Efficient computations of $\mathfrak{sl}_3$-s-invariants for knots and links

Dirk Schütz

ICERM, Links in Dimensions 3 and 4

14/05/25

Durham University

s-invariants

The s-invariant of a knot was introduced by Rasmussen in analogy to the Ozsvath–Szabo τ-invariant, and used to give a combinatorial proof of the Milnor conjecture on the 4-genus of torus knots.

Other applications include

- Piccirillo's proof that the Conway knot is not slice involves the calculation of an s-invariant of a knot with over 40 crossings.
- Freedman–Gompf–Morrison–Walker proposed the s-invariant as a way to detect exotic 4-spheres.
- Dunfield–Gong have an ongoing project to classify slice knots with up to 19 crossings.

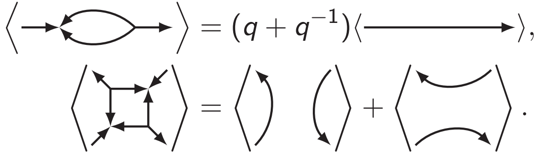
$\mathfrak{sl}_3$-link homology

In between defining Khovanov homology and $\mathfrak{sl}_n$-Khovanov–Rozansky homology, Khovanov defined $\mathfrak{sl}_3$-link homology as a categorification of the quantum $\mathfrak{sl}_3$-link invariant of Reshetikhin–Turaev, following the approach of Kuperberg.

A *closed web* Γ is a finite trivalent oriented graph in $\mathbb{R}^2$, possibly with vertex-less loops, such that at each vertex all edges are either incoming, or all are outgoing.

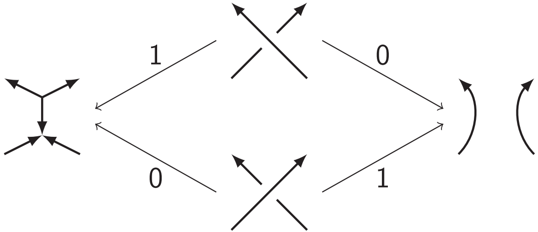
The *Kuperberg bracket* $\langle\Gamma\rangle$ of a web Γ is the Laurent polynomial in one variable q determined by the relations

$$\langle \Gamma \sqcup \bigcirc \rangle = (q^2 + 1 + q^{-2})\langle \Gamma \rangle,$$



$\mathfrak{sl}_3$-link homology

An oriented link diagram D with n crossings gives rise to 2^n closed webs by resolving each crossing in two different ways as in

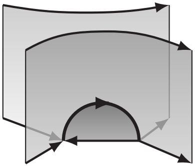


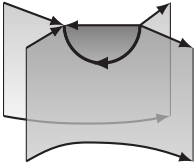
The quantum $\mathfrak{sl}_3$-link polynomial is then given by an appropriate sum of the Kuperberg brackets of the webs arising from the diagram.

Khovanov used foams and a rank 3 Frobenius system $\mathcal{F}_{\mathrm{Kh}}$ to construct graded free abelian groups $\mathcal{F}_{\mathrm{Kh}}(\Gamma)$ for a web Γ such that the graded rank of this abelian group is $\langle \Gamma \rangle$.

$\mathfrak{sl}_3$-link homology

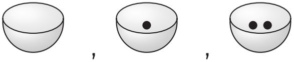
To map between $\mathcal{F}_{\mathrm{Kh}}(\Gamma_0)$ and $\mathcal{F}_{\mathrm{Kh}}(\Gamma_1)$, where Γ_0, Γ_1 are adjacent resolutions from the link diagram, we need the unzip and zip foams.





Roughly, $\mathcal{F}_{\mathrm{Kh}}(\Gamma_0)$ is the abelian group generated by dotted foams from $\emptyset$ to Γ_0 subject to relations coming from evaluations of closed foams.

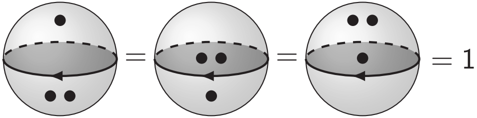
For example,



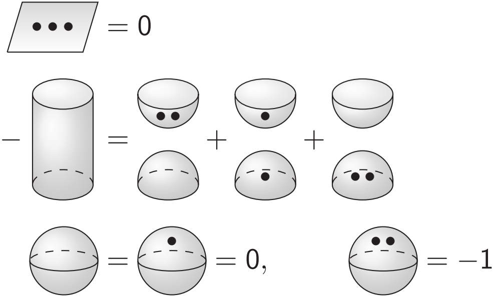
generate $\mathcal{F}_{\mathrm{Kh}}(O) \cong \mathbb{Z}^3$.

$\mathfrak{sl}_3$-link homology

Closed foam evaluation is determined by the *theta-foam* evaluation

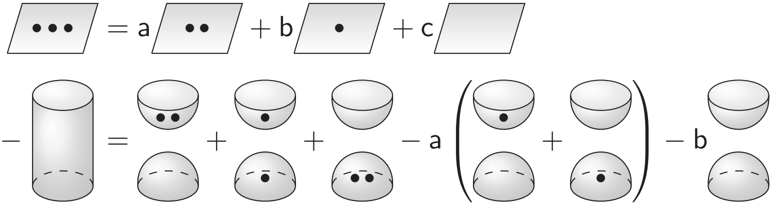


and the following relations



Universal $\mathfrak{sl}_3$-link homology

Mackaay–Vaz ('07) generalized this to work over $\mathbb{Z}[a, b, c]$ using relations



Passing to field coefficients $\mathbb{F}$ and $a, b, c \in \mathbb{F}$ one gets a link homology $H_{\mathfrak{sl}_3}(L; \mathcal{F}_{f(X)})$, where $f(X) = X^3 - aX^2 - bX - c$.

Mackaay–Vaz showed that if f has 3 different roots over an algebraically closed field, then $H_{\mathfrak{sl}_3}$ has total dimension $3^{|L|}$, where $|L|$ is the number of components.

s-invariants

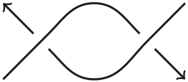
This can be used to generalize Rasmussen's s-invariant to $\mathfrak{sl}_3$-link homology, but at the time a different approach using Khovanov–Rozansky $\mathfrak{sl}_n$-link homology was pursued by Lobb ('09) and Wu ('09).

But the case $n = 3$ is much more amenable for computations: Lewark ('13) demonstrated that $\mathfrak{sl}_3$-link homology can be computed using a Bar-Natan-style fast algorithm which just about works for $T(6,7)$. Furthermore, he was able to get enough information for s-invariants to show that the $\mathfrak{sl}_3$-version behaves quite different from the Rasmussen invariant. His computations rely on a spectral sequence collapsing in a predictable way, which can lead to ambiguities for some 14-crossing knots.

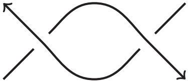
Unlike in Khovanov homology, the polynomials $X^3 - 1$ and $X^3 - X$ give rise to different spectral sequences.

Bipartite Links

A link is called *bipartite*, if it admits a diagram built out of the following pieces



and



Krasner ('09) showed that $\mathfrak{sl}_n$-link homology for a bipartite link simplifies significantly. This was used by Lewark–Lobb ('16) to observe all sorts of unusual behaviour of spectral sequences associated to $\mathfrak{sl}_n$-homology.

In particular, they showed that the corresponding slice obstructions depend on the polynomial, and in order to get a homomorphism from the concordance group to the integers, you need the polynomial $X^n - 1$ (over $\mathbb{C}$).

$X^3 - 1$ vs $X^3 - X$ for 12^n_{340}

$q \backslash h$	−8	−7	−6	−5	−4	−3	−2	−1	0
24	1								
22	1	1							
20		1	1						
18		1	1	1					
16		1	1	2	1				
14			1	2	1				
12				1	2	2			
10					2	2	1		
8					1	1	1	1	
6						1	2		
4							1	1	1
2								1	2
0									1

$\mathfrak{sl}_3$-s-invariants in any characteristic

For $\mathbb{F}$ a field of characteristic $p \neq 3$ the polynomial $X^3 - 1$ gives rise to a homomorphism $s^p_{\mathfrak{sl}_3}$ from the smooth knot concordance group to the integers.

For $p = 3$ the polynomial $X^3 - X$ works (S.'25).

If we normalize the homomorphism so that

$$s^p_{\mathfrak{sl}_3}(T(2,3)) = 2,$$

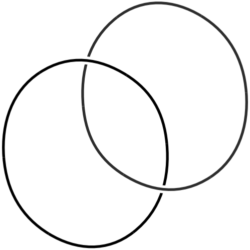
then $s^p_{\mathfrak{sl}_3}$ surjects on $\mathbb{Z}$ for $p \neq 3$, while $s^3_{\mathfrak{sl}_3}$ has image $2\mathbb{Z}$.

Characteristic 3 seems to produce a 'worse' slice obstruction than the other characteristics, although this can be remedied by considering the polynomial $X^3 - X^2 - 1$ (which does not produce an obvious homomorphism).

s-invariants for links

Beliakova and Wehrli extended the definition of the (Rasmussen) s-invariant to links using a q-grading of specific generators. This can be generalized to the $\mathfrak{sl}_3$ case.

q \ h	-2	-1	0
10	1		
8	2		
6	2		
4	1		1
2			1
0			1



Sometimes (as with the Hopf link) it can be easily read off from the E_∞ page of the Lee spectral sequence.

Mackaay–Vaz generators

Let $\mathbb{F}$ be a field which contains an element $\rho \neq 1$ with $\rho^2 + \rho + 1 = 0$. Then with $f(X) = X^3 - 1$ we have

$$\mathbb{F}[X]/(f(X)) \cong \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

as an $\mathbb{F}$-algebra. Moreover, $1 \in \mathbb{F}[X]/f(X)$ satisfies

$$1 = I_1 + I_\rho + I_{\rho^2},$$

where

$$\begin{aligned} I_1 &= \frac{1}{3}(X-\rho)(X-\rho^2) = \frac{1}{3}(X^2+X+1) \\ I_\rho &= \frac{\rho}{3}(X-1)(X-\rho^2) \\ I_{\rho^2} &= \frac{\rho^2}{3}(X-1)(X-\rho) \end{aligned}$$

are mutually orthogonal idempotents.

Mackaay–Vaz generators

Any oriented link diagram has a unique smoothing which consists entirely of circles, and it lies in homological degree 0. The corresponding summand in the $\mathfrak{sl}_3$-complex is isomorphic to $\mathbb{F}[X]/(f(X))^{\otimes k}$, where k is the number of circles in the smoothing.

The *Mackaay–Vaz* generators

$$I_\alpha(L) = I_\alpha \otimes I_\alpha \otimes \cdots \otimes I_\alpha$$

for $\alpha \in \{1, \rho, \rho^2\}$ are cocycles which generate a 3-dimensional subspace of $H^0_{\mathfrak{sl}_3}(L; \mathcal{F}_{X^3-1})$. Since this homology group is filtered, each element has a q-grading, and

$$|I_1(L)|_q = |I_\rho(L)|_q = |I_{\rho^2}(L)|_q.$$

Furthermore, if L is a knot, this value is equal to $2s^p_{\mathfrak{sl}_3}(L) + 2$.

$\mathfrak{sl}_3$-s-invariants for links

Given a link L, we define

$$s^p_{\mathfrak{sl}_3}(L) = -\left(\frac{|I_1(\overline{L})|_q - 2}{2}\right),$$

where $\overline{L}$ is the mirror of L.

The definition is so that it agrees with the Beliakova–Wehrli s-invariant for unlinks and Hopf links. The basic properties of the Beliakova–Wehrli invariant carry over, in particular

Lemma

Let $L \subset S^3$ be bounded by an oriented smooth connected surface $P \subset B^4$ of genus 0 (L is slice in the weak sense*), then*

$$|s^p_{\mathfrak{sl}_3}(L)| \leq |L| - 1,$$

where $|L|$ is the number of components of L.

Borromean Rings

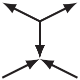
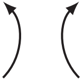
For the Borromean Rings the E_∞ page is given by

$q \backslash h$	-3	-2	-1	0	1	2	3
8							
6							
4				2			
2				7			
0				9			
-2				7			
-4				2			
-6							
-8							

So, we have $s^p_{\mathfrak{sl}_3}(B) \in \{-1, 0, 1\}$. None of these values obstructs weak sliceness, but one can imagine situations where one of the values would, but the others do not.

Bar-Natan scanning

Let us recall how the Bar-Natan scanning algorithm works. For a diagram D with n crossings one chooses an order of them, then starts with the tangle complex

$C_1 = u^{-1}q^3$  $\xrightarrow{U} u^0 q^2$ 

After k crossings, we have a complex C_k. To obtain C_{k+1}, we

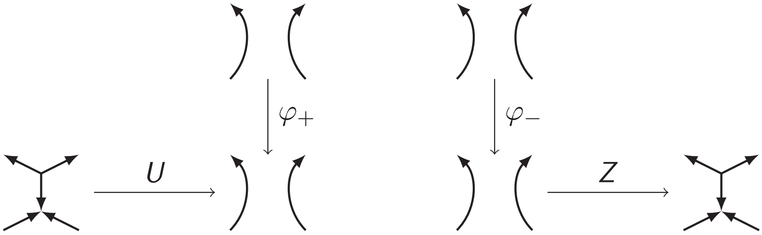
- Tensor C_k with the next crossing complex.
- Deloop any circles, digons and squares.
- Perform Gaussian eliminations to obtain C_{k+1}.

The complex C_n is chain homotopy equivalent to $C_{\mathfrak{sl}_3}(D; \mathcal{F}_{X^3-1})$, but usually much smaller. Obtaining the spectral sequence from C_n is straightforward, but how do we recover the Mackaay–Vaz generators?

Recovering the Mackaay–Vaz generators

We can think of the cocycle $I_1(L) \in C_{\mathfrak{sl}_3}(D; \mathcal{F}_{X^3-1})$ as the image of a cochain map $\varphi\colon C \to C_{\mathfrak{sl}_3}(D; \mathcal{F}_{X^3-1})$, where C is a cochain complex concentrated in homological degree 0, with $C^0 = \mathbb{F}$.

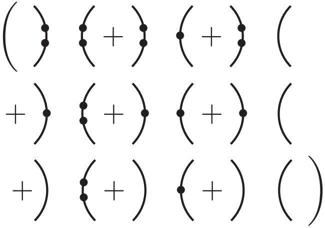
If T is a tangle consisting of a single crossing, we can look at a cochain map $\varphi\colon C_T \to C_{\mathfrak{sl}_3}(T; \mathcal{F}_{X^3-1})$, with C_T concentrated in homological degree 0. Rather than using $C_T^0 = \mathbb{F}$, we let C_T^0 be generated by the orientation preserving smoothing of T.



Recovering the Mackaay–Vaz generators

Both maps φ_+ and φ_- are represented by the same linear combination of dotted foams, namely

$$\varphi_\pm = \frac{1}{9}$$



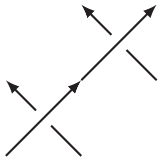
This is motivated by

$$I_1 \otimes I_1 = \frac{1}{9}\,(X^2 + X + 1) \otimes (X^2 + X + 1).$$

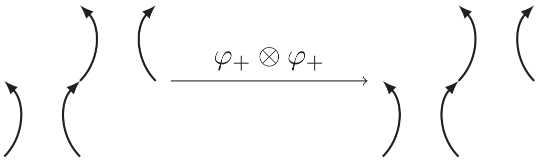
One has to check that $Z \circ \varphi_- = 0$, but this holds, at least if one uses the right relations in the definition of the tangle category.

Recovering the Mackaay–Vaz generators

When tensoring two tangle complexes, we can also tensor the cochain maps. For example, the tangle



leads to



It appears that this has 81 summands, but because of three dots equalling no dots and $I_1 \cdot I_1 = I_1$, this simplifies to 27 summands.

Recovering the Mackaay–Vaz generators

Delooping and Gaussian elimination lead to chain homotopic cochain complexes with known chain homotopy equivalences, so we can alter the cochain map $\varphi\colon C_{T_k} \to C_k$ as we go through the scanning complex (with T_k the tangle after k crossings).

We can also deloop in C_{T_k} if any arcs turn into circles.

After going through all crossings, we have a cochain map $\varphi\colon C_D \to C_n$ with C_D isomorphic to $C_{\mathfrak{sl}_3}(U; \mathcal{F}_{X^3-1})$, where U is an unlink diagram without crossings. The image of φ is a cocycle representing $I_1(L)$.

The spectral sequence is worked out via Gaussian elimination, so we can again keep track of φ in these steps. At the final step we can read off the q-degree of $I_1(L)$.

A recent preprint of Sano has a similar approach for the Bar-Natan complex.

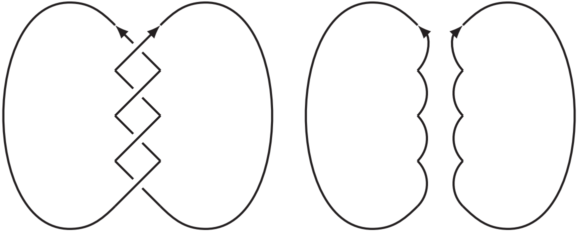
Notes on Complexity

- We can deloop C_{T_k} as we go along, if any loops appear. Rather than tripling the number of generators with every delooping, we only need to keep one of the generators. The final complex will only have one generator then.
- The element $I_1(L)$ is well defined over the prime field, so we do not have to make computations over fields such as $\mathbb{C}$ or $\mathbb{Q}(\rho)$, we can just work over the prime field.
- Since the image of each $\varphi_\pm$ consists of nine summands, the image of φ after k steps may have quite a few summands. In practice, any $\mathfrak{sl}_3$-link homology calculations are likely to fail if there are prolonged periods of 10 or more endpoints on the tangles.
- If we work in characteristic 3, we define $I_1(L)$ using the idempotent $-X^2 - X$, so the image of $\varphi_\pm$ only consists of four summands.

The Beliakova–Wehrli invariant

The original s-invariant for links coming from Khovanov homology can be calculated in a similar way.

Given an oriented link diagram, there is a unique smoothing S which preserves the orientation. This smoothing sits in homological degree 0. There is an equivalence relation on the circles in S with two equivalence classes, such that if any two circles are connected by a surgery arc coming from the diagram, these two circles are not equivalent.



Lee generators

The Lee complex uses the Frobenius algebra $A = \mathbb{F}[X]/(X^2 - 1)$, and the summand in the Lee complex corresponding to the smoothing S is $A^{\otimes k}$, where we assume that S has k circles (denoted $S_1, \ldots, S_k$). Fixing an equivalence class $\mathcal{O}$, the Lee generator $I_{\mathcal{O}}(L)$ is given by

$$I_{\mathcal{O}}(L) = w_1 \otimes w_2 \otimes \cdots \otimes w_k,$$

with

$$w_i = \begin{cases} \frac{1}{2}(1 - X) & S_i \in \mathcal{O} \\ \frac{1}{2}(1 + X) & S_i \notin \mathcal{O} \end{cases}$$

The s-invariant of the link L is then given by

$$s(L) = |I_{\mathcal{O}}(L)|_q + 1,$$

and the algorithm to compute it works as before, see also Sano ('25).

Using the Bar-Natan complex

The Bar-Natan complex is similar, but uses the Frobenius algebra $\mathbb{F}[X]/(X^2 - X)$. It has the advantage that it works over any characteristic, and even over $\mathbb{Z}$.

The only difference is that we use

$$w_i = \begin{cases} 1 - X & S_i \in \mathcal{O} \\ X & S_i \notin \mathcal{O} \end{cases}$$

Another advantage of this is that the image of $\varphi_\pm$ only has two summands.

This algorithm has been implemented in `knotjob`. When applied to knots it is similar in speed to the usual method.

s-invariants via the Bar-Natan complex

Exact computations for s-invariants of links are somewhat sparse.

- For quasi-alternating links it agrees with the signature.
- Whenever the 0-th homology group only has two non-trivial q-degrees, we can read it off from the BLT-spectral sequence.
- Ren ('24) computed s-invariants for torus links with arbitrary orientations.

In all of these examples, the s-invariant is $1+$ the minimal non-trivial q-degree.

We have examples of 11 crossing links where this is not the case.

The $(5,5)$-torus link

$q \backslash h$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
13													5
11													9
9													5
7													1
5									4				
3									5				
1									1				
-1													
-3													
-5													
-7	1												
-9	1												

The E_∞-page of the $(5,5)$-torus link with one orientation changed.