Network Coding
Lecture 1

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ICERM
Graduate Workshop on Linear Algebra over Finite Fields & Applications

- Facts about Finite Fields.
- From Channels to Networks
- One Source Networks
  - The Unicast Network
  - The Multicast Network

What is a field?

A field is a nonempty set $\mathbb{F}$ with two operations, addition $(+)$ and multiplication $(\cdot)$ such that for all $a, b, c \in \mathbb{F}$:

- ✓ $(\mathbb{F}, +)$ is an abelian group
  - $+$ is associative: $a + (b + c) = (a + b) + c$
  - there is an additive ~~unit~~ neutral element: $a + 0 = 0 + a = a$
  - there is an additive inverse element: $a + (-a) = (-a) + a = 0$
  - $+$ is commutative: $a + b = b + a$

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- ✓ $\cdot$ is associative: $a(bc) = (ab)c$
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Example of fields: $\mathbb{Q}$, $\mathbb{F}_2$, $\mathbb{Z}_p$ p prime, $\mathbb{F}_q$

Finite Fields

Definition

A **finite field** is a field with a finite number of elements.

- The cardinality of a finite field is a power of a prime, meaning that if $\mathbb{F}_q$ is a finite field with q elements, then $q = p^t$ for some prime p.

Why?

- If p is a prime, then $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ and is called **prime field.**

Example: $\mathbb{F}_3$ = {0, 1, 2}

- If $q = p^t$ with p prime, then $\mathbb{F}_q \simeq \mathbb{F}_p[x]/(\mu)$ where $\mu \in \mathbb{F}_p[x]$ is irreducible of degree t.

Example: $\mathbb{F}_9$ x^2+x-1 x^2+1 $\mathbb{F}_3[x]/(x^2+1) =$

Finite Fields

- Let $\mathbb{F}$ be a finite field containing a subfield $\mathbb{K}$ with q elements. Then $\mathbb{F}$ has q^m elements, where $m = [F : K]$. Moreover, for all $\mu \in \mathbb{K}[x]$ irreducible with degree m such that $\mathbb{F} \simeq \mathbb{K}[x]/(\mu)$.

- If $\mathbb{F}$ is a finite field with q elements, then $a^q = a$ for all $a \in \mathbb{F}$.

- If $\mathbb{F}$ is a finite field with q elements and $\mathbb{K}$ is a subfield of $\mathbb{F}$, then the polynomial $x^q - x \in \mathbb{K}[x]$ factors in $F[x]$ as

  $$x^q - x = \prod_{a \in \mathbb{F}} (x - a)$$

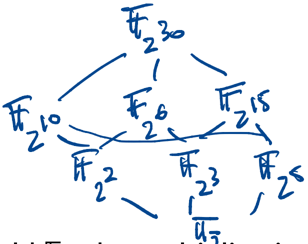
  and F is a splitting field of $x^q - x$ over $\mathbb{K}$.
  Example: $x^9 - x \in \mathbb{F}_3$

Finite Fields

- **Subfield Criterion.** Let $\mathbb{F}_q$ be the finite field with $q = p^t$ elements. Then every subfield of $\mathbb{F}_q$ has order p^m, where m is a positive divisor of t. Conversely, if m is a positive divisor of t, then there is exactly one subfield of $\mathbb{F}_q$ with p^m elements.

Example: Diagram of $\mathbb{F}_{2^{30}}$

$30 = 2 \cdot 3 \cdot 5$



- For every finite field $\mathbb{F}_q$ the multiplicative group $\mathbb{F}_q^*$ of nonzero elements of $\mathbb{F}_q$ is cyclic. A generator of $\mathbb{F}_q^*$ is called a primitive element of $\mathbb{F}_q$.

Example $\mathbb{F}_9^*$ $\quad \mathbb{F}_3[\alpha]/(\alpha^2+1) \qquad \langle \alpha + 1 \rangle = \mathbb{F}_9^*$

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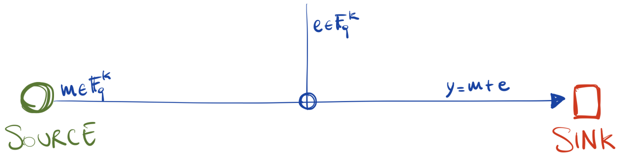
Noisy-Channel Coding Theorem - Shannon 1948)



Over $\mathbb{F}_2$:

$m = 1$

$e = 1$

$y = 0$

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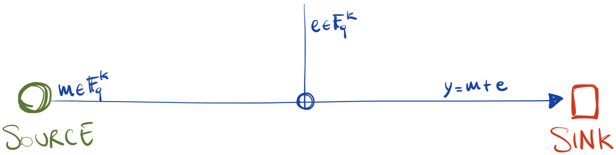

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Theorem (Noisy-Channel Coding Theorem - Shannon - 1948)

“In communication theory, any channel, however affected by noise, possesses a specific channel capacity - a rate of conveying information that can never be exceeded without error, but that can, in principle, always be attained with an arbitrarily small probability of error."

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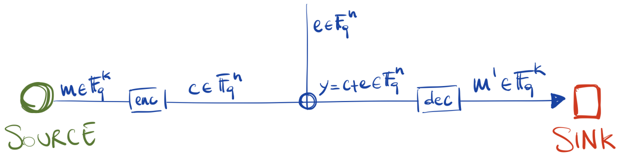

Over $\mathbb{F}_2$:
$m = 1$
$c = (111)$
$e = (010)$
$y = (101)$

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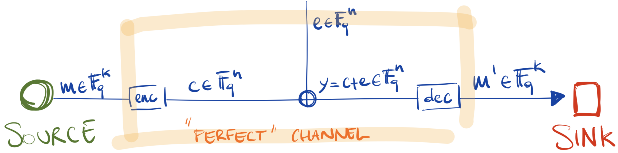

Over $\mathbb{F}_2$:
$m = 1$
$c = (111)$
$e = (010)$
$y = (101)$
$m' = 1$

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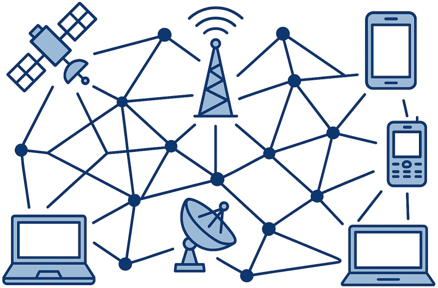
Turbo codes (LTE networks), Polar & LDPC codes (5G networks)

Example of (Communication) Networks



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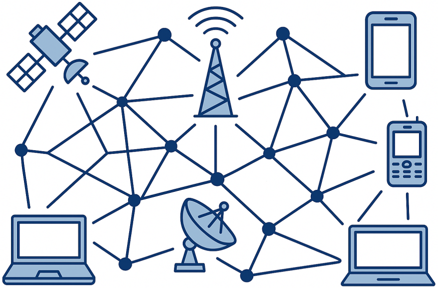

Question

Is routing the **best** communication strategy on a network?

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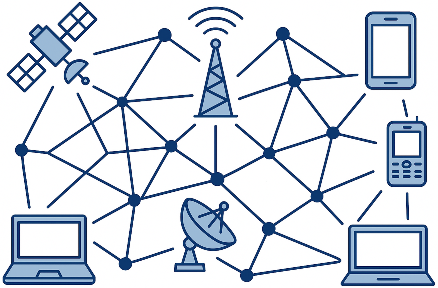

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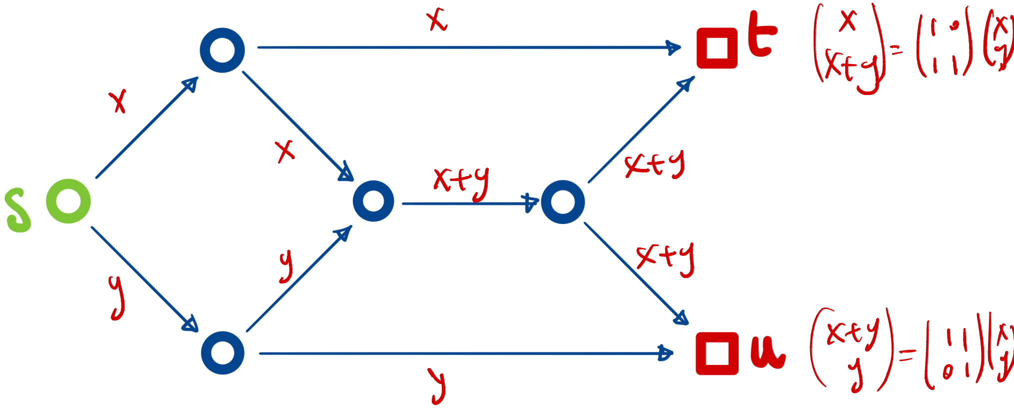
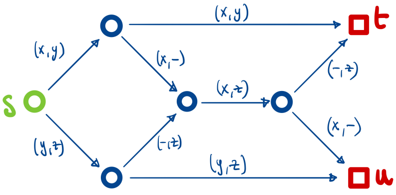
Notes adapted from [1, 2].

The Butterfly Network

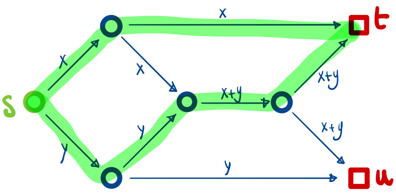


Made with Goodnotes

The Butterfly Network



(a) Rounting



(b) Network Coding

Rates (ρ):

Routing: $\frac{3}{2} = 1{,}5$

Network coding: $\frac{2}{1} = 2$

Can we do better?

Network representation

Disclaimer: We will consider the alphabet to be $\mathbb{F}_q$.

A **network** is a 4-tuple $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{T})$ where:

- ✓ $G = (\mathcal{V}, \mathcal{E})$ is a a finite directed acyclic multigraph with $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the multiset of directed edges;
- ✓ $\mathcal{S} \subset \mathcal{V}$ is the set of sources;
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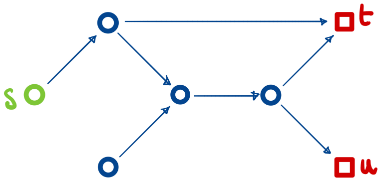
- ✓ $(u, v) \in \mathcal{E}$ is a perfect unit capacity channel from u to v.

Examples of non networks



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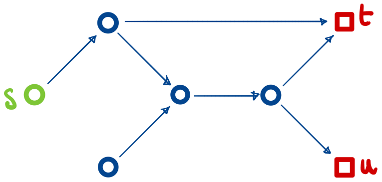

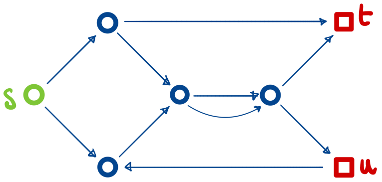

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