Network Coding
Lecture 3

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ICERM
Graduate Workshop on Linear Algebra over Finite Fields & Applications

Project idea and inspiration

F.R. Kschischang, FM, A. Ravagnani, and K. Savary, External codes for multiple unicast networks via interference alignment, Designs, Codes and Cryptography, 2024

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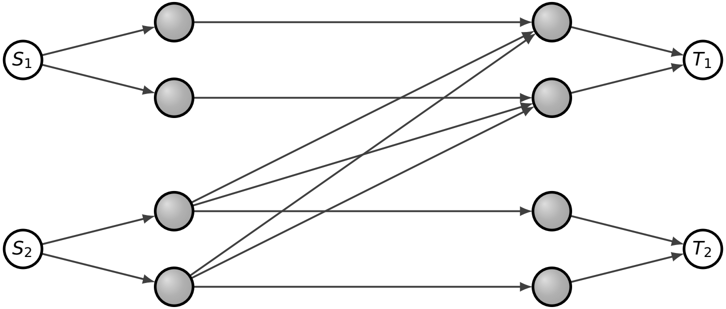
- Inspired by *Interference Alignment*.

- ✓ Communication Model and Problem Formulation
- ✓ Achievable Rate Regions and Their Properties
- ✓ An Outer Bound for the Achievable Rate Region
- ✓ The Role of the Field Characteristic

Network example

- Network Alphabet: $\mathbb{F}_q$.
- Linear network coding with fixed channel gains.
- Terminal T_i is interested in decoding only the symbols emitted by source S_i.

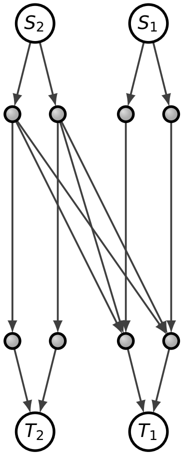


Multiple unicast network

Definition

A **multiple unicast network** is a 4-tuple
$\mathcal{N} = (\mathcal{V}, \mathcal{E}, (S_1, \ldots, S_n), (T_1, \ldots, T_n))$, where:

- $(\mathcal{V}, \mathcal{E})$ is a finite, directed, acyclic multigraph;
- $n \geq 1$ is an integer;
- $S_1, \ldots, S_n \in \mathcal{V}$ are distinct vertices called **sources**;
- $T_1, \ldots, T_n \in \mathcal{V}$ are distinct vertices called **terminals**.
- $\{S_1, \ldots, S_n\} \cap \{T_1, \ldots, T_n\} = \emptyset$;
- for any $i \in [n]$, there exists a directed path in $(\mathcal{V}, \mathcal{E})$ connecting S_i to T_i;
- sources no incoming edges and terminals no outgoing edges;
- For all $V \in \mathcal{V}$, there exists a directed path S_i-V and V-T_j, for some $i, j \in [n]$.



(Fixed) Network code

A **network code** for $\mathcal{N}$ is a tuple of matrices

$$\mathcal{F} = \left(\mathcal{F}_V \in \mathbb{F}_q^{\partial^-(V) \times \partial^+(V)} \mid V \in \mathcal{V} \setminus (\{S_1, \ldots, S_n\} \cup \{T_1, \ldots, T_n\})\right).$$

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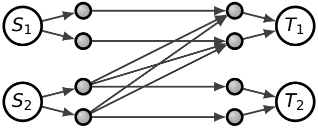
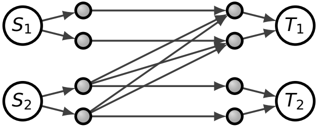


$$\mathcal{F}_1 = \left((1),(1),\begin{pmatrix}1 & 1 & 1\end{pmatrix},\begin{pmatrix}1 & 1 & 1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},(1),(1)\right)$$

$$\mathcal{F}_2 = \left((1),(1),\begin{pmatrix}1 & 1 & 1\end{pmatrix},\begin{pmatrix}1 & 1 & 1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix},(1),(1)\right)$$

Transfer Matrices



$\mathcal{F}_1$: $F_{1,1} = F_{2,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $F_{1,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $F_{2,1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

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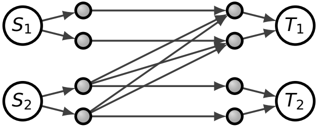

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q-ary Linear Multiple Uncast Channel

Definition **(q-LMUC)**

A **q-ary linear multiple unicast channel** (in short, **q-LMUC**) is a 4-tuple $\mathcal{L} = (n, \boldsymbol{s}, \boldsymbol{t}, F)$, where $n \in \mathbb{N}$ is a positive integer, $\boldsymbol{s} = (s_1, \dots, s_n), \boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{N}^n$, and $F \in \mathbb{F}_q^{s \times t}$, where $s = \sum_{i=1}^n s_i$ and $t = \sum_{i=1}^n t_i$. F is the **transfer matrix**

$$F = \begin{pmatrix} F_{1,1} & \cdots & F_{1,n} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,n} \end{pmatrix},$$

where block $F_{i,j}$ has size $s_i \times t_j$.

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Channel input: $x = (x_1, \dots, x_n) \in \mathbb{F}_{q^m}^{s_1} \times \cdots \times \mathbb{F}_{q^m}^{s_n} = \mathbb{F}_{q^m}^{s}$.

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Channel output: $y = (y_1, \ldots, y_n) \in \mathbb{F}_{q^m}^{t_1} \times \cdots \times \mathbb{F}_{q^m}^{t_n} = \mathbb{F}_{q^m}^{t}$, where

$$y_i = x_i F_{i,i} + \sum_{j \neq i} x_j F_{j,i} \tag{1}$$

$y = xF$

From q-LMUC to multiple unicast network $\mathcal{N}$

Given any q-LMUC, it is always possible to construct a multiple unicast network $\mathcal{N}$ and a network code $\mathcal{F}$ for $\mathcal{N}$ that induces the given transfer matrix.

Example

11-LMUC $\mathcal{L} = \left(2, (1,2), (2,2), \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 4 & 5 & 0 \\ 6 & 7 & 0 & 0 \end{pmatrix}\right)$

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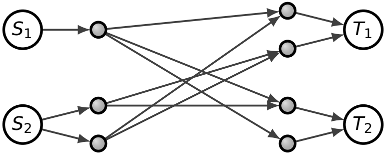
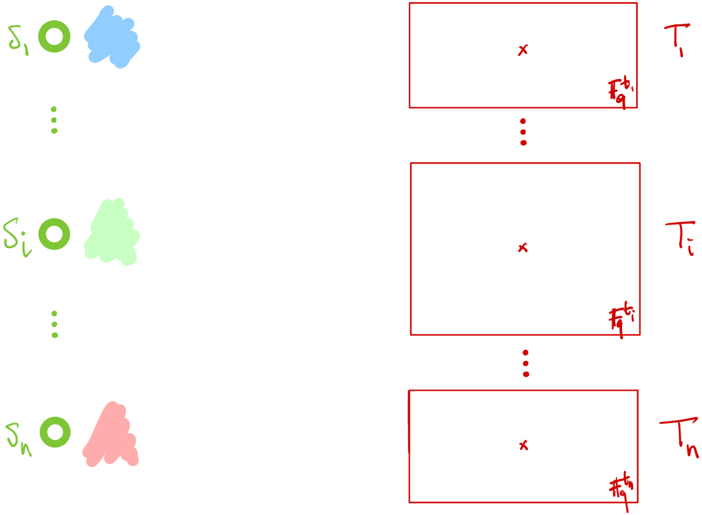


A network code that induces these matrices is

$$\mathcal{F} = \left(\begin{pmatrix}1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\end{pmatrix}, \begin{pmatrix}1 \\ 6\end{pmatrix}, \begin{pmatrix}4 \\ 7\end{pmatrix}, \begin{pmatrix}2 \\ 5\end{pmatrix}, \begin{pmatrix}3\end{pmatrix}\right), \textit{or}$$

$$\mathcal{F} = \left(\begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}4 & 5\end{pmatrix}, \begin{pmatrix}6 & 7\end{pmatrix}, \begin{pmatrix}1 \\ 1\end{pmatrix}, \begin{pmatrix}1 \\ 1\end{pmatrix}, \begin{pmatrix}1 \\ 1\end{pmatrix}, \begin{pmatrix}1\end{pmatrix}\right),$$

A Picture is Worth 1000 Words



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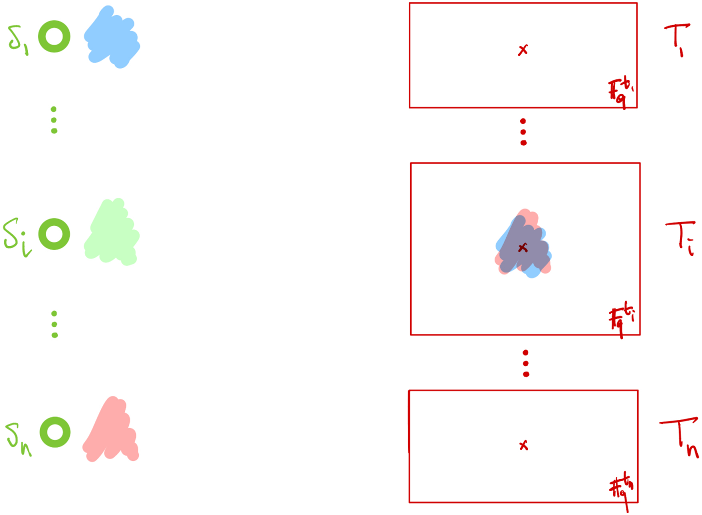

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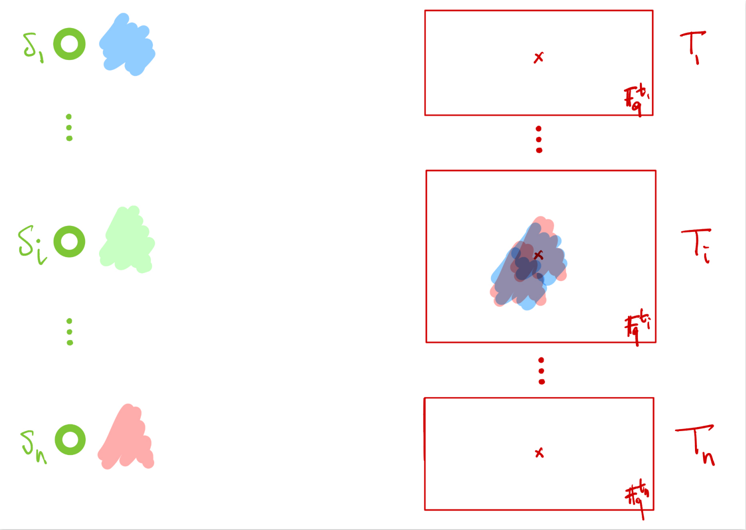

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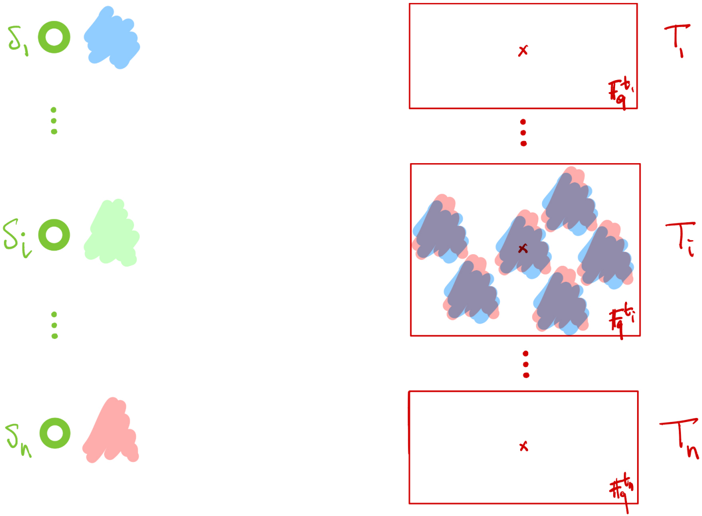

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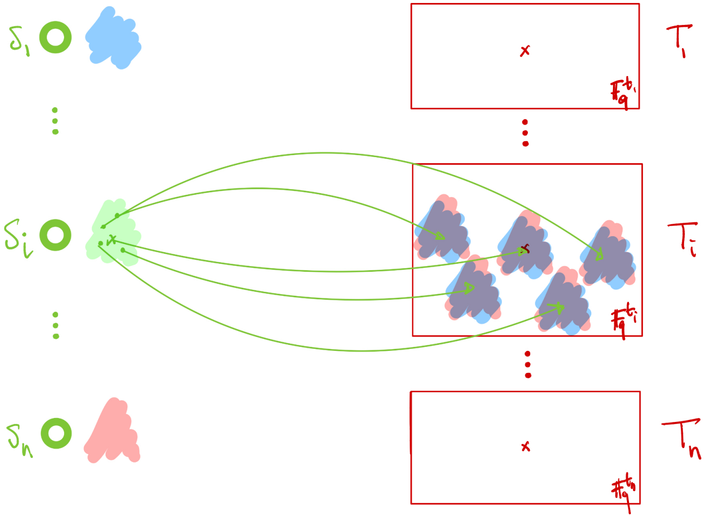


Codes and codewords and fan-out sets for a q-LMUC

Definition

Let $\mathcal{L} = (n, \boldsymbol{s}, \boldsymbol{t}, F)$ be a q-LMUC and let $m \geq 1$ be an integer. An **m-code** for $\mathcal{L}$ is a Cartesian product $C = C_1 \times \cdots \times C_n$, where $C_i \subseteq \mathbb{F}_{q^m}^{s_i}$ for all $i \in \{1, \ldots, n\}$. The elements of each C_i are called **codewords**.

Definition

Let $\mathcal{L} = (n, \boldsymbol{s}, \boldsymbol{t}, F)$ be a q-LMUC, $i \in \{1, \ldots, n\}$, C an m-code for $\mathcal{L}$, and $x \in C_i$. We denote by

$$\mathrm{Fan}_i(x, C) := \{\pi_i((x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)F) \mid x_j \in C_j \text{ for all } j \neq i\}$$

the **i-th fan-out set** of x with respect to terminal i and the code C. The **i-th fan-out set** of C is $\mathrm{Fan}_i(C) = \cup_{x \in C_i} \mathrm{Fan}_i(x, C) \subseteq \mathbb{F}_{q^m}^{t_i}$, for all $i \in \{1, \ldots, n\}$.

Interference and Successful Communication

Definition

Let $\mathcal{L}$, m, and C. We define the **interference set** of C at terminal T_i as

$$\mathrm{IS}_i(C) = \mathrm{Fan}_i(0, C) = \left\{ \sum_{j \neq i} x_j F_{j,i} \mid x_j \in C_j \right\}.$$

$$\mathrm{Fan}_i(x, C) = xF_{i,i} + \mathrm{IS}_i(C) = \{ xF_{i,i} + y \mid y \in \mathrm{IS}_i(C) \}.$$

Unambiguous Codes

Definition

Let $\mathcal{L}$, m, and C. We say that C is **unambiguous** for $\mathcal{L}$ if for all $i \in \{1, \ldots, n\}$ and for all codewords $x_1, x_2 \in C_i$ with $x_1 \neq x_2$, we have

$$\mathrm{Fan}_i(x_1, C) \cap \mathrm{Fan}_i(x_2, C) = \emptyset.$$

Achievable Rate Region

Definition

The m**-shot achievable rate region** of a q-LMUC $\mathcal{L}$ is the set

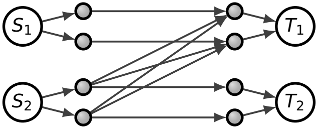
$$\mathcal{R}_m(\mathcal{L}) = \{\alpha \in \log_{q^m}(\mathbb{N}^n) \mid \exists\, C = C_1 \times \cdots \times C_n \text{ unambiguous } m\text{-code for } \mathcal{L} \\ \text{with } \log_{q^m}(|C_i|) = \alpha_i \;\forall\, 1 \leq i \leq n\} \subseteq \mathbb{R}^n_{\geq 0}.$$

The **achievable rate region** of $\mathcal{L}$ is the set

$$\mathcal{R}(\mathcal{L}) = \overline{\bigcup_{m \geq 1} \mathcal{R}_m(\mathcal{L})},$$

where the overline indicates the closure operator with respect to the Euclidean topology on $\mathbb{R}^n$. The elements of $\mathcal{R}(\mathcal{L})$ elements are called **achievable rates**.

Network Code Dependency of the Achievable Rate Region



Consider the 3-LMUC

$$\mathcal{L}_1 = \left(2, (2,2), (2,2), \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \right), \quad \mathcal{L}_2 = \left(2, (2,2), (2,2), \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \right).$$

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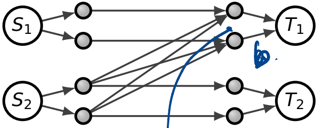

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- $(1,2) \in \mathcal{R}_1(\mathcal{L})$ since $\langle (1,2) \rangle_{\mathbb{F}_3} \times \mathbb{F}_3^2$
- $(1,2) \notin \mathcal{R}_1(\mathcal{L})$

- ✓ Communication Model and Problem Formulation
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The Outer Bound

Proposition (Unicast Bound - tight)

Let $\mathcal{L} = (n, \mathbf{s}, \mathbf{t}, F)$ be a q-LMUC and let $m \geq 1$ be an integer. If $n = 1$, then $\mathcal{R}_m(\mathcal{L}) = \{\alpha \in \log_q(\mathbb{N}) \mid 0 \leq \alpha \leq \operatorname{rank} F\}$.

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Theorem

Let $\mathcal{L}$ be a q-LMUC, $m \geq 1$, and $(\alpha_1, \ldots, \alpha_n) \in \mathcal{R}_m(\mathcal{L})$. Then for all non-empty $I \subseteq \{1, \ldots, n\}$ and $j \in I$, we have $\sum_{i \in I} \alpha_i \leq \operatorname{rank}(F_{I,j}) - \operatorname{rank}(F_{I \setminus \{j\}, j}) + \sum_{\substack{k \in I \\ k \neq j}} s_k$. Therefore,

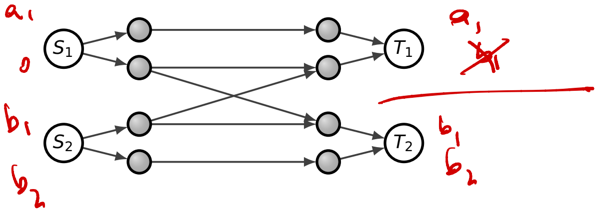
for all non-empty $I \subseteq \{1, \ldots, n\}$, we have

$$\sum_{i \in I} \alpha_i \leq \min_{j \in I} \left\{ \operatorname{rank}(F_{I,j}) - \operatorname{rank}(F_{I \setminus \{j\}, j}) + \sum_{\substack{k \in I \\ k \neq j}} s_k \right\}.$$

Examples

Let q be arbitrary and consider the q-LMUC $\mathcal{L} = (2, (2, 2), (2, 2), F)$, where

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$



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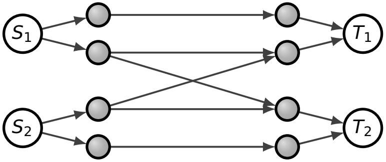
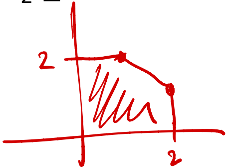


$$\operatorname{rank}\begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix} - \operatorname{rank} F_{21} + s_2 = \operatorname{rank}\begin{pmatrix} F_{12} \\ F_{22} \end{pmatrix} - \operatorname{rank} F_{12} + s_1 = 2 - 1 + 2 = 3$$

Other bound: for all $m \geq 1$ and for all $(\alpha_1, \alpha_2) \in \mathcal{R}_m(\mathcal{L})$ we have $\alpha_1 + \alpha_2 \leq 3$.



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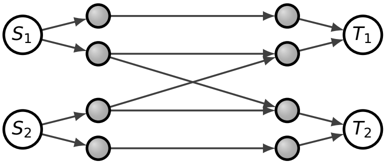

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For $m = 1$, the 1-codes $C = \left(\mathbb{F}_q^2, \langle(0,1)\rangle\right)$ and $C = \left(\langle(1,0)\rangle, \mathbb{F}_q^2\right)$ are both unambiguous, meaning that the rates $(2, 1)$ and $(1, 2)$ are achievable in one shot.

✓ Communication Model and Problem Formulation

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✓ An Outer Bound for the Achievable Rate Region

✓ The Role of the Field Characteristic

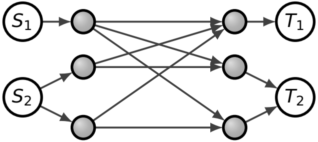
A Modified Problem

Remark

Any field contains the neutral elements for addition and multiplication and that in general those are denoted by 0 and 1. This implies that any matrix with entries only in $\{0, 1\}$ can be the transfer matrix of a q-LMUC for any prime power q.

It is natural to look into the achievable rates regions across different fields for these types of q-LMUC.

A Very Specific Example



Theorem

Let $\mathcal{L} = (2, (1,2), (1,2), F)$ be the q-LMUC with $F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

If q is odd, then $(1,1) \in \mathcal{R}_1(\mathcal{L}) \subseteq \mathcal{R}(\mathcal{L})$. If q is even, then for any $m \geq 1$ and any $(\alpha_1, \alpha_2) \in \mathcal{R}_m(\mathcal{L})$ we have

$$2\alpha_1 + \alpha_2 \leq 2.$$

In particular, $(1,1) \notin \mathcal{R}(\mathcal{L})$.

Proof

- if q is odd, then the 1-code $C = \mathbb{F}_q \times \langle(1,-1)\rangle$ is unambiguous $\Rightarrow (1,1) \in \mathcal{R}_1(\mathcal{L})$
- if q is even, $C = \mathbb{F}_q \times \langle(1,1)\rangle$ is the only candidate for unambiguity, but

$$F_{2,2}\langle(1,1)\rangle = \langle(1,1)\rangle = \mathrm{IS}_2(C)$$

but $\mathrm{Fan}_2(x_1, C) = \mathrm{Fan}_2(x_2, C)$ for all $x_1, x_2 \in \langle(1,1)\rangle$, so it is not ambiguous.

Tightness

Proposition

Let $\mathcal{L} = (2, (1, 2), (1, 2), \mathbb{F}_{2^m})$ be the 2^m-LMUC from the previous theorem. Then for any $n \leq m$ we have $\left(\frac{n}{m}, 2\left(1 - \frac{n}{m}\right)\right) \in \mathcal{R}(\mathcal{L})$.

Let $\{x_1, \ldots, x_m\}$ be an ordered basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Define

- $C_1 = \langle x_1, \ldots, x_n \rangle_{\mathbb{F}_2}$, and
- $C_2 = \langle (x_i, 0), (0, x_i) \mid i = n + 1, \ldots, m \rangle_{\mathbb{F}_2}$.

Then $C = C_1 \times C_2$ is unambiguous.

Conclusion

- We introduce a formal framework for investigating multi-shot interference alignment problems over finite fields.
- We establish an outer bound for the achievable rate regions in the context outlined above and provide examples where the bound is sharp.
- We show how the field characteristic plays a crucial role in the solution to this problem.

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- We establish an outer bound for the achievable rate regions in the context outlined above and provide examples where the bound is sharp.
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Thank you.