Linking of 2-spheres in 4-space

ICERM May 2025

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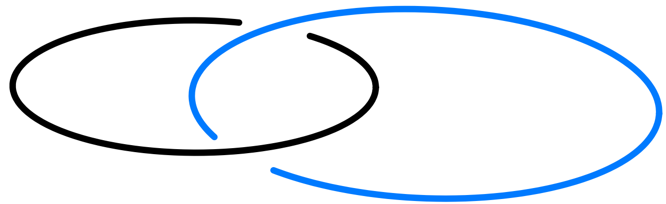
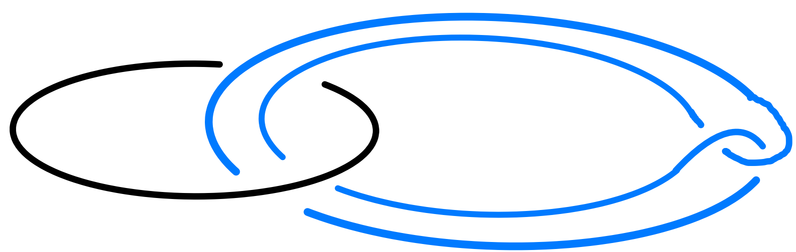
— Introduced for classical links by Milnor (1954) to study “Linking modulo knotting”.

EXAMPLES:

A Hopf link is <u>not</u> link homotopically trivial.



A Whitehead link <u>is</u> link homotopically trivial.



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— Habegger-Lin (1990): A link homotopy classification of classical links via \mu-invariants of “string links” (based links) and a “partial conjugation” equivalence relation.

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So from now on “link map” means “spherical link map”…

2-component link homotopy of 2-spheres in the 4-sphere

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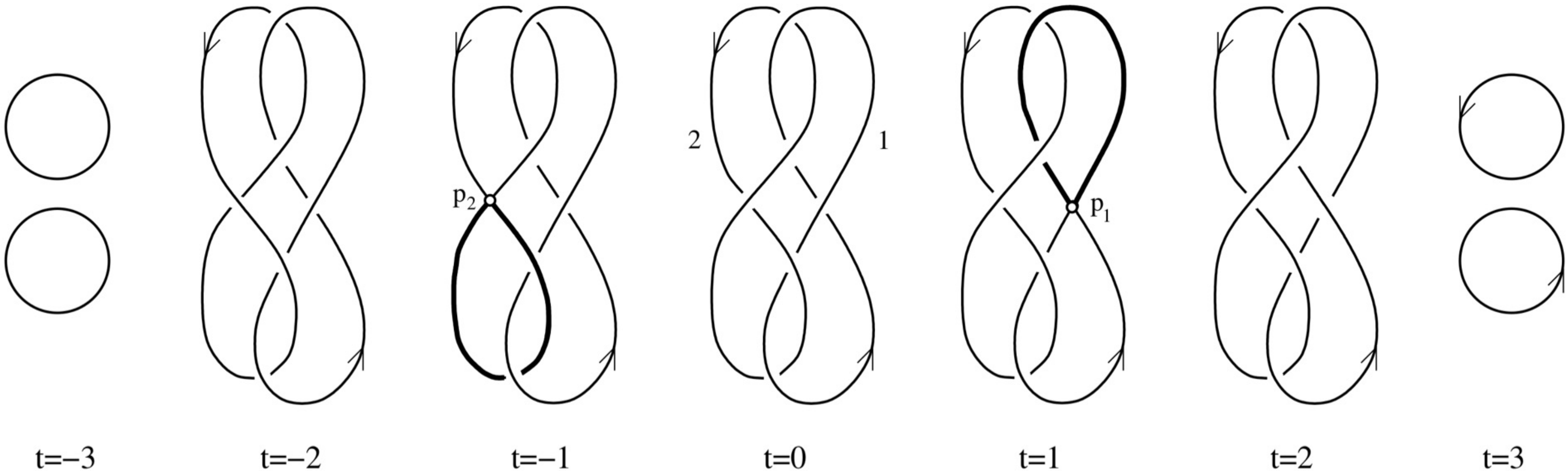
In fact, codimension-2 embedded (spherical) link maps with any number of components are link homotopically trivial above the classical dimension.

2-component link homotopy of 2-spheres in the 4-sphere

Bartels-Teichner (1999): no 4-dimensional Hopf link!

Fenn-Rolfsen (1986): exists a 2-component immersed link map of 2-spheres in the 4-sphere which is *not* link homotopically trivial.

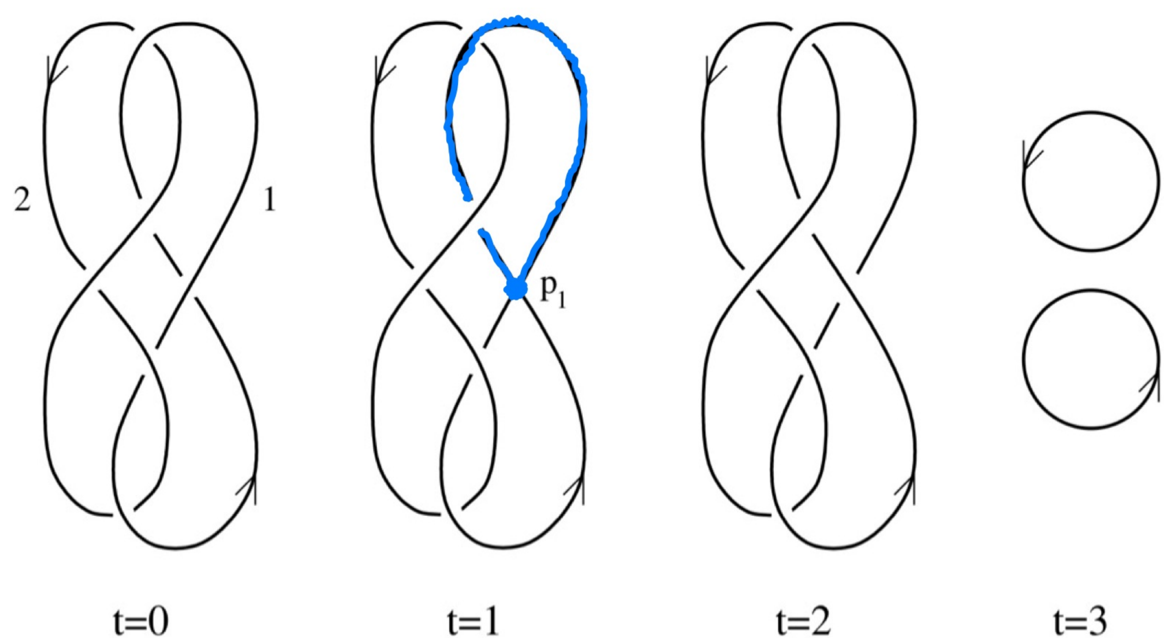
A “movie” of the Fenn-Rolfsen link map FR:



(Figure courtesy of Audoux—Meilhan—Yasuhara)

One half of the Fenn-Rolfsen link map *FR*:

A double point loop thru the self intersection in the first component links the second component:



The *Kirk invariant* of a link map "counts" linking numbers of double point loops on one component with the other component:

$$(f_1, f_2) : S^2 \amalg S^2 \to S^4 \rightsquigarrow \sigma_i(f_1, f_2) := \lambda(f_i, f_i) \in \mathbb{Z}[\mathbb{Z}]$$

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Kirk (1988): 2-component link homotopy classes map onto an infinitely generated Abelian group.

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Corollary: Kirk’s invariant classifies 2-component link maps in the 2-sphere up to link homotopy.

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Proof strategy:

Express link homotopy to the trivial link in terms of *Z[Z]*-intersections between one component and “metabolic” Whitney disks on the other component.

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More on the proof later; for now some ingredients…

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— Freedman’s technology (!) to find certain framed embedded topological Whitney disks

3-component link homotopy of 2-spheres in the 4-sphere

Stirling (2023): There exist infinitely many distinct link homotopy classes of Brunnian 3-component link maps.

“Brunnian” means that deleting any one component yields a link which is link homotopically trivial.

3-component link homotopy of 2-spheres in the 4-sphere

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Proofs later; first the invariant…

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Work modulo *regular* (link) homotopy, with

$$e(\nu(f_i)) = 0$$

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$$M^i := \text{free Milnor group on meridians to } f_j \text{ and } f_k$$

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– mod out indeterminacies...?

Definition:

The *free Milnor group* M^i is the quotient of the free group on x_j and x_k by the "self-finger move" relations:

$$[x_j, x_j^g] = 1 = [x_k, x_k^g]$$

Lemma: Each *g* in M^i can be written as

$$g = x_j^u x_k^v [x_j, x_k]^w$$

for unique integers *u*, *v*, and *w*.

Here x_j and x_k are meridian generators to f_j and f_k, with $\pi_1(S^4 \setminus (f_j \cup f_k)) \cong M^i$.

$$\sigma_i(f_1, f_2, f_3) := \lambda(f_i, f_i)$$

$$= \sum\nolimits_{(u,v,w)} a^i_{uvw} x^u_j x^v_k c^w_{jk} \in \mathbb{Z}[M^i]$$

$$(u, v, w) \in \mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}$$

$$a^i_{uvw} \in \mathbb{Z}, \text{ with } c_{jk} := [x_j, x_k]$$

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Depends on basing choice defining generators!

Need to mod out by effect of basing change…

Basing change:

$$x_j^u x_k^v c_{jk}^w \in M^i$$

$$\updownarrow$$

$$(x'_j)^u (x'_k)^v (c'_{jk})^{(\ell_{jk}-\ell_{ik})u+(\ell_{ij}-\ell_{kj})v+w} \in (M^i)'$$

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Need to quotient by these relations in the target of

$$\sigma_i(f_1, f_2, f_3) := \lambda(f_i, f_i)$$

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For example, in the case *i* = *1:*

$$\sigma_1(f_1, f_2, f_3) := \lambda(f_1, f_1) \in \frac{\mathbb{Z}M^1}{\text{Relations}}$$

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$$\sum_{(u,v,w)} a^1_{uvw} x_2^u x_3^v c_{23}^w$$

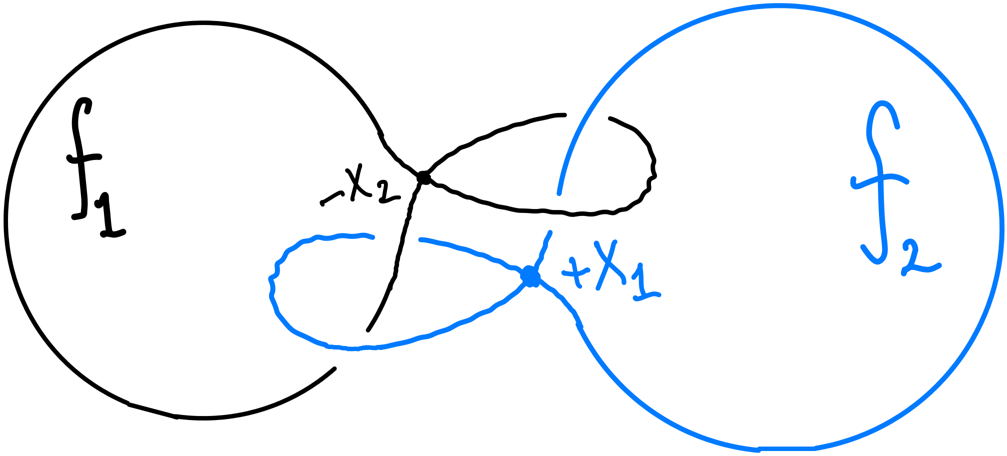
$$\updownarrow$$

$$\sum_{(u,v,w)} a^1_{uvw} x_2^u x_3^v c_{23}^{pu+qv+w}$$ for arbitrary *p,q* $\in \mathbb{Z}$

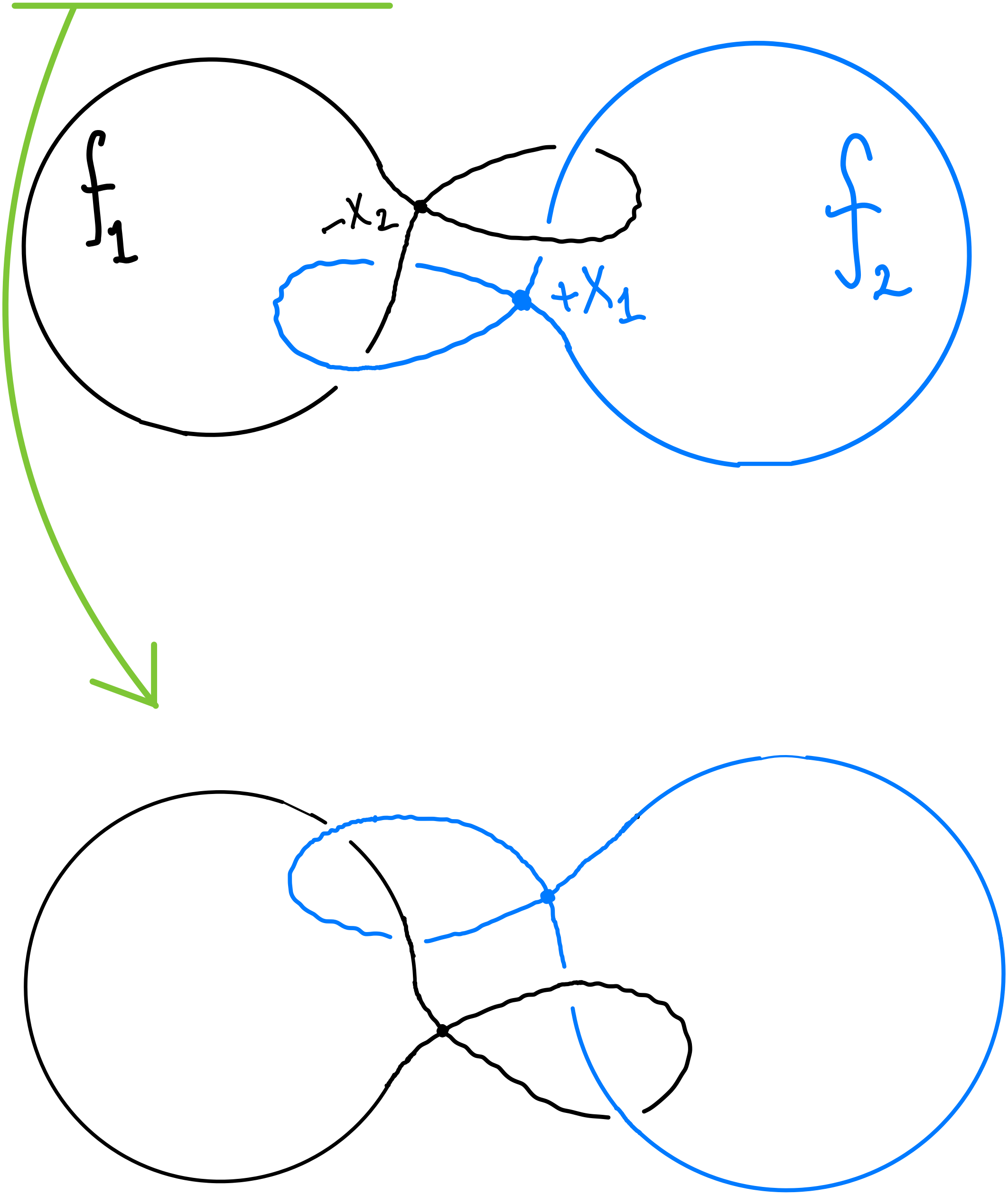
Stirling (2023): There exist infinitely many distinct link homotopy classes of Brunnian 3-component link maps.

Construction to follow…

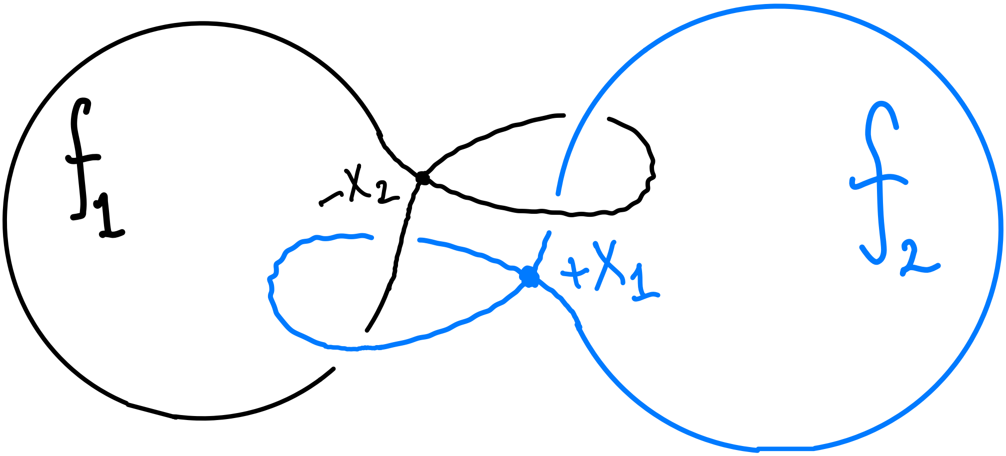
Schematic picture of *FR* in S^4:

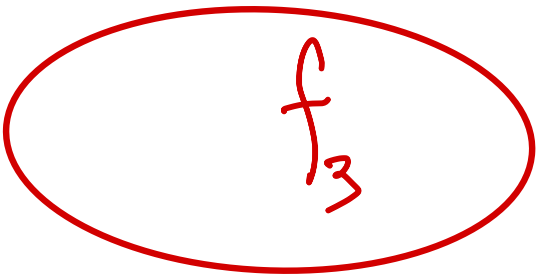


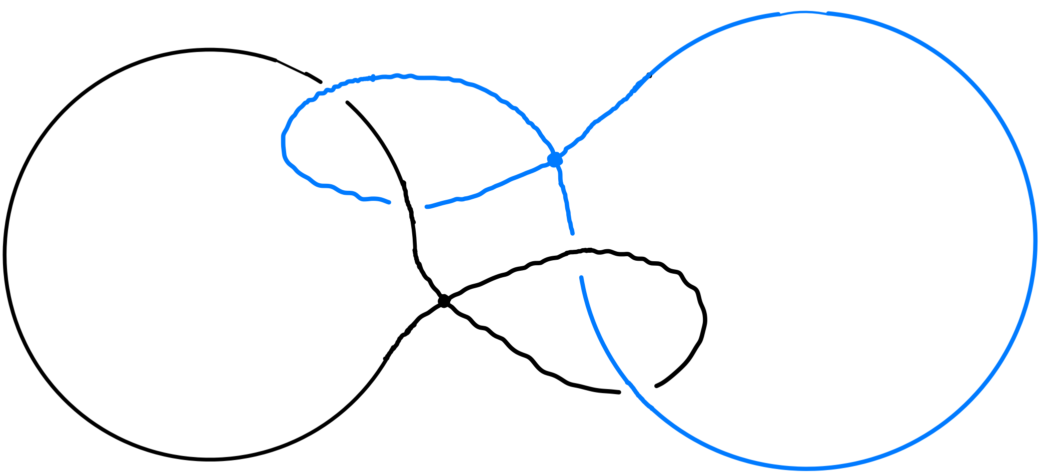
Disjoint union of *FR* with its reflection *FR'* in *S^4*:



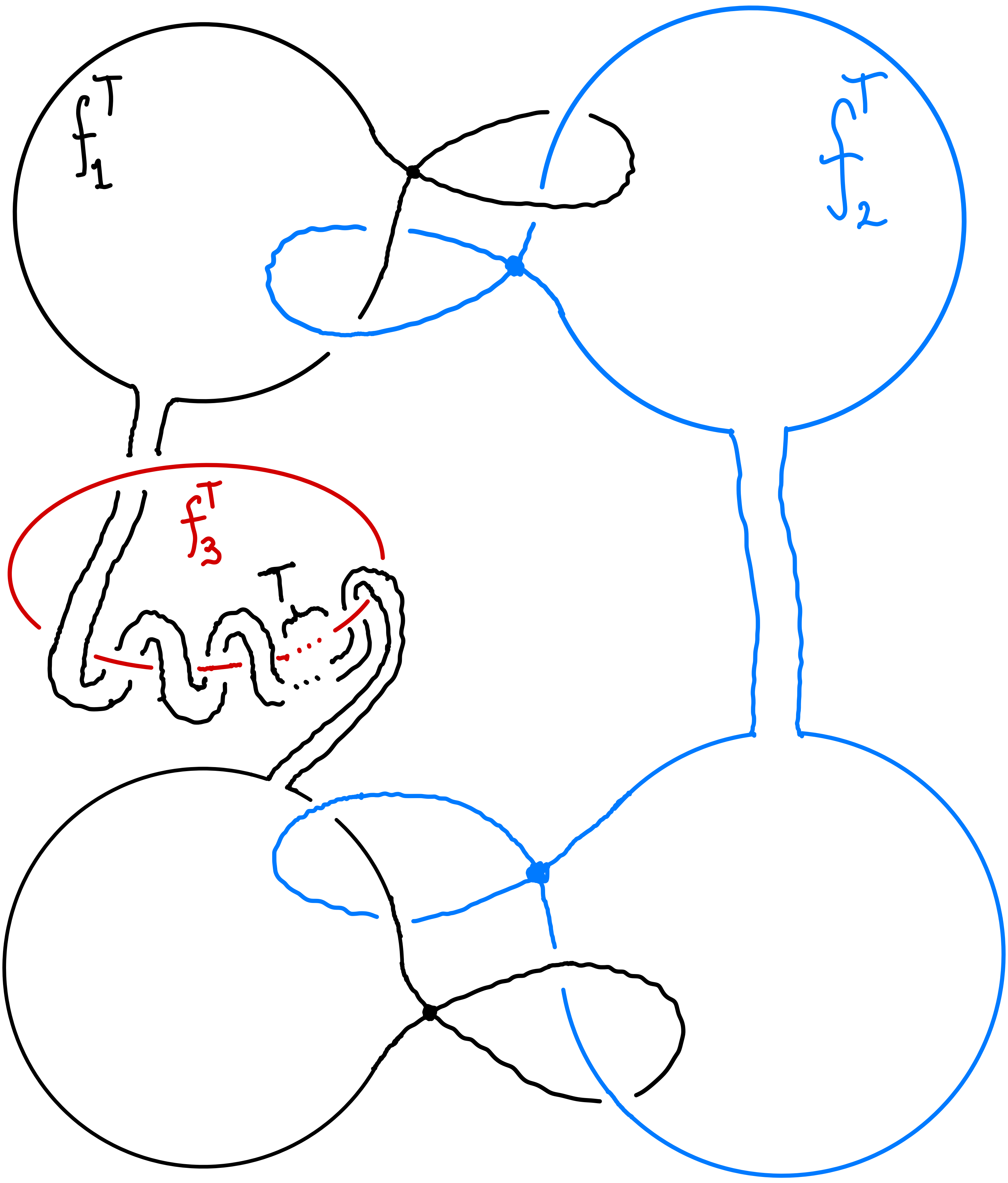
$FR \cup FR' \cup$ unknotted $f_3(S^2) \subset S^4$:







$f^T = (f_1^T, f_2^T, f_3^T) :=$ T twists of f_1 thru f_3



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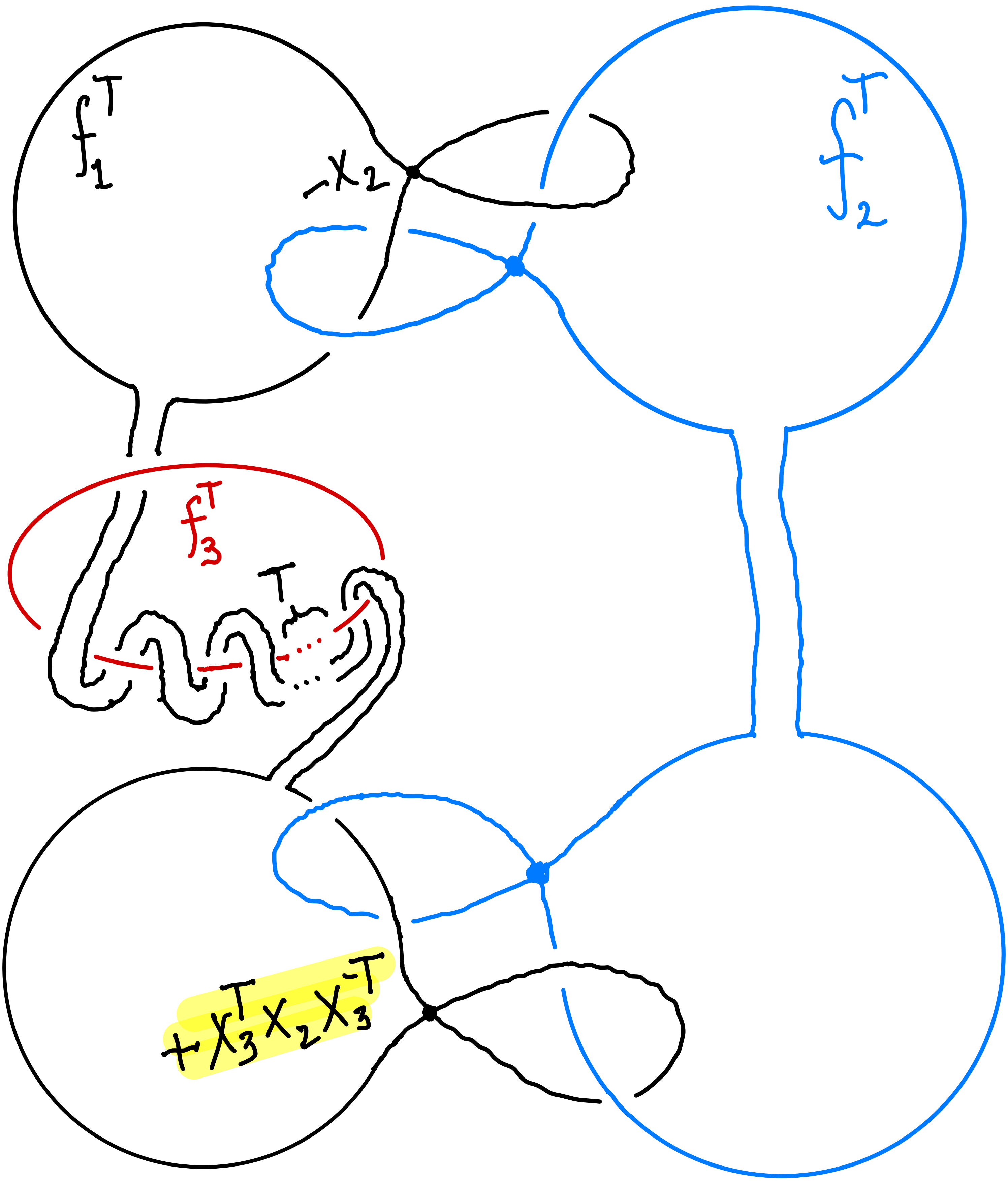
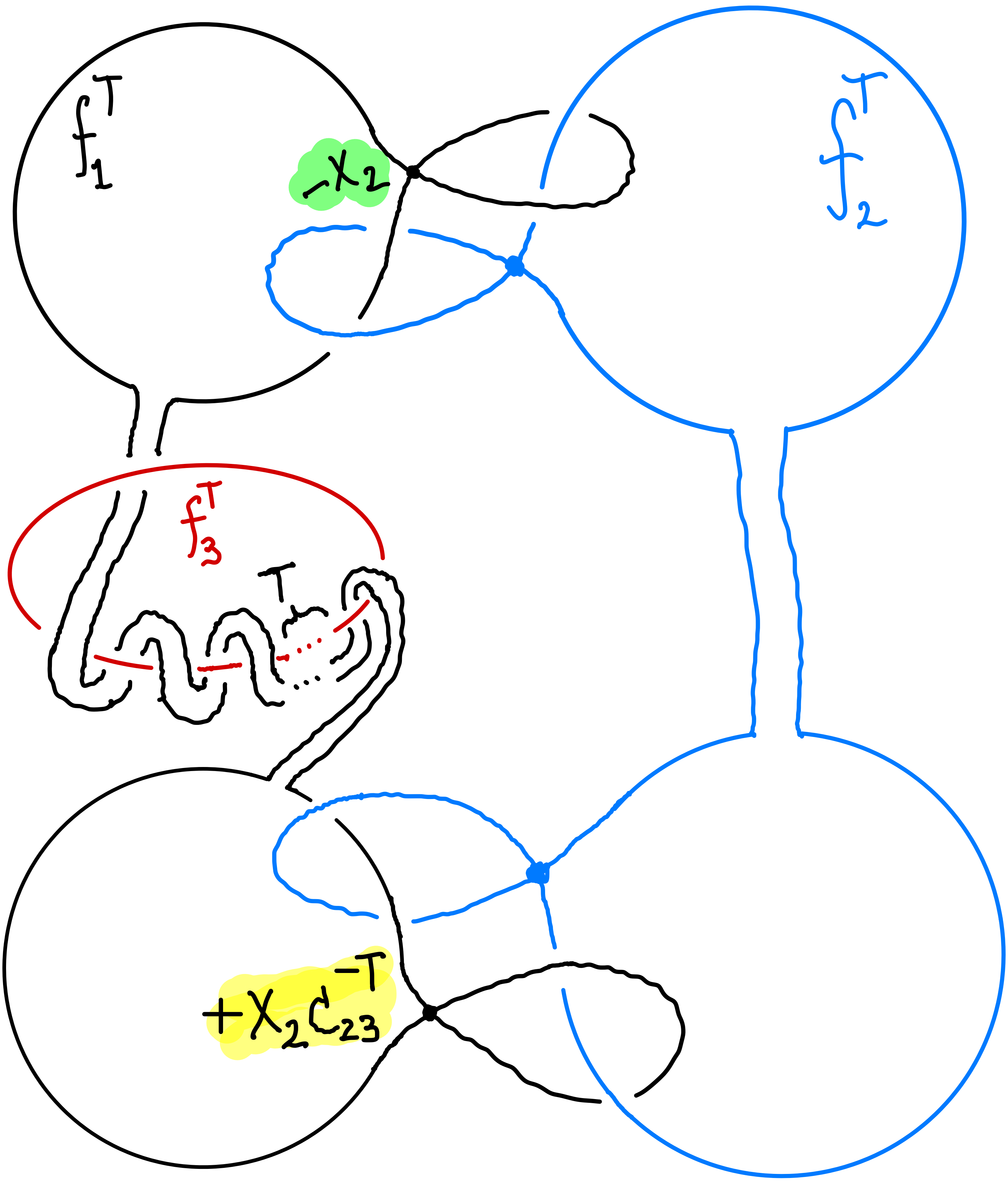


$$\mu f_1^T = -X_2 + X_2 c_{23}^{-T} \in \mathbb{Z}M^1$$



$$\sum_{(u,v,w)} a^1_{uvw} x_2^u x_3^v c_{23}^w$$

$$\updownarrow$$

Relations

$$\sum_{(u,v,w)} a^1_{uvw} x_2^u x_3^v c_{23}^{pu+qv+w}$$

$$\underbrace{\boxed{-x_2^1 + x_2^1 c_{23}^{-T}}}_{\mu f_1^T} = -x_2^1 c_{23}^{P} + x_2^1 c_{23}^{P-T}$$

$$= x_2 \left(c_{23}^{P-T} - c_{23}^{P} \right)$$

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$$\Downarrow$$

$$\mu f_1^T = -X_2 + X_2 C_{23}^{-T} \neq 0 \in \frac{\mathbb{Z} M^1}{\text{Relations}}$$

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Similarly, have:

$$T_1 \neq T_2 \iff \mu f_1^{T_1} \neq \mu f_1^{T_2}$$

$\square$

Stirling (2023): There exist infinitely many pairs of 3-component link maps with equal images that are not link homotopic.

Construction:

Flip orientation of unknotted component f_3^T of $f^T = (f_1^T, f_2^T, f_3^T)$

Check that invariant changes…

3-component link homotopy questions:

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—Does Stirling’s triple invariant detect link homotopy to the unlink?

—Techniques for computing in the target of Stirling’s triple invariant?

—Classify 3-component link homotopy?

S—Teichner (2017):
Kirk's invariant is injective.

Proof uses:

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— Elementary link homotopies which shrink Whitney spheres

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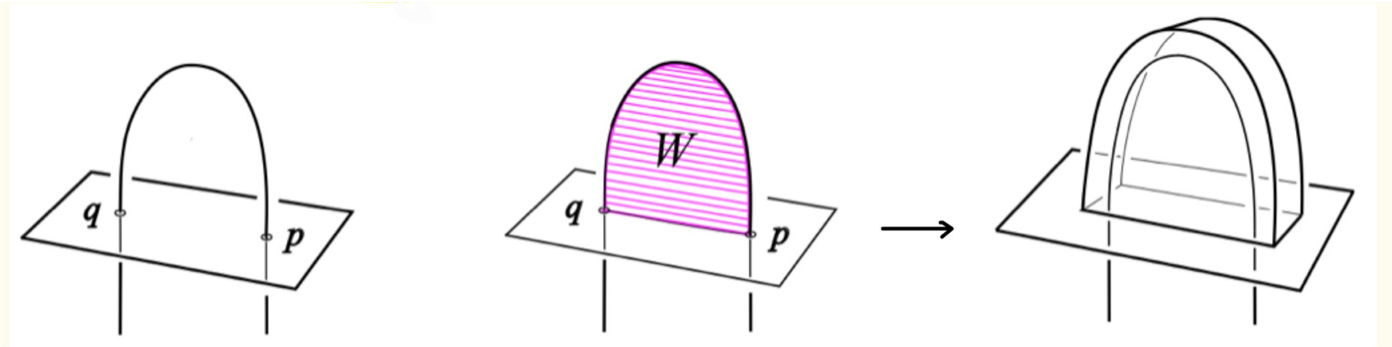
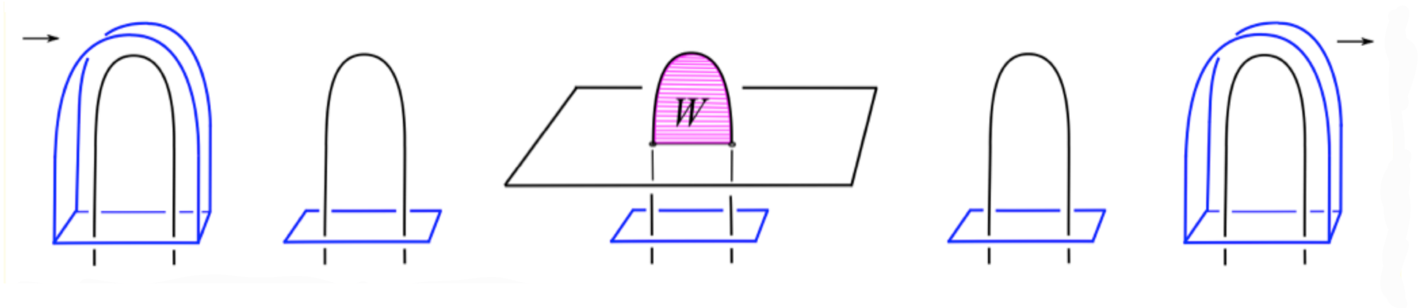
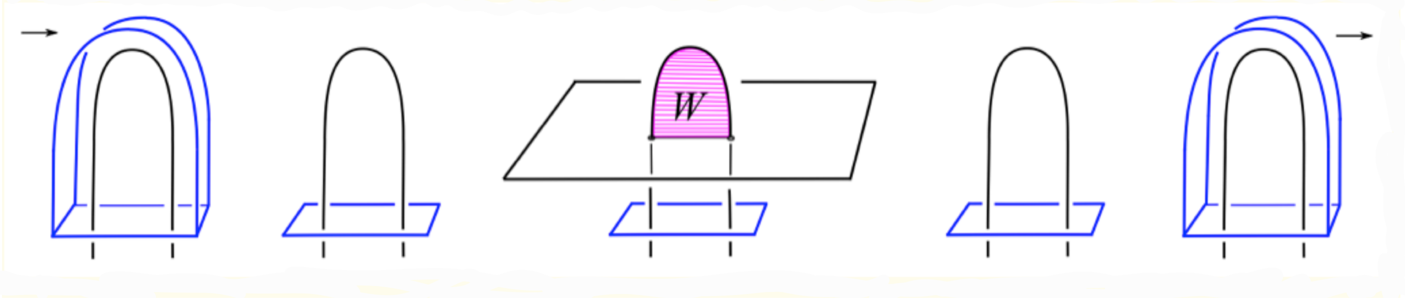
— Freedman's technology (!) to find certain framed embedded topological Whitney disks

Whitney move:

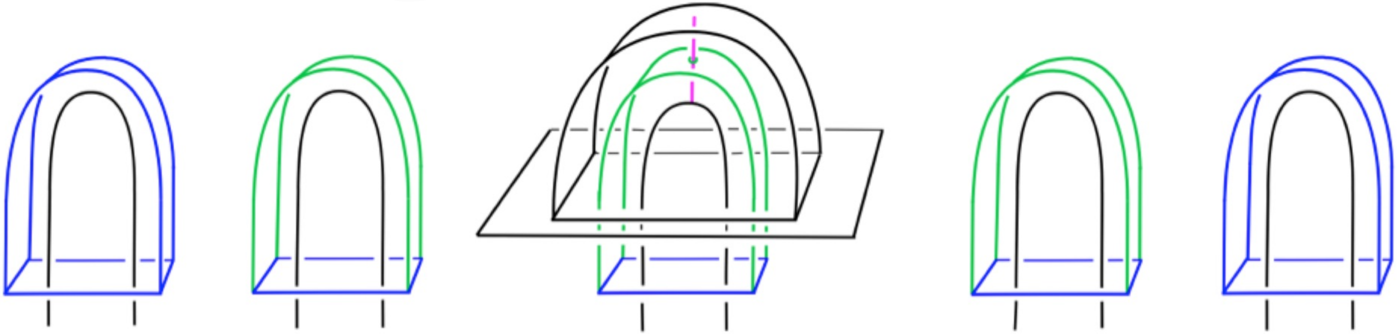


Whitney sphere:





Whitney sphere bounds ball after Whitney move:



Accessory spheres and Whitney spheres: