Green's Function for a General System of Hyperbolic-Parabolic Balance Laws with Application

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Motivation and Background

Cauchy problem of a system of hyperbolic-parabolic balance laws:

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{1}$$

$$w(x,0) = w_0(x). \tag{2}$$

$w \in \mathbb{R}^n$ — unknown density function (components are mass density, momentum density, etc.)

$f \in \mathbb{R}^n$ — flux function

$B \in \mathbb{R}^{n\times n}$ — viscosity matrix (viscosity, heat conduction, species diffusion, etc.)

$r \in \mathbb{R}^n$ — external forces, relaxation, chemical reaction, etc.

Important facts:

- The flux function f satisfies an entropy condition so that the corresponding inviscid form

$$w_t + f(w)_x = 0 \tag{3}$$

is hyperbolic.

- The entropy condition is extended to include B and r in that f', B and r' (restricted to the equilibrium manifold) are symmetrizable simultaneously.

- B and r' are rank deficient.

Examples

Example 1. Polyatomic gas flows in both translational and vibrational non-equilibrium (see [Clarke & McChesney, Vincenti &Kruger]),

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + ((\rho u^2 + p)_x = (\mu u_x)_x, \\ (\rho E)_t + (\rho E u + p u)_x = (\mu u u_x + \kappa T_{1x} + \nu \rho e_{2x})_x, \\ (\rho e_2)_t + (\rho e_2 u)_x = (\nu \rho e_{2x})_x + \rho \frac{e_2^* - e_2}{\tau}. \end{cases} \tag{4}$$

ρ — density $\quad u$ — velocity $\quad p$ — pressure

$E = e + \frac{1}{2}u^2$ — total specific energy with $\quad e = e_1 + e_2$

e_2 — non-equilibrium vibrational energy

e_1 — the rest of internal energy

Equ (4) is derived from Boltzmann equation. Dissipation mechanisms are introduced to compensate the non-equilibrium in the translational mode through Chapman-Enskog expansion:

μ — viscosity coefficient κ — thermal conductivity

ν — self-diffusion coefficient

On the other hand, the vibrational mode is singled out as the non-equilibrium mode. In its relaxation,

e_2^* — the local equilibrium value of e_2

τ — the relaxation time scale

Such a setup is based on the fact that the relaxations of translational mode and of vibrational mode are in very different time scales.

Two sets of thermal dynamic variables:

- "1" for the translational mode (and all other internal modes that are in equilibrium)
- "2" for the vibrational mode

Two thermodynamic equations:

$$
\begin{aligned}
T_1 ds_1 &= de_1 + p dv, \qquad v = 1/\rho, \\
T_2 ds_2 &= de_2.
\end{aligned}
$$

Equ (4) is a system of 4 equations for 4 unknowns (u and three dynamic variables, two with index "1" and one with index "2").

Equ (4) is an example of the general system (1).

Example 2. Two-temperature Navier-Stokes type model, which is another model for polyatomic gas flows, proposed by Aoki et al (*Phys. Rev. E*, **102** (2020), 023104)

$$
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + \rho R T_1)_x = (\mu_{\mathrm{tr}} u_x)_x, \\
[\rho(\frac{3+\delta}{2} RT + \frac{1}{2} u^2)]_t + [\rho u(\frac{3+\delta}{2} RT + RT_1 + \frac{1}{2} u^2)]_x \\
\qquad\qquad = [\kappa_{\mathrm{tr}}(\frac{3+\delta}{2} T_x + T_{1x}) + \mu_{\mathrm{tr}} u u_x]_x, \\
(\rho T_2)_t + (\rho u T_2)_x = \frac{2}{\delta R}(\kappa_{\mathrm{int}} T_{2x})_x + \theta A_c(T)\rho^2(T - T_2).
\end{cases}
\tag{5}
$$

$$
p = \rho R T_1, \qquad T = \frac{3T_1 + \delta T_2}{3+\delta}
$$

$$
e = \frac{3+\delta}{2} RT = \frac{3}{2} RT_1 + \frac{\delta}{2} RT_2 \equiv e_1 + e_2
$$

$\delta \geq 2$ — the number of the internal degrees of freedom

$\theta \in (0, 1]$ — constant parameter $A_c(T)\rho$ — the collision frequency

The system is derived from the polyatomic version of the ellipsoidal-statistical model (a simplified kinetic model) rather than the original Boltzmann equation.

The setup of (5) is different from that of (4). The translational mode is separated from δ (at least two) relaxing internal modes.

However, (4) and (5) turn out to be very similar. Both are in the form of (1), with some discrepancies in the formulation of dissipation parameters.

A much simpler example than these two is a chemotaxis model, which is 2x2.

These examples give us motivation to study (1).

Hyperbolic-parabolic balance laws

One space dimension

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{1}$$

Multi space dimensions

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w)w_{x_k}\right]_{x_j} + r(w) \tag{6}$$
$$x = (x_1, \dots, x_m)^t \in \mathbb{R}^m,\ t \in \mathbb{R}^+$$

Hyperbolic-parabolic conservation laws

$$\textit{One-D:} \qquad w_t + f(w)_x = [B(w)w_x]_x$$

$$\textit{Multi-D:} \qquad w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w)w_{x_k}\right]_{x_j} \tag{7}$$

Hyperbolic balance laws

$$\textit{One-D:} \qquad w_t + f(w)_x = r(w)$$

$$\textit{Multi-D:} \qquad w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = r(w) \tag{8}$$

Cauchy problems near equilibrium states in the space $C([0,\infty); H^s(\mathbb{R}^m)) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}^m)) \quad (s > \frac{m}{2}+1)$

- hyperbolic-parabolic conservation laws:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w) w_{x_k}\right]_{x_j} \tag{7}$$

  - Early works on isentropic gas flows (Nash, Itaya, Kazhikhov-Shelukhin)
  - Navier-Stokes equations via energy method (Matsumura-Nishida)
  - Complete theory for the general system (Kawashima for structural conditions and Kawashima-Shizuta for KS condition). The theory includes global existence of unique solution and L^2 time decay rates to a constant state.
  - For one space dimension, Green's function of the linearization in the physical space, pointwise (in x and t) asymptotic behavior, and hence L^p $(1 \le p \le \infty)$ asymptotic behavior (Z, Liu-Z)
  - For two and three space dimensions, Green's function of the linearized Navier-Stokes equations in the physical space (Wang, Lin and others)

- hyperbolic balance laws:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = r(w) \tag{8}$$

  - Early works on 2×2 systems (Nishida, Liu)
  - Structural conditions for global existence and L^2 time decay rates (Chen-Levermore-Liu, Hanouzet-Natalini, Yong, Kawashima-Yong, Xu-Kawashima, Z)
  - Green's function in one space dimension (Z, Bianchini-Hanouzet-Natalini)
  - Pointwise asymptotic behavior in one space dimension (Z, Z-chen)
  - Green's function for thermal non-equilibrium flows in three space dimensions (Z)

- hyperbolic-parabolic balance laws:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w) w_{x_k}\right]_{x_j} + r(w) \qquad (6)$$

  - Structural conditions for global existence and L^2 time decay rates (Z)
  - Asymptotic behavior in multi-space dimensions (Z)
  - Green's function in one space dimension (Z, the topic of this talk)
  - Pointwise asymptotic behavior in one space dimension (to be done)
  - Green's function in multi-space dimensions for specific physical examples (open)

Green's function for the Linear System

We consider the linearized system of (1) around a constant equilibrium state $\bar{w}$, with $r(\bar{w}) = 0$. It reads

$$w_t + Aw_x = Bw_{xx} + Lw, \tag{9}$$

where

$$A = f'(\bar{w}), \quad B = B(\bar{w}), \quad L = r'(\bar{w}).$$

The challenge is the intertwining of the matrices A, B and L. These matrices together with their coupling determine various parameters in the leading terms of the Green's function. The parameters are to be found employing ideas from Kato's perturbation theory (*Perturbation theory for linear operators*, 2nd edn., Springer, New York (1976)) and from Chapman-Enskog expansions.

$$w_t + Aw_x = Bw_{xx} + Lw \tag{9}$$

$$w \in \mathbb{R}^n \qquad A, B, L \in \mathbb{R}^{n\times n}$$

Structural Conditions:

(i) There exists a symmetric, positive definite matrix A_0, such that A_0A is symmetric, A_0B is symmetric, semi-positive definite, and A_0L is symmetric, semi-negative definite.

(ii) There are n_1 conservation laws in (9). That is, there is a partition $n = n_1 + n_2$, $n_1, n_2 \geq 0$, such that

$$L = \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ L_{21} & L_{22} \end{pmatrix}, \tag{10}$$

where $L_{21} \in \mathbb{R}^{n_2\times n_1}$ and $L_{22} \in \mathbb{R}^{n_2\times n_2}$ is nonsingular (if $n_2 > 0$).

(iii) (Kawashima-Shizuta condition) Let $\mathbb{N}_1$ be the null space of B and $\mathbb{N}_2$ be the null space of L. Then $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of A.

The Green's function for the Cauchy problem of (9) is the solution matrix $G(x, t)$ that satisfies

$$\begin{cases} G_t + AG_x = BG_{xx} + LG, \\ G(x, 0) = \delta(x)I, \end{cases} \tag{11}$$

where δ is the Dirac δ-function, and I is the $n \times n$ identity.

Notation for a heat kernel:

$$H(x, t; \lambda, \mu) \equiv \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{(x-\lambda t)^2}{4\mu t}}, \tag{12}$$

where $\mu > 0$ and λ are constants.

Theorem

Under the structural conditions, for $x \in \mathbb{R}$ and $t > 0$, the Green's function G of the Cauchy problem of (9) satisfies

$$
\begin{aligned}
G(x,t) = & \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j)P_j \\
& + O(1)(t+1)^{-\frac{1}{2}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},C) \\
& + \sum_{j=1}^{m'} e^{-\beta_j t}\delta(x-\alpha_j t)Q_j .
\end{aligned}
\tag{13}
$$

Here m and m' are nonnegative integers; $\lambda_j^{(r)}$ and $\mu_j > 0$, $1 \le j \le m$, are constants; α_j and $\beta_j > 0$, $1 \le j \le m'$, are constants; P_j, $1 \le j \le m$, and $Q_j(t)$, $1 \le j \le m'$ are constant projections; , $C > 0$ is a constant. The parameters are determined from the coefficient matrices A, B and L through the procedure below and have explicit formulation.

To simplify the formulation of parameters we introduce new variables according to (ii) of the structural conditions. Let

$$\begin{aligned} w &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad w_1 \in \mathbb{R}^{n_1},\ w_2 \in \mathbb{R}^{n_2}; \\ \tilde{w} &= \begin{pmatrix} w_1 \\ L_{21} w_1 + L_{22} w_2 \end{pmatrix} \equiv \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}. \end{aligned} \tag{14}$$

The corresponding Jacobian matrices are

$$\begin{aligned} \tilde{w}_w &= \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ L_{21} & L_{22} \end{pmatrix}, \\ w_{\tilde{w}} = \tilde{w}_w^{-1} &= \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ -L_{22}^{-1} L_{21} & L_{22}^{-1} \end{pmatrix}. \end{aligned} \tag{15}$$

The equation for $\tilde{w}$ is

$$\tilde{w}_t + \tilde{A}\tilde{w}_x = \tilde{B}\tilde{w}_{xx} + \tilde{L}\tilde{w}, \tag{16}$$

where

$$\tilde{A} = \tilde{w}_w A w_{\tilde{w}}, \quad \tilde{B} = \tilde{w}_w B w_{\tilde{w}}, \quad \tilde{L} = \tilde{w}_w L w_{\tilde{w}}. \tag{17}$$

One can verify that $\tilde{A}, \tilde{B}$ and $\tilde{L}$ satisfy the structural conditions, while

$$\tilde{L} = \begin{pmatrix} 0_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & L_{22} \end{pmatrix}.$$

Since the Green's functions for (9) and for (16) are related by

$$\tilde{G}(x,t) = \tilde{w}_w G(x,t) w_{\tilde{w}},$$

we formulate parameters for $\tilde{G}(x,t)$ instead.

Thus,

$$w_t + Aw_x = Bw_{xx} + Lw \tag{9}$$

under Structural Conditions:

(i) ...

(ii′)

$$L = \mathrm{diag}(0_{n_1\times n_1}, L_{22}),$$

where $L_{22} \in \mathbb{R}^{n_2\times n_2}$ is invertible.

(iii) ...

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$.

Let

$$A^{(r)} = A_{11} \in \mathbb{R}^{n_1 \times n_1}, \tag{18}$$

which is the part of A related to the reduced system (the equilibrium system).

Procedure to find parameters

Step 1. All eigenvalues of $A^{(r)}$ are real. Assume they are simple,

$$\lambda_1^{(r)} < \lambda_2^{(r)} < \cdots < \lambda_{n_1}^{(r)}.$$

These are $\lambda_j^{(r)}$ in (13), and $m = n_1$ therein. Let the left eigenvector (row vector) and right eigenvector of $A^{(r)}$ corresponding to $\lambda_j^{(r)}$ be $l_j^{(r)}$ and $r_j^{(r)}$, respectively,

$$A^{(r)} r_j^{(r)} = \lambda_j^{(r)} r_j^{(r)}, \qquad l_j^{(r)} A^{(r)} = \lambda_j^{(r)} l_j^{(r)}, \qquad 1 \le j \le n_1,$$
$$l_i^{(r)} r_j^{(r)} = \delta_{ij}, \quad 1 \le i, j \le n_1.$$

Then corresponding to each $\lambda_j^{(r)}$, P_j in (13) is given by

$$P_j = \operatorname{diag}\left(r_j^{(r)} l_j^{(r)}, 0_{n_2 \times n_2}\right), \qquad 1 \le j \le n_1. \tag{19}$$

Step 2. L has spectral decomposition

$$L = \sum_{j=1}^{s} -\sigma_j Q_j, \tag{20}$$

with $\sigma_j > 0$, $1 \leq j \leq s$. Here each $-\sigma_j$ is a negative eigenvalue of L, with the corresponding eigen-projection Q_j. Let Q_0 be the eigen-projection corresponding to the eigenvalue zero,

$$Q_0 = I - \sum_{j=1}^{s} Q_j = \mathrm{diag}(I_{n_1 \times n_1}, 0_{n_2 \times n_2}). \tag{21}$$

Define

$$S = \sum_{j=1}^{s} \frac{1}{\sigma_j} Q_j. \tag{22}$$

Then $-S$ is the value at zero of the reduced resolvent of L with respect to the eigenvalue zero. One can show that

$$S = \mathrm{diag}(0_{n_1 \times n_1}, S_{22}), \tag{23}$$

where S_{22} is symmetric, positive definite.

Now for each heat kernel in the first term of (13), corresponding to $\lambda_j^{(r)}$ we have

$$\mu_j = (l_j^{(r)}, 0_{1\times n_2})(ASA + B)\begin{pmatrix} r_j^{(r)} \\ 0_{n_2\times 1}\end{pmatrix}, \qquad 1 \le j \le n_1. \tag{24}$$

One can show $\mu_j > 0$, $1 \le j \le n_1$.

Step 3. B has distinct eigenvalues

$$0 = \sigma_0' < \sigma_1' < \cdots < \sigma_{s'}',$$

with corresponding eigen-projections $Q_0', Q_1', \ldots, Q_{s'}'$, and hence B has spectral decomposition

$$B = \sum_{j=1}^{s'} \sigma_j' Q_j'. \tag{25}$$

Now we form a matrix $Q_0' A Q_0'$, which is diagonalizable. Next, we have its spectral decomposition in the range of Q_0',

$$Q_0' A Q_0' = \sum_{k=1}^{s''} \alpha_k \mathcal{Q}_k, \qquad \sum_{k=1}^{s''} \mathcal{Q}_k = Q_0'. \tag{26}$$

The distinct eigenvalues α_k are those appearing in the third term of (13).

Step 4. Related to (25) we define a matrix

$$S' = \sum_{j=1}^{s'} \frac{1}{\sigma_j'} Q_j', \tag{27}$$

which is the value at zero of the reduced resolvent of B with respect to the eigenvalue zero. Then for each $\mathcal{Q}_k$ associated with a distinct α_k in (26) we form a matrix $\mathcal{Q}_k(AS'A - L)\mathcal{Q}_k$ and find its spectral decomposition in the range of $\mathcal{Q}_k$,

$$\mathcal{Q}_k(AS'A - L)\mathcal{Q}_k = \sum_{k'=1}^{s_k} \beta_{kk'}\mathcal{Q}_{kk'}, \qquad \sum_{k'=1}^{s_k} \mathcal{Q}_{kk'} = \mathcal{Q}_k. \tag{28}$$

One can show that all $\beta_{kk'}$ are positive. Now we collect all distinct pairs $(\alpha_k, \beta_{kk'})$, $1 \leq k \leq s''$, $1 \leq k' \leq s_k$. The total number of those pairs is m' in the third term of (13), with $(\alpha_k, \beta_{kk'})$ being (α_j, β_j) and $\mathcal{Q}_{kk'}$ being Q_j therein.

Example 3. We linearize the polyatomic gas flow model in Example 1 around a constant equilibrium state $\bar{w} = (\bar{\rho}, 0, \bar{\rho}\bar{e}, \bar{\rho}\bar{e}_2)^t$ to have

$$\tilde{w}_t + A\tilde{w}_x = B\tilde{w}_{xx} + L\tilde{w} \tag{9}$$

where $\tilde{w} = w - \bar{w}$,

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \\ \rho e_2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -v^2 p_v - v p_{e_1} e_1 & 0 & v p_{e_1} & -v p_{e_1} \\ 0 & e + vp & 0 & 0 \\ 0 & e_2 & 0 & 0 \end{pmatrix}_{\bar{w}}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu v & 0 & 0 \\ -\kappa v(v T_{1v} + T_{1e_1} e_1) - \nu e_2 & 0 & \kappa v T_{1e_1} & -\kappa v T_{1e_1} + \nu \\ -\nu e_2 & 0 & 0 & \nu \end{pmatrix}_{\bar{w}}$$

$$L = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \left[\nabla_w \left(\rho \frac{e_2^* - e_2}{\tau} \right) \right]_{\bar{w}}$$

We can verify those matrices satisfy our structural conditions. We further change variables to make entries of L zero except the one in (4,4) location. The change of variables only affects the formulation of projections but not heat kernels or δ-functions.

- The reduced matrix $A^{(r)}$ is 3×3 and have 3 eigenvalues,

$$\lambda_-^{(r)} = -\bar{c} \qquad \lambda_0^{(r)} = 0 \qquad \lambda_+^{(r)} = \bar{c}$$

  where $\bar{c}$ is the equilibrium sound speed taken at $\bar{w}$. These are the direction of the three heat kernels.

- We can formulate the projections associated with the heat kernels as well.
- The diffusion coefficients for the heat kernels are

$$\mu_0 = -p_v(v, T_1)v^2 T_{1e_1}[\kappa v + \nu e_2'(T_1)]/(a\bar{c}^2)\Big|_{\bar{w}} > 0$$

$$\mu_\pm = v^2 p_{e_1}^2 T_1[\tau \bar{c}^2 e_2'(T_1) + \kappa v + \nu e_2'(T_1)]/(2a^2\bar{c}^2) + \frac{1}{2}\mu v\Big|_{\bar{w}} > 0$$

with

$$a = e_2'(T_1)T_{1e_1} + 1\Big|_{\bar{w}}$$

- The singular part is an exponentially decaying δ-function along the particle path,

$$e^{p_v(v,T_1)vt/\mu}\delta(x)Q_0'.$$

- One can set some dissipation parameters as zero while Kawashima-Shizuta condition still holds. In those cases we have more δ-functions.
  - "$\kappa > 0, \mu > 0, \nu = 0$" or "$\nu > 0, \mu > 0, \kappa = 0$": two δ-functions along the particle path.
  - $\kappa > 0$, $\nu > 0$ and $\mu = 0$: two δ-functions along opposite directions.
  - $\kappa > 0$ and $\mu = \nu = 0$: three δ-functions, one along the particle path and two along opposite directions.
  - $\nu > 0$ and $\kappa = \mu = 0$: three δ-functions, one along the particle path and two along opposite directions. In this case, the opposite directions are frozen acoustic directions.
  - $\nu = \kappa = \mu = 0$: inviscid case; hyperbolic system; Kawashima-Shizuta condition fails; four δ-functions, two along the frozen acoustic directions and two along the particle path with one does not decay in time.

Thank You