Roots of Alexander polynomials of random positive 3-braids

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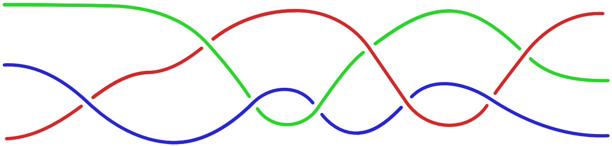
joint with Giulio Tiozzo

Based on: arXiv:2402.06771

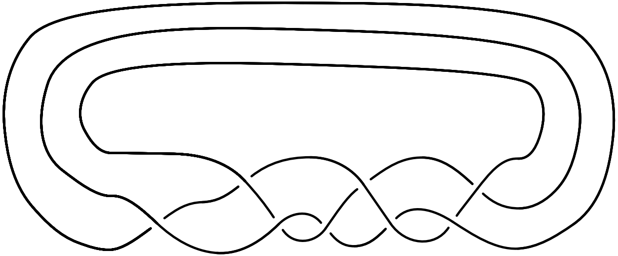
Slides at: https://dunfield.info/slides/ICERM2025.pdf

3-strand braid group: $\mathrm{Br}_3 = \langle\, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle$

$w = \sigma_1\sigma_2\sigma_1^{-2}\sigma_2\sigma_1^2\sigma_2^{-1}$:



Braid closure: $\widehat{w}$



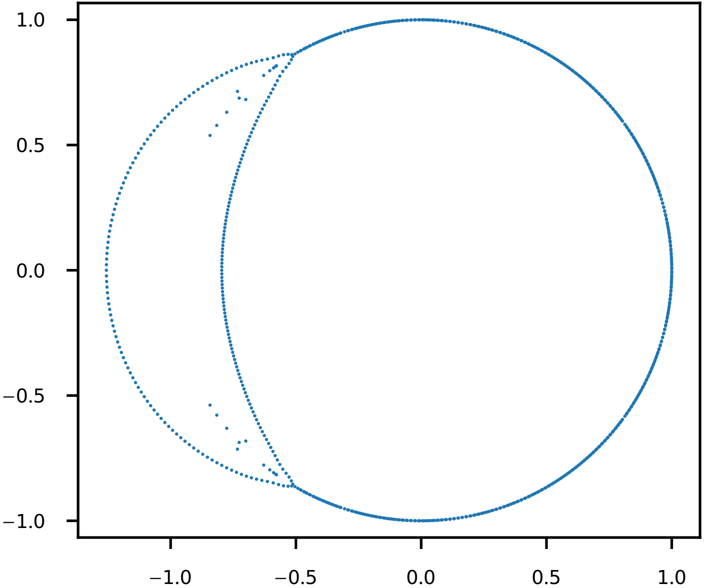
Alexander polynomial in $\mathbb{Z}[t^{\pm 1}]$

$$\Delta_{\widehat{w}}(t) = t^4 - 2t^3 + 3t^2 - 2t + 1$$

Positive braid:

σ_1, σ_2 only, no σ_1^{-1}, σ_2^{-1}.

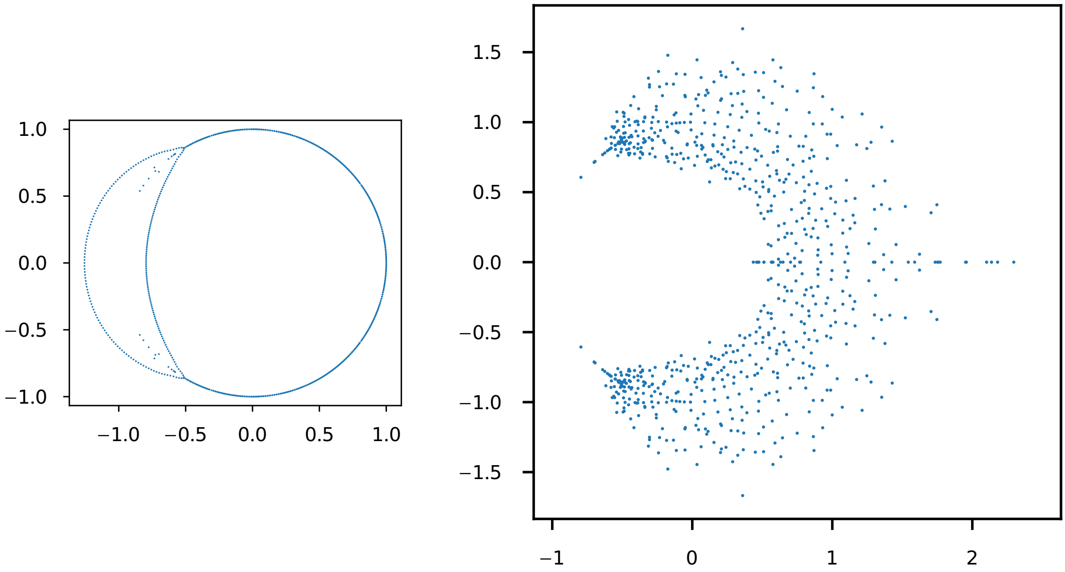
Dehornoy (2015): roots of $\Delta_{\widehat{w}}$ for positive $w \in \mathrm{Br}_3$ are highly structured.



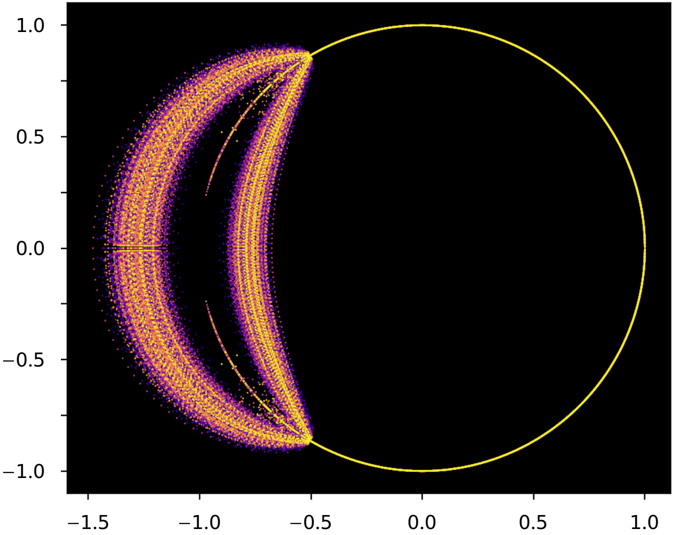
$\deg \Delta_{\widehat{w}} = \#w - 2 \approx 760$

69.3% of roots on S^1

Comparison to roots of $\Delta_{\hat{w}}$ for a random braid in $\{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}\}$.



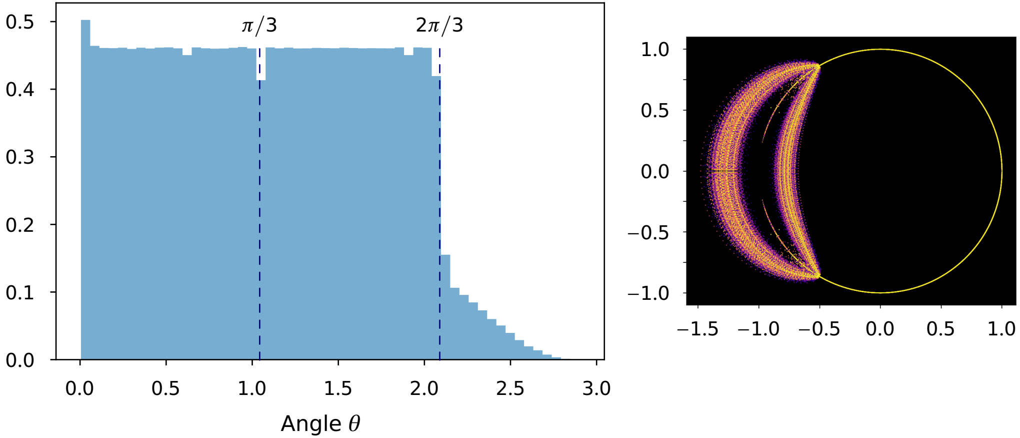
Roots of $\Delta_{\widehat{w}}$ for 2,500 positive braids with mean $\#w \approx 500$ and std. dev. 170.



w chosen randomly with σ_1 and σ_2 having probability 1/2

1.2 million roots shown

Distribution of roots of $\Delta_{\widehat{w}}$ on the top half of the circle.



For $w \in \mathrm{Br}_3$ let ν_w be the prob. measure on $\mathbb{C}$ unif. supported on the roots of $\Delta_{\widehat{w}}$.

Generate a random walk $w_n := g_1 g_2 \cdots g_n$ by picking $(g_i)_{i \in \mathbb{N}} \in \{\sigma_1, \sigma_2\}^{\mathbb{N}}$ with respect to the uniform measure.

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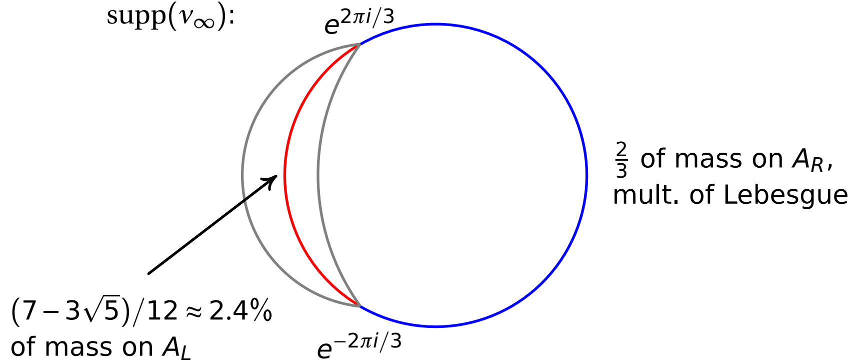
Conj. There exists a compactly supported measure ν_∞ on $\mathbb{C}$ such that for a.e. w_n one has $\nu_{w_n} \to \nu_\infty$ weakly. Moreover, ν_∞ has the following properties:

$\mathrm{supp}(\nu_\infty)$:



Burau rep $B_t : \mathrm{Br}_3 \to \mathrm{GL}_2\mathbb{Z}[t^{\pm 1}]$ defined by $\sigma_1 \mapsto \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$

$$\Delta_{\widehat{w}}(t) = \det(B_t(w) - 1) \Big/ (t^2 + t + 1)$$

For $\overline{\mathbb{D}} = \{\ |z| \leq 1\ \}$, take $\rho_w : \overline{\mathbb{D}} \to \mathbb{R}_{\geq 0}$ to be the max abs. val. of an eig. val. of $B_z(w)$.

Key: $\rho_w = 1$ at any root of $\Delta_{\widehat{w}}$ in $\overline{\mathbb{D}}$.

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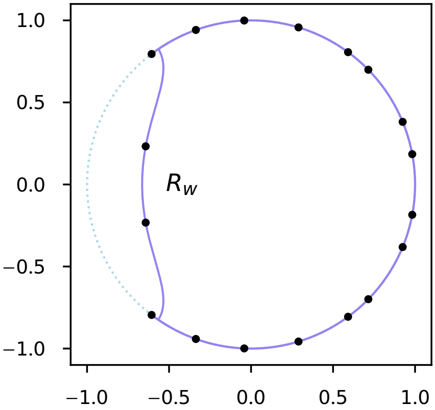
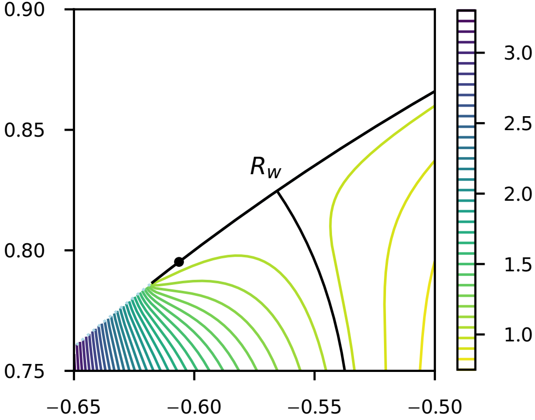
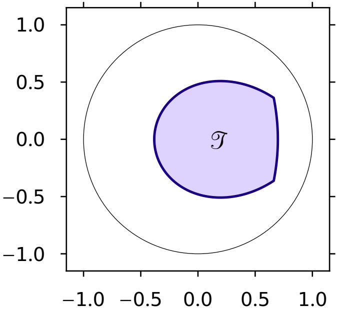
Key: $\rho_w = 1$ at any root of $\Delta_{\widehat{w}}$ in $\overline{\mathbb{D}}$. Set $R_w = \left\{ z \in \overline{\mathbb{D}} \;\middle|\; \rho_w(z) = 1 \right\}$.





Thm. For any positive $w \neq \sigma_i^k$, the set R_w contains the arc $A_R := \left\{ t = e^{i\theta} \;\middle|\; |\theta| < 2\pi/3 \right\}$, is disjoint from the set $\mathscr{T}$, and meets $(-1, 1)$ in a single point.



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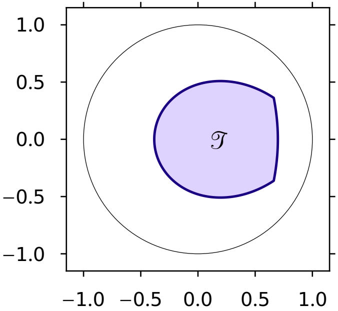


Thm. For any positive $w \neq \sigma_i^k$, at least $\frac{2}{3}\left(\deg(\Delta_{\widehat{w}}) - 1\right)$ of the roots of $\Delta_{\widehat{w}}$ occur on the arc A_R.

Thm. Set $A_L := \left\{ t = e^{i\theta} \;\middle|\; |\theta - \pi| < \pi/3 \right\}$. For a.e. random walk w_n, asymptotically the portion of the roots of $\Delta_{\widehat{w}_n}$ on A_L is $\geq (7 - 3\sqrt{5})/12 \approx 2.4\%$.

Thm. The signature $\left|\sigma_{\widehat{w}}(-1)\right|$ obeys a central limit theorem with positive drift $(5 - \sqrt{5})/4$.

Lyapunov exponent $\lambda(t) := \lim_{n\to\infty} \frac{1}{n} \cdot \text{Mean}\{ \log \|B_t(g)\| \mid g \text{ word in } \sigma_1, \sigma_2 \text{ of len } n\}$.

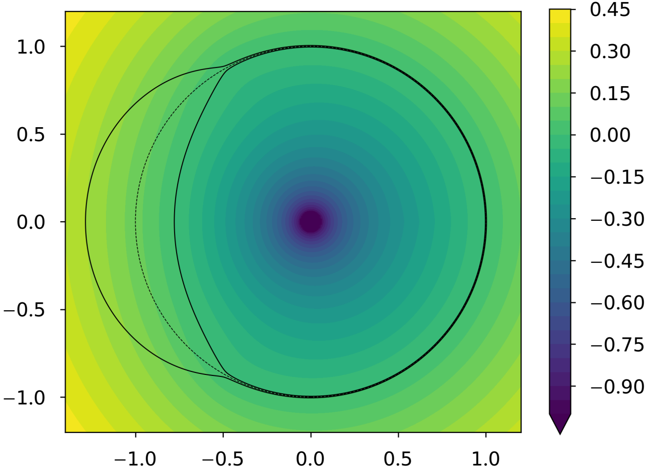
$\chi(t) := \max\left\{\lambda(t), \log|t|, 0\right\}$.

Bifurcation measure: $\nu_{bif} := \Delta\chi$

Conj. $\nu_{w_n} \to \nu_{bif}$

Motivated by Deroin-Dujardin.

We have some partial results towards this conjecture.

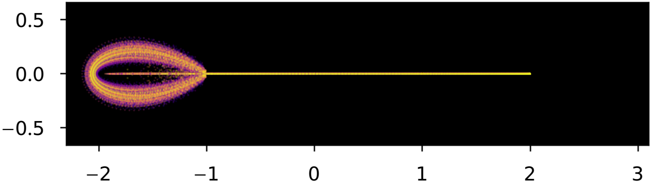


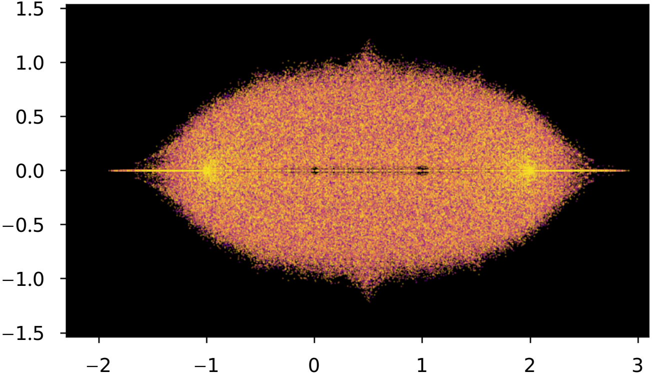
Open questions:

Prove the conjecture!

n-strand braids

Non-positive braids?





Ribbon concordances and slice obstructions: experiments and examples

Forthcoming work with Sherry Gong

- 352 million knots with ≤ 19 crossings [Burton]
- 1.6 million are slice
- 350.4 million are not slice
- $< 13{,}000$ unknown (0.004%)