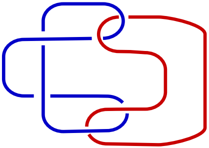
How many crossing changes does it take to get a homotopy trivial link?

Christopher William Davis University of Wisconsin - Eau Claire
Joint with: **Anthony Bosman** Andrews University **Taylor Martin** Sam Houston State University **Carolyn Otto** The University of Wisconsin - Eau Claire, and **Katherine Vance** Simpson College.

Links in dimensions 3 (mostly) and 4 (a little bit),
ICERM.

May 16, 2025

The Gordian distance, and a tractable variant



Question

Given a link (or knot) L, how many crossing changes are needed to transform L to the unlink?

This is the **Gordian distance** from L to U, or the **unlinking number** of L.

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Alexander the Great cuts the Gordian Knot by Jean-Simon Berthélemy (1743–1811)

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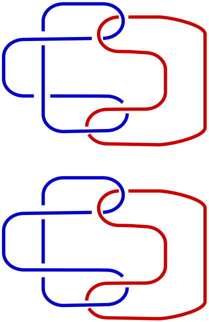

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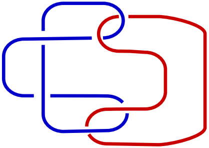

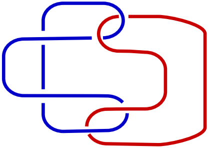

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- Lots of tools: signatures [Mur65], Blanchfield form [BF15], Gauge theory [CL86, KroMro93], Heegaard-Floer homology [Ras03], etc.
- [KT21] Computing the unlinking number is NP hard.

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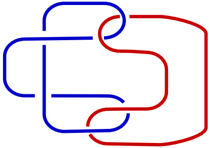

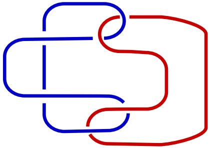

$u(L) = 1,\ n_h(L) = 0$

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Question (A simpler question)

Given a link L, how many crossing changes are needed to transform L to a homotopy trivial link (a link that can be undone via self-crossing changes)?

Link homotopy.

$n_h(L) =$ "homotopy trivializing number" of a link, L.

Question

Given a link L, can you determine $n_h(L)$?

- Why is it still interesting?
- Why is it easier?
  - (Milnor [Mi54])
  - (Goldsmith [Go73] Habegger-Lin [HL90])
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  - (Kotorii-Mizisawa [KotMiz23]) Link homotopy of $n \leq 5$-component links is classified. To go further solve a system of Diophantine equations

Intuition, an answer

Linking number gives a lower bound,

$$n_h(L) \geq \Lambda(L) := \sum_{i<j} |\operatorname{lk}(L_i, L_j)|$$

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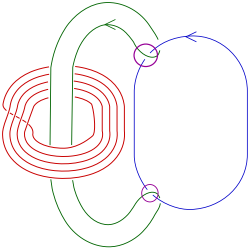

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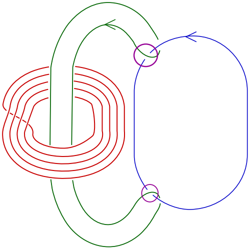

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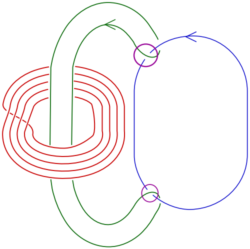

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Problem Compute C_n.

Controlling C_n, today's main theorem

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- $10 \leq C_5 \leq 12$. To do:
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How to prove these results?

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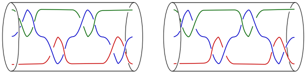
  - Use extremal graph theory to do some book-keeping.

Links, string links, link homotopy

Why string links?



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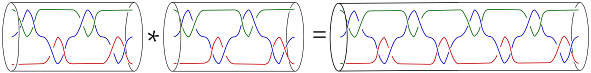

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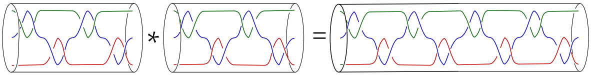

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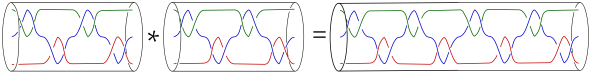

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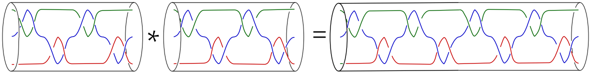

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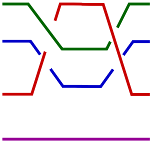

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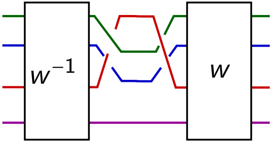

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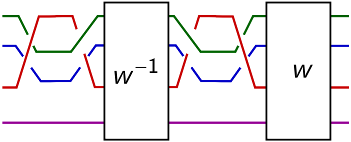

The reduced free group and the Habegger-Lin classification

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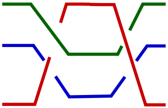
$\phi(x_i) = x_{i,n} =$ 		

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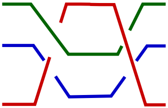
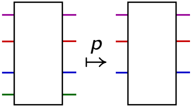
$\phi(x_i) = x_{i,n} =$ 		

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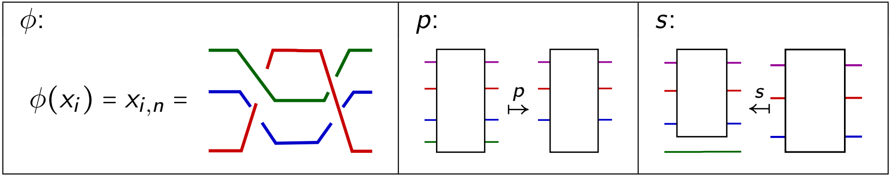

Strategy: $L = \phi(x) * s(p(L))$
Bound $n_h(\phi(x)) =$

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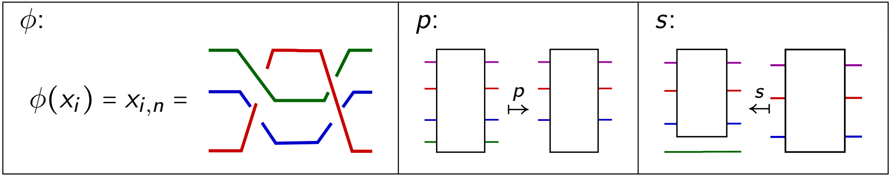

Strategy: $L = \phi(x) * s(p(L))$
Bound $n_h(\phi(x))$ = how many x_i's must be deleted to transform $x \in RF$ into 1.
$n_h(s(p(L)))$. Induction.

Homotopy trivializing in $RF(n-1)$.

Theorem (BDMOV)

Any $x \in RF(n-1)$ can be realized as

$$x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1} [x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$$

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Theorem (BDMOV)

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Consequence: If L is in the image of ϕ (aka $L_1 \cup \ldots L_{n-1} = U$) then $n_h(L) \leq \Lambda(L) + 2(n-2)$.

Consequence: Any n-component Bruunian link is only $2(n-2)$ crossing changes from homotopy trivial.

Homotopy trivializing in $RF(n-1)$.

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$$L = \underbrace{\phi(x)}_{n_h \leq \Lambda + 2(n-2)} * (\underbrace{L_1 \cup \ldots L_{n-1}}_{n-1 \text{ components}} \cup U) =$$

Homotopy trivializing in $RF(n-1)$.

Theorem (BDMOV)

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$n_h(L) - \Lambda(L) \leq (n-1)(n-2)$

Conclusion:

Homotopy trivializing in $RF(n-1)$.

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$$L = \underbrace{\phi(x)}_{n_h \leq \Lambda + 2(n-2)} * (\underbrace{L_1 \cup \ldots L_{n-1}}_{n-1 \text{ components}} \cup U) =$$

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$$n_h(L) - \Lambda(L) \leq (n-1)(n-2)$$

Conclusion: $C_n \leq (n-1)(n-2)$.

Homotopy trivializing in $RF(n-1)$. (Proof)

Theorem (BDMOV)

Any $x \in RF(n-1)$ *is* $x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$

Properties of $RF(n-1)$**:**

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Homotopy trivializing in $RF(n-1)$. (Proof)

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Strategy: Push the realization into the lower central series quotients.

Homotopy trivializing in $RF(n-1)$. (Proof)

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Homotopy trivializing in $RF(n-1)$. (Proof)

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Homotopy trivializing in $RF(n-1)$. (Proof)

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$$\equiv x_{ab} \cdot \prod_{i=1}^{n-2}\left[x_i, \prod_{j=i+1}^{n-1} x_j^{a_{ij}}\right] =$$

Homotopy trivializing in $RF(n-1)$. (Proof)

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$$\equiv x_{ab} \cdot \prod_{i=1}^{n-2} \left[x_i, \prod_{j=i+1}^{n-1} x_j^{a_{ij}} \right] = x_{ab} \cdot \prod_{i=1}^{n-2} [x_i, w_i]$$

Realization theorem in $RF(n-1)$ (proof ctd.)

Goal: $x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$

- ${}^{RF_p}/_{RF_{p+1}}$ is an Abelain group generated by $x_{i_1,\ldots,i_p} = [x_{i_1}, [x_{i_2}, \ldots [x_{i_{p-1}}, x_{i_p}]]]$ with $1 \le i_1 < (i_2, \ldots, i_{p-1}) < i_p$
- If $a \in RF_i$, $b \in RF_j$ and $c \in RF_k$ then $[a,b][a,c] = [a, bc]$ mod RF_{i+j+k}.
  - So

In RF/RF_4: $x = x_{ab} \prod_{i=1}^{n-2} [x_i, w_i] y$. $y \in RF_3/RF_4$.

$y \equiv$

RF/RF_p: Same argument.

Realization theorem in $RF(n-1)$ (proof ctd.)

Goal: $x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$

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$$y \equiv \prod_I x_I^{a_I} \equiv \qquad \equiv$$

RF/RF_p: Same argument.

Realization theorem in $RF(n-1)$ (proof ctd.)

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Realization theorem in $RF(n-1)$ (proof ctd.)

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$$y \equiv \prod_I x_I^{a_I} \equiv \prod_{i=1}^{n-2}\prod_J [x_i, x_J^{a_{iJ}}] \equiv \qquad \equiv$$

RF/RF_p: Same argument.

Realization theorem in $RF(n-1)$ (proof ctd.)

Goal: $x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1]\ldots[x_{n-2}, w_{n-2}]$

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In RF/RF_4: $\quad x = x_{ab} \prod_{i=1}^{n-2} [x_i, w_i] y$. $\; y \in RF_3/RF_4$.

$$y \equiv \prod_I x_I^{a_I} \equiv \prod_{i=1}^{n-2} \prod_J [x_i, x_J^{a_{iJ}}] \equiv \prod_{i=1}^{n-2} [x_i, \prod_J x_J^{a_{iJ}}] \equiv$$

RF/RF_p: Same argument.

Realization theorem in $RF(n-1)$ (proof ctd.)

Goal: $x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$

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$$x \equiv$$

RF/RF_p: Same argument.

Realization theorem in $RF(n-1)$ (proof ctd.)

Goal: $x = x_{n-1}^{a_{n-1}} \cdot \ldots x_1^{a_1}[x_1, w_1] \ldots [x_{n-2}, w_{n-2}]$

- ${}^{RF_p}/_{RF_{p+1}}$ is an Abelain group generated by $x_{i_1,\ldots,i_p} = [x_{i_1}, [x_{i_2}, \ldots [x_{i_{p-1}}, x_{i_p}]]]$ with $1 \le i_1 < (i_2, \ldots, i_{p-1}) < i_p$
- If $a \in RF_i$, $b \in RF_j$ and $c \in RF_k$ then $[a,b][a,c] = [a, bc]$ mod RF_{i+j+k}.
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In RF/RF_4: $x = x_{ab} \prod_{i=1}^{n-2} [x_i, w_i] y$. $y \in RF_3/RF_4$.

$$\begin{aligned} y &\equiv \prod_I x_I^{a_I} \equiv \prod_{i=1}^{n-2} \prod_J [x_i, x_J^{a_{iJ}}] \equiv \prod_{i=1}^{n-2} [x_i, \prod_J x_J^{a_{iJ}}] \equiv \prod_{i=1}^{n-2} [x_i, D_i] \\ x &\equiv x_{ab} \prod_{i=1}^{n-2} [x_i, w_i] \prod_{i=1}^{n-2} [x_i, D_i] \equiv \qquad \equiv \end{aligned}$$

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Determining $n_h(L)$ for all 4-component links.

$n = 3$:

(DOP22) $n_h(L) = \begin{cases} \Lambda(L) & \text{if } \Lambda(L) \neq 0 \\ 2 & \text{if } \Lambda(L) = 0 \text{ and } \mu_{123}(L) \neq 0 \\ 0 & \text{otherwise} \end{cases}$

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$n = 4$:

$\binom{4}{2} = 6$ linking numbers, $\binom{4}{3} = 4$ pairwise linking numbers, 2 independent length 4 Milnor invariants.

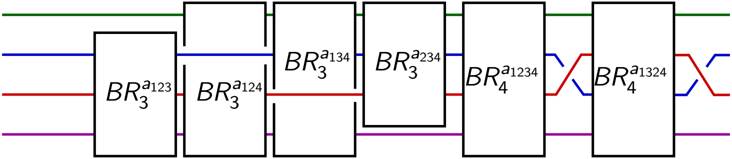
Any formula we get will be complicated looking

Computing the homotopy trivializing number of every 4-component link

[Mi54, KotMiz23] Any $\alpha \in \mathcal{H}_4$ has the form

$$\alpha = x_{12}^{a_{12}} x_{13}^{a_{13}} x_{14}^{a_{14}} x_{23}^{a_{23}} x_{24}^{a_{24}} x_{34}^{a_{34}} x_{123}^{a_{123}} x_{124}^{a_{124}} x_{134}^{a_{134}} x_{234}^{a_{234}} x_{1234}^{a_{1234}} x_{1324}^{a_{1324}}$$

$x_{ijk} = [x_{ik}, x_{ij}]$ $x_{ijk\ell} = [x_{i\ell}, [x_{ik}, x_{ij}]]$



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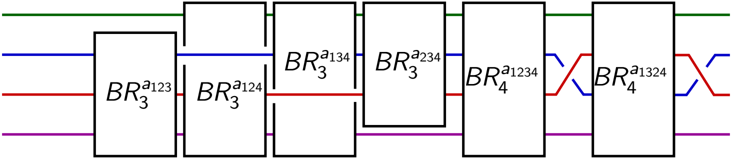

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Idea: $n_h(L) = 1$ iff $L = \alpha x_{ij} \alpha^{-1}$ for some α. Compute $\alpha x_{ij} \alpha^{-1}$ for each x_{ij}:

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Conjugate of x_{12}: $x_{12} x_{123}^{a} x_{124}^{b} x_{1234}^{c} x_{1324}^{-ab}$, $a, b, c \in \mathbb{Z}$.
Conjugate of x_{13} : $x_{13} x_{123}^{a} x_{134}^{b} x_{1234}^{ab} x_{1324}^{c}$, $a, b, c \in \mathbb{Z}$.
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Conjugate of x_{23}: $x_{23} x_{123}^{a} x_{234}^{b} x_{1234}^{c} x_{1324}^{d}$ $c + d = ab$.
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Consequence:

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Consequence: L can be undone in a single (positive) L_1, L_2 crossing change iff $\mathrm{lk}(L_1, L_2) = 1$, $\mu_{134} = \mu_{234} = 0$ and $\mu_{1324} = -\mu_{123}\mu_{124}$.

All 4-component links

L can be reduced to homotopy trivial in k crossing changes iff L is a product of k conjugates of $x_{ij}^{\pm 1}$'s. Compute products of the terms in the pervious page / their inverses:

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Let $\Lambda(L) = 0$:

- L can be undone in two L_1, L_2 crossing changes iff
- L can be undone in two L_1, L_2 and two L_1, L_3 crossing changes iff
- L can be undone in two L_1, L_2 and two L_3, L_4crossing changes iff
- $C_4 \leq 6$: "Everything else" takes six crossing changes.

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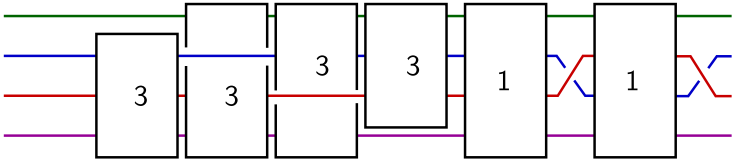
Constructing link with large homotopy trivializing number, some graphical book-keeping

Let J be an n-component link whose every 4-component sublink has $n_h(L) = 6$. **Goal:** Determine $n_h(J)$.



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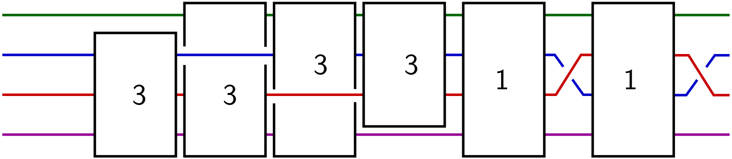

Build an edge-weighted graph.

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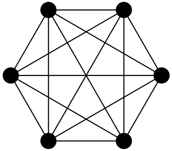
1 vertex per component. $\text{wt}(e_i, e_j) = \frac{1}{2} \cdot \#(\ i, j$ crossing changes). Every 4-vertex subgraph of Γ has wt ≥ 3.



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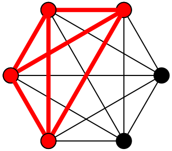

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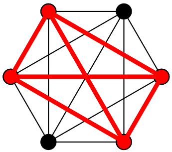

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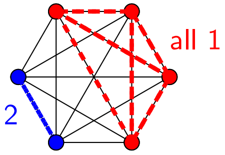

- (BDMOV) The minimal weight amongst all such n-vertex graphs is $\left\lceil \frac{1}{3} n(n-2) \right\rceil$.
- $C_n \geq n_h(J) \geq 2 \left\lceil \frac{1}{3} n(n-2) \right\rceil$

Thanks for listening!

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