem.*

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Note that $(\mathbb{Z}, \cdot)$ respectively $(\mathbb{Z}/n\mathbb{Z}, \cdot)$, respectively $(\mathbb{Z}/p\mathbb{Z}, \cdot)$, p a prime, is a semigroup but not a group. This has been one of the main reasons to look immediately at semigroup actions and not to restrict to group actions as considered in the recent literature [ADFMP20]

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- Their common secret key is then

$$a(bx) = (a \cdot b)x = (b \cdot a)x = b(ax)$$

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- Bob can decrypt the message using

$$m = (a(bx))^{-1} \circ c_2 = (bc_1)^{-1} \circ c_2.$$

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- Prover chooses randomly elements $b_i \in G$ and computes $z_i := b_i y = b_i ax$ for $i = 1, \ldots, n$.
- For each index i Verifier can either ask $\log_x z_i = b_i a$ or $\log_y z_i = b_i$.

Chebyshev action

Definition

$$T_n(x) = \cos(n \cos^{-1} x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} (1-x^2)^k$$

is called the nth Chebyshev polynomial.

Theorem

$T_{nm}(x) = T_n(T_m(x))$ in $\mathbb{Z}[x]$. In particular if R is any finite semiring then $T_n(r)$ can be efficiently computed for any $r \in R$ and $n \in \mathbb{N}$.

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Action on Endomorphism Ring

Example

Any abelian group H comes with its ring of endomorphisms $\operatorname{End} H$ where addition is defined pointwise and multiplication via composition of maps. There is a natural action of $\operatorname{End} H$ on H as follows :

$$\begin{aligned} \operatorname{End} H \times H &\longrightarrow H \\ (\varphi, h) &\longmapsto \varphi(h) \end{aligned}$$

For a given $\varphi \in \operatorname{End} H$, the subring $\mathbb{Z}[\varphi]$ of $\operatorname{End} H$ is commutative and yields to a Diffie-Hellman protocol.

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Special situation

Let $\mathbb{F}_p$ be a prime finite field ($p > 3$), $\overline{\mathbb{F}_p}$ its algebraic closure and $E\ :\ y^2 = x^3 + ax + b$ an ordinary elliptic curve over $\mathbb{F}_p$ with complex multiplication. In this case, it is known that $\mathrm{End}\, E(\overline{\mathbb{F}_p}) \cong \mathbb{Z} \oplus \mathbb{Z}\varphi$, where φ is the Frobenius endomorphism:

$$\begin{array}{rcl} \varphi : E(\overline{\mathbb{F}_p}) & \longrightarrow & E(\overline{\mathbb{F}_p}) \\ (x, y) & \longrightarrow & (x^p, y^p) \end{array}$$

Actions on semi-modules

Let R be a semiring, not necessarily finite.
(Two operations '$+$' and '$\cdot$' which are distributive and associative. We assume also that '$+$' is commutative. No neutral elements assumed.)
Let M be a finite semi-module over R. With this we mean that M has the structure of a finite semigroup and there is an action $R \times M \longrightarrow M$ such that

$$\begin{aligned} r(sm) &= (rs)m, \\ (r+s)m &= rm + sm, \\ r(m+n) &= rm + rn. \end{aligned}$$

for all $r, s \in R$ and $m, n \in M$.

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Actions on semi-modules

Let $Mat_{n\times n}(R)$ be the set of all $n \times n$ matrices with entries in R. The semiring structure on R induces a semiring structure on $Mat_{n\times n}(R)$. Moreover the semi-module structure on M lifts to a semi-module structure on M^n via the matrix multiplication:

$$\begin{aligned} Mat_{n\times n}(R) \times M^n &\longrightarrow M^n \\ (A, x) &\longmapsto Ax. \end{aligned} \tag{1}$$

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One readily verifies that $Mat_{n\times n}(R) \times M^n \longrightarrow M^n$ is an action by a semigroup, indeed one readily computes that $A(Bg) = (AB)g$.

Commutative semigroups

Let $R[t]$ be the polynomial ring in the indeterminant t and let $A \in Mat_{n \times n}(R)$ be a fixed matrix. Let $C \subset R$ be the center of R.

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Commutative semigroups

Let $R[t]$ be the polynomial ring in the indeterminant t and let $A \in Mat_{n\times n}(R)$ be a fixed matrix. Let $C \subset R$ be the center of R.If

$$p(t) = r_0 + r_1 t + \cdots + r_k t^k \in C[t]$$

then we define in the usual way $p(A) = r_0 I_n + r_1 A + \cdots + r_k A^k$, where $r_0 I_n$ is the $n \times n$ diagonal matrix with entry r_0 in each diagonal element.
Consider the semigroup

$$G := C[A] := \{p(A) \mid p(t) \in C[t]\}.$$

Clearly G has the structure of an abelian semigroup.

Diffie-Hellman protocol

Alice and Bob agree on an R-module $\mathcal{M}$, an element $b \in \mathcal{M}^n$ and a matrix $A \in Mat_{n\times n}(R)$.
Alice chooses secretly $p(t) \in C[t]$ and computes $p(A)b$ and sends the result to Bob. Bob chooses secretly $q(t) \in C[t]$ and computes $q(A)b$ and sends the result to Alice.
As a common secret key serves $k := p(A)q(A)b$

Nota Bene:

It should be difficult to find $\tilde{p}(t) \in C[t]$ such that

$$\tilde{p}(A)b = p(A)b.$$

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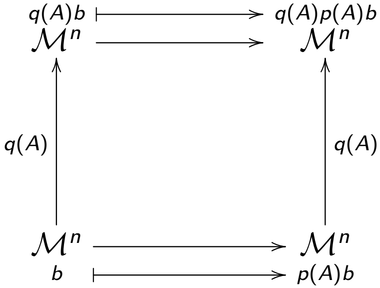
In Diagram:


$q(A)b \longmapsto q(A)p(A)b$
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Lattice isomorphism problem (LIP)

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Group action:

$$\begin{aligned} \varphi: \quad Gl_n(\mathbb{Z}) \times \mathrm{Sym}_n(\mathbb{Z}) &\longrightarrow \mathrm{Sym}_n(\mathbb{Z}) \\ (U, T) &\longmapsto U^t T U. \end{aligned}$$

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Remark

It should be possible to build signature schemes if one allows general quadratic forms, not necessarily positive definite.

Code equivalence problem

$\mathrm{Grass}(k, \mathbb{F}^n)$ the Grassman variety of k-dimensional subspaces inside the vector space $\mathbb{F}^n$.

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Group action:

$$\begin{aligned} \varphi : \quad M_n \times \mathrm{Grass}(k, \mathbb{F}^n) &\longrightarrow \mathrm{Grass}(k, \mathbb{F}^n) \\ (U, \mathrm{rowsp}(G)) &\longmapsto \mathrm{rowsp}(GU) \end{aligned}$$

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Above SAP describes the linear code equivalence problem heavily studied for building signature algorithms [BBPS23, BBP$^+$24].

Further interesting cryptographic group actions

Isogeny-based Cryptography

The study of isogeny-based cryptography was initiated by Couveignes [Cou06]. Couveignes already pointed out that some protocols can be seen as a group action on a set.

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Semidirect Discrete Logarithm Problem (SDLP)

Battarbee e.a. show [BKS24] that the SDLP can be viewed as a group action and the underlying problem is hence also a SAP.

Generic Algorithms for the SAP

Given a semigroup action $\varphi : \ G \times X \longrightarrow X$.

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In the survey article on semigroup actions [GZ24] Gnilke and Zumbrägel focused also on the generic complexity.

They explain that for group actions the generic complexity has both a square-root lower bound and a square-root upper bound. For proper semigroup actions one is lacking inversion in the group and the situation is less clear what the generic complexity is concerned.

Semirings

Definition

A semiring R is a non-empty set together with two associative operations $+$ and $\cdot$. with regard to addition $(R, +)$ is a commutative semigroup. The following distributive laws hold:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (a + b) \cdot c = a \cdot c + b \cdot c.$$

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Example

Consider the finite ring $R = \mathbb{Z}_6$. Consider the semigroup $G := \mathrm{Mat}_{n \times n}(R)$ consisting of $n \times n$ matrices with entries in R.

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Reduction modulo 2 and modulo 3 reduces the problem to two simpler instances which can be solved efficiently.

Remark

For above reason it is advisable to consider somehow 'simple rings'.

Simple semirings

Definition

A **congruence relation** *on a semiring R is an equivalence relation $\sim$ that also satisfies*

$$x_1 \sim x_2 \Rightarrow \begin{cases} c + x_1 & \sim & c + x_2, \\ x_1 + c & \sim & x_2 + c, \\ cx_1 & \sim & cx_2, \\ x_1 c & \sim & x_2 c, \end{cases}$$

for all $x_1, x_2, c \in R$. A semiring R that admits no congruence relations other than the trivial ones, id_R and $R \times R$, is said to be **congruence-simple**, *or* **c-simple**.

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Results on simple semirings

Theorem (Monico [Mon02])

Let R be a finite, additively commutative, congruence-simple semiring. Then one of the following holds:

1. $|R| = 2$.
2. $R \cong \mathrm{Mat}_{n\times n}(\mathbb{F}_q)$ *for some finite field* $\mathbb{F}_q$ *and some* $n \geq 1$.
3. R *is a zero multiplication ring of prime order.*
4. R *is additively idempotent.*
5. *There is an infinite element* ∞ *having the property that* $\infty r = r\infty = \infty + r = r + \infty = \infty$.

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Theorem (Zumbraegel [Zum08])

A finite semiring of order > 2 with zero which is not a ring is congruence-simple if and only if it is isomorphic to a "dense" subsemiring of the endomorphism semiring of a finite idempotent commutative monoid.

Some simple semirings of small order

A Simple Semiring of order 2

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	1	1

Some simple semirings of small order

A Simple Semiring of order 2

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	1	1

A Simple Semiring of order 3

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

*	0	1	2
0	0	0	0
1	0	1	2
2	2	2	2

A simple semiring of order 6, called S_6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	5
2	2	1	2	1	2	5
3	3	1	1	3	3	5
4	4	1	2	3	4	5
5	5	5	5	5	5	5

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	2	0	0	5
3	0	3	4	3	4	3
4	0	4	4	0	0	3
5	0	5	2	5	2	5

Example of DLP in a matrix group over S_6

Assume a matrix is given as:

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

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Example of DLP in a matrix group over S_6

What exponent results in the matrix

$$\begin{pmatrix} 2 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 3 & 2 \\ 3 & 2 & 3 & 3 & 3 & 2 & 1 & 3 & 2 & 3 \\ 0 & 5 & 2 & 1 & 5 & 5 & 5 & 0 & 5 & 5 \\ 5 & 0 & 5 & 2 & 1 & 5 & 1 & 5 & 0 & 5 \\ 5 & 5 & 5 & 5 & 2 & 5 & 1 & 5 & 5 & 5 \\ 3 & 3 & 3 & 4 & 3 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 3 \\ 0 & 3 & 0 & 4 & 3 & 3 & 2 & 0 & 3 & 3 \\ 3 & 0 & 3 & 0 & 4 & 0 & 4 & 2 & 0 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 2 & 3 \end{pmatrix}$$

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Semigroup action on itself

$G := \mathrm{Mat}_{n \times n}(R)$ be the semigroup consisting of $n \times n$ matrices over some simple semiring R.

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$$\begin{array}{rcl} G \times G & \longrightarrow & G \\ (A, X) & \longmapsto & AX = Y. \end{array}$$

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Remark

Over a field this is a trivial linear algebra problem. Over a non-commutative simple semiring where neither multiplicative nor additive inverses exist in general, we do not know how to solve the problem efficiently.

A two-sided abelian group action

Alice and Bob agree on a simple semiring R having center $C \subset R$ and agree on three matrices

$$A, B, M \in \mathrm{Mat}_{n \times n}(R).$$

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Alice chooses secretly $p_1(t), p_2(t) \in C[t]$ and computes $p_1(A)Mp_2(B)$ and sends the result to Bob. Bob chooses secretly $q_1(t), q_2(t) \in C[t]$ and computes $q_1(A)Mq_2(B)$ and sends the result to Alice.

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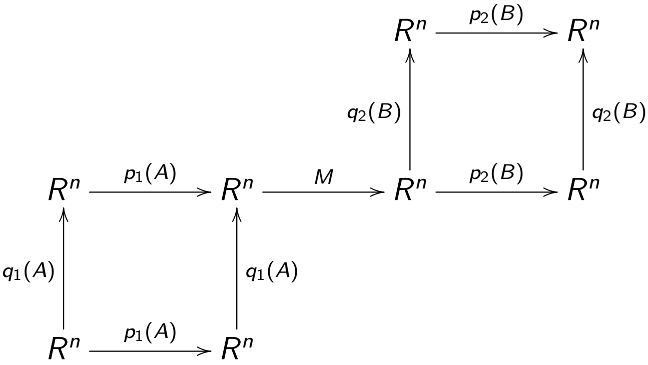
As a common secret key serves

$$k := p_1(A)q_1(A)Mq_2(B)p_2(B)$$

which both can easily compute.

In Diagram:



As a concrete choice let assume that $n = 20$. Consider the matrices

$$A = \begin{bmatrix} 10000000000000000000 \\ 00100000000000000000 \\ 00010000000000000000 \\ 00001000000000000000 \\ 01000000000000000000 \\ 00000010000000000000 \\ 00000002000000000010 \\ 00000100000000000000 \\ 00000000010000000000 \\ 00000000001000000000 \\ 00000000000200000000 \\ 00000000000010000000 \\ 00000000100000000000 \\ 00000000000000100000 \\ 00000000000000010000 \\ 00000000000000001000 \\ 00000000000000000100 \\ 00000000000000000010 \\ 00000000000000000001 \\ 00000000000001000000 \end{bmatrix} \quad B = \begin{bmatrix} 00000000000000000010 \\ 00000000000100000000 \\ 00000010000000000000 \\ 00100000000000000000 \\ 00000000000000000004 \\ 00000000000000010000 \\ 01000000000000000000 \\ 00000000000000000100 \\ 00010000100000000000 \\ 00000000000310000000 \\ 00000000000000200000 \\ 00010000000000000100 \\ 00000000001000000000 \\ 00000100000000000000 \\ 00000000010000000000 \\ 00000001000000000000 \\ 10000000000000000000 \\ 00001000000000000000 \\ 00000000000000001000 \\ 00000000000001000000 \end{bmatrix}$$

Example

$$M = \begin{bmatrix} 00200000000000000100 \\ 01000000010001000000 \\ 00000001000000000030 \\ 20020000000010000000 \\ 00000010000000001000 \\ 00000005000100000001 \\ 00000000200010000001 \\ 01000000030000000003 \\ 00000002000000010001 \\ 01000100000010000000 \\ 00000000000050100000 \\ 00000000000004000000 \\ 00000000000000100500 \\ 00300000002000100000 \\ 00001000000200001000 \\ 00000002000000000100 \\ 00002000001000000000 \\ 00100000000100000000 \\ 00020001000000000030 \\ 10000001000010000001 \end{bmatrix} \qquad T = \begin{bmatrix} 02020000000204000200 \\ 00111411002100241114 \\ 30111011002000240134 \\ 12000020020200202034 \\ 22111424020100201110 \\ 12222020022220222212 \\ 11111014222124211122 \\ 21111014222124222124 \\ 00222020022022200200 \\ 00002000022020220000 \\ 00222020020000200200 \\ 00000000022022200000 \\ 00002000000000020200 \\ 03333404021324040300 \\ 02202420020020001010 \\ 01111014000104040104 \\ 32000020020220000034 \\ 11111014020104211104 \\ 31333424021124040334 \\ 12202420020000211014 \end{bmatrix}.$$

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Example

$$M = \begin{bmatrix} 00200000000000000100 \\ 01000000010001000000 \\ 00000001000000000030 \\ 20020000000010000000 \\ 00000010000000001000 \\ 00000005000100000001 \\ 00000000200010000001 \\ 01000000030000000003 \\ 00000002000000010001 \\ 01000100000010000000 \\ 00000000000050100000 \\ 00000000000004000000 \\ 00000000000000100500 \\ 00300000002000100000 \\ 00001000000200001000 \\ 00000002000000000100 \\ 00002000001000000000 \\ 00100000000100000000 \\ 00020001000000000030 \\ 10000001000010000001 \end{bmatrix} \quad T = \begin{bmatrix} 02020000000204000200 \\ 00111411002100241114 \\ 30111011002000240134 \\ 12000020020200202034 \\ 22111424020100201110 \\ 12222020022220222212 \\ 11111014222124211122 \\ 21111014222124222124 \\ 00222020022022200200 \\ 00002000022020220000 \\ 00222020020000200200 \\ 00000000022022200000 \\ 00002000000000020200 \\ 03333404021324040300 \\ 02202420020020001010 \\ 01111014000104040104 \\ 32000020020220000034 \\ 11111014020104211104 \\ 31333424021124040334 \\ 12202420020000211014 \end{bmatrix}.$$

The task of Eve will be to find $p_1(t), p_2(t) \in C[t]$ such that $p_1(A)Mp_2(B) = T$. See Steinwandt and Suárez Corona, [SSC11] and Otero and Lopez Ramos [ALR25].

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