Structure-Preserving Learning of High-Dimensional Lagrangian and Hamiltonian Systems

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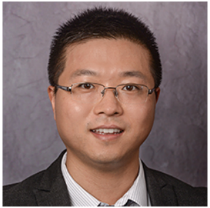

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- Office of Naval Research: *Nonlinear Data-driven and Structure-Preserving Hamiltonian Model Reduction*

Motivating thoughts

- With the ever-increasing data volumes we can access, there is a growing demand to **learn computationally efficient surrogate models** of **high-dimensional dynamical systems** for optimization, uncertainty quantification, and long-term prediction

- Unconstrained learning for nonlinear finite-dimensional systems (reduced or full-order)

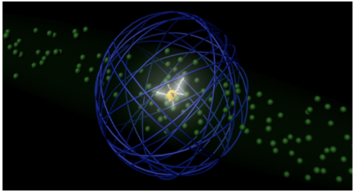
$$\dot{\mathbf{x}}(t,\mu) = \mathbf{f}(\mathbf{x}(t),\mu) + \mathbf{g}(\mathbf{x})\mathbf{u}(t) \quad \in \mathbb{R}^n$$

  a la machine learning may be very expressive, but often fails to extrapolate in time.

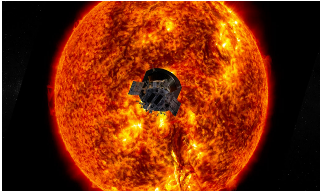
- In general, if n is very large (e.g., $n \geq 1,000$) then almost all methods require dimension reduction (linear or nonlinear)

- **Incorporating model knowledge** into learning framework is imperative for **predictions**: geometric/mechanical structure, nonlinear terms, inputs, etc.

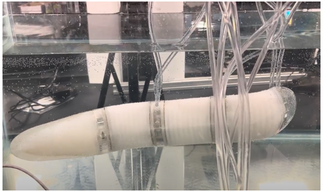
Structured physical systems are everywhere!



(a) Bose-Einstein condensate



(b) Solar plasma



(c) Soft-robotic fish

- Physical systems have interesting properties like conservation laws, symplecticity, reversibility or configuration space structure
- Structure-preserving methods preserve underlying geometric structure
  - Conserve discrete quantities which are close to continuous quantity
  - Reproduce long-time behavior
- Long-time numerical simulation of large-scale systems using structure-preserving methods is computationally prohibitive

Need for physics-preserving reduced-order models

Part I: Learning Hamiltonian reduced models via structure-preserving optimization

- Hamiltonian operator inference: physics-preserving learning of reduced-order models for Hamiltonian systems. *Sharma/Wang/K.*, Physica D: Nonlinear Phenomena, Vol. 431, 133122, 2022.
- Gradient preserving operator inference: data-driven reduced-order models for equations with gradient structure. *Geng/Singh/Ju/K./Wang*, Computer Methods in Applied Mechanics and Engineering, Vol. 427, 117033, 2024.

Gradient systems

- General infinite-dimensional Hamiltonian system

$$\frac{\partial y(x,t)}{\partial t} = \mathcal{L}\frac{\delta \mathcal{H}}{\delta y}$$

where $\mathcal{L}$ is a linear differential operator, $\frac{\delta \mathcal{H}}{\delta y}$ is the variational derivative of

$$\mathcal{H}[y] = \int \Big(\underbrace{H_{\mathsf{quad}}(y, y_x, \dots)}_{\text{quadratic terms}} + \underbrace{H_{\mathsf{nl}}(y)}_{\text{spatially local terms}} \Big) \, \mathrm{d}x$$

1. If $\mathcal{L}$ **is skew adjoint**, $\mathcal{H}$ is referred to as the Hamiltonian, is constant, and the PDE is **conservative**.
2. If $\mathcal{L}$ **is negative semi-definite** (resp. definite), $\mathcal{H}$, referred to as a Lyapunov/energy function, is nonincreasing (resp. monotonically decreasing) and the PDE is **dissipative**.

- Structure-preserving space discretization leads to finite-dimensional Hamiltonian models

$$\dot{\mathbf{y}} = \mathbf{D}\nabla_{\mathbf{y}} H_{\mathrm{d}}(\mathbf{y})$$

where H_{d} is the space-discretized Hamiltonian function.

Goal: Learn low-dimensional Hamiltonian systems from trajectory data $\mathbf{y}(t_1), \dots, \mathbf{y}(t_f)$.

Special forms of gradient systems

(i) When $\mathbf{D}$ is skew-symmetric (a.k.a. skew-adjoint), $\mathbf{D} = -\mathbf{D}^\intercal$, the system is Hamiltonian. A special case is when $\mathbf{D} = \left[\begin{smallmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{smallmatrix}\right]$ the system is a canonical Hamiltonian system and the solution flow is symplectic. Thus, the internal energy of the system, $H(\mathbf{y})$, is conserved, e..g, for any $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$H(\mathbf{y}(t_2)) - H(\mathbf{y}(t_1)) = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{y}(t))\,\mathrm{d}t = \int_{t_1}^{t_2} (\nabla_{\mathbf{y}} H(\mathbf{y}))^\intercal\, \mathbf{D}\, \nabla_{\mathbf{y}} H(\mathbf{y})\,\mathrm{d}t = 0.$$

(ii) When $\mathbf{D}$ is negative semi-definite, the system represents a gradient flow and is dissipative:

$$H(\mathbf{y}(t_2)) - H(\mathbf{y}(t_1)) = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{y}(t))\,\mathrm{d}t = \int_{t_1}^{t_2} (\nabla_{\mathbf{y}} H(\mathbf{y}))^\intercal\, \mathbf{D}\, \nabla_{\mathbf{y}} H(\mathbf{y})\,\mathrm{d}t \le 0.$$

Note that if $\mathbf{D}$ is negative definite, $H(\mathbf{y})$ is strictly decreasing.

Numerical schemes to solve these equations recognize the special gradient structure in time discretization, e.g., geometric integrators and average vector field methods.

This talk: Canonical Hamiltonian Systems

- We focus on canonical Hamiltonian systems (for other gradient systems, see[1])

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{J}_{2n} \nabla_{\mathbf{y}} H_{\mathrm{d}}(\mathbf{q}, \mathbf{p}) = \begin{bmatrix} \nabla_{\mathbf{p}} H_{\mathrm{d}}(\mathbf{q}, \mathbf{p}) \\ -\nabla_{\mathbf{q}} H_{\mathrm{d}}(\mathbf{q}, \mathbf{p}) \end{bmatrix}$$

- Key features
  1. **Canonical Hamiltonian structure**, i.e. $\mathbf{J}_{2n} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0} \end{bmatrix}$
  2. State vector $\mathbf{y} \in \mathbb{R}^{2n}$ can be partitioned as $\mathbf{y} = [\mathbf{q}^\top, \mathbf{p}^\top]^\top$ where $\mathbf{q}, \mathbf{p} \in \mathbb{R}^n$ both have distinct **physical interpretation**
  3. **Symmetry** in linear FOM operators due to structure-preserving space discretization

Assumption: Functional form of $H_{\mathsf{nl}}(q, p)$ is known, whereas $H_{\mathsf{quad}}(q, p)$ and details about spatial discretization are unavailable.

[1] Gradient preserving operator inference: data-driven reduced-order models for equations with gradient structure. *Geng/Singh/Ju/K./Wang*, CMAME 427, 117033, 2024.

Problem formulation: FOM data

Given: Snapshot data matrices from Hamiltonian FOM simulations

$$\mathbf{Q} = \left[\mathbf{q}_1 \cdots \mathbf{q}_K\right] \in \mathbb{R}^{n \times K}, \qquad \mathbf{P} = \left[\mathbf{p}_1 \cdots \mathbf{p}_K\right] \in \mathbb{R}^{n \times K}$$

- Use knowledge about Hamiltonian functional to define nonlinear forcing snapshot matrices

$$\mathbf{F}_{\mathbf{q}} = \left[\mathbf{f}_{\mathbf{q}}(\mathbf{y}_1) \cdots \mathbf{f}_{\mathbf{q}}(\mathbf{y}_K)\right] \in \mathbb{R}^{n \times K}, \qquad \mathbf{F}_{\mathbf{p}} = \left[\mathbf{f}_{\mathbf{p}}(\mathbf{y}_1) \cdots \mathbf{f}_{\mathbf{p}}(\mathbf{y}_K)\right] \in \mathbb{R}^{n \times K}$$

- Build snapshot matrices of time-derivative data via finite difference

$$\dot{\mathbf{Q}} = \left[\dot{\mathbf{q}}_1 \cdots \dot{\mathbf{q}}_K\right] \in \mathbb{R}^{n \times K}, \qquad \dot{\mathbf{P}} = \left[\dot{\mathbf{p}}_1 \cdots \dot{\mathbf{p}}_K\right] \in \mathbb{R}^{n \times K}$$

Next step: Project FOM data onto low-dimensional symplectic subspaces

Problem formulation: symplectic projection

- Symplectic projection via proper symplectic decomposition (PSD)

$$\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} \approx \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{bmatrix}$$

  where $\mathbf{V}_{\mathbf{q}} = \mathbf{V}_{\mathbf{p}} = \mathbf{\Phi} \in \mathbb{R}^{n \times r}$ are obtained via cotangent the lift algorithm.[2]
- Projecting FOM data to obtain
  1. Reduced snapshot data
$$\hat{\mathbf{Q}} = \mathbf{V}_{\mathbf{q}}^{\top} \mathbf{Q} \in \mathbb{R}^{r \times K}, \qquad \hat{\mathbf{P}} = \mathbf{V}_{\mathbf{p}}^{\top} \mathbf{P} \in \mathbb{R}^{r \times K}$$
  2. Reduced time-derivative data
$$\dot{\hat{\mathbf{Q}}} = \mathbf{V}_{\mathbf{q}}^{\top} \dot{\mathbf{Q}} \in \mathbb{R}^{r \times K}, \qquad \dot{\hat{\mathbf{P}}} = \mathbf{V}_{\mathbf{p}}^{\top} \dot{\mathbf{P}} \in \mathbb{R}^{r \times K}$$
  3. Reduced nonlinear forcing data
$$\hat{\mathbf{F}}_{\mathbf{q}} = \mathbf{V}_{\mathbf{p}}^{\top} \mathbf{F}_{\mathbf{q}} \in \mathbb{R}^{r \times K}, \qquad \hat{\mathbf{F}}_{\mathbf{p}} = \mathbf{V}_{\mathbf{q}}^{\top} \mathbf{F}_{\mathbf{p}} \in \mathbb{R}^{r \times K}$$

Next step: Fit reduced operators to the projected trajectories in a structure-preserving way

[2]Peng L, Mohseni K. Symplectic model reduction of Hamiltonian systems. SIAM Journal on Scientific Computing. 2016;38(1):A1–A27

Problem formulation: model form for learning

- Reduced Hamiltonian in terms of the inferred reduced operators $\hat{\mathbf{D}}_{\mathbf{q}} \in \mathbb{R}^{r \times r}$ and $\hat{\mathbf{D}}_{\mathbf{p}} \in \mathbb{R}^{r \times r}$

$$\hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = \frac{1}{2}\hat{\mathbf{q}}^\top \hat{\mathbf{D}}_{\mathbf{q}} \hat{\mathbf{q}} + \frac{1}{2}\hat{\mathbf{p}}^\top \hat{\mathbf{D}}_{\mathbf{p}} \hat{\mathbf{p}} + \hat{H}_{\mathsf{nl}}(\hat{\mathbf{q}}, \hat{\mathbf{p}})$$

- Model form for learning Hamiltonian ROMs based on $\hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}})$

$$\dot{\hat{\mathbf{q}}} = \frac{\partial \hat{H}}{\partial \hat{\mathbf{p}}} = \hat{\mathbf{D}}_{\mathbf{p}} \hat{\mathbf{p}} + \mathbf{V}_{\mathbf{p}}^\top \mathbf{f}_{\mathbf{q}}(\mathbf{V}_{\mathbf{q}} \hat{\mathbf{q}}, \mathbf{V}_{\mathbf{p}} \hat{\mathbf{p}})$$

$$\dot{\hat{\mathbf{p}}} = -\frac{\partial \hat{H}}{\partial \hat{\mathbf{q}}} = -\hat{\mathbf{D}}_{\mathbf{q}} \hat{\mathbf{q}} - \mathbf{V}_{\mathbf{q}}^\top \mathbf{f}_{\mathbf{p}}(\mathbf{V}_{\mathbf{q}} \hat{\mathbf{q}}, \mathbf{V}_{\mathbf{p}} \hat{\mathbf{p}})$$

Hamiltonian Operator Inference

- Constrained optimization problem[3] to compute $\hat{\mathbf{D}}_{\mathbf{q}}$ and $\hat{\mathbf{D}}_{\mathbf{p}}$

$$\min_{\substack{\hat{\mathbf{D}}_{\mathbf{q}}=\hat{\mathbf{D}}_{\mathbf{q}}^{\top},\\ \hat{\mathbf{D}}_{\mathbf{p}}=\hat{\mathbf{D}}_{\mathbf{p}}^{\top}}} \left\| \begin{bmatrix} \dot{\hat{\mathbf{Q}}} - \hat{\mathbf{F}}_{\mathbf{q}}(\hat{\mathbf{Q}}, \hat{\mathbf{P}}) \\ \dot{\hat{\mathbf{P}}} + \hat{\mathbf{F}}_{\mathbf{p}}(\hat{\mathbf{Q}}, \hat{\mathbf{P}}) \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \hat{\mathbf{D}}_{\mathbf{p}} \\ -\hat{\mathbf{D}}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Q}} \\ \hat{\mathbf{P}} \end{bmatrix} \right\|_F$$

  where symmetric constraints on $\hat{\mathbf{D}}_{\mathbf{q}}$ and $\hat{\mathbf{D}}_{\mathbf{p}}$ ensure that the learned reduced operators retain symmetric property of full-model operators
- Separate, symmetric linear least-squares problems of the form

$$\min_{\hat{\mathbf{D}}_{\mathbf{p}}=\hat{\mathbf{D}}_{\mathbf{p}}^{\top}} \left\| \underbrace{\dot{\hat{\mathbf{Q}}} - \hat{\mathbf{F}}_{\mathbf{q}}(\hat{\mathbf{Q}}, \hat{\mathbf{P}})}_{\mathbf{R}_{\mathbf{p}}} - \hat{\mathbf{D}}_{\mathbf{p}}\hat{\mathbf{P}} \right\|_F \rightarrow (\hat{\mathbf{P}}\hat{\mathbf{P}}^{\top})\hat{\mathbf{D}}_{\mathbf{p}} + \hat{\mathbf{D}}_{\mathbf{p}}(\hat{\mathbf{P}}\hat{\mathbf{P}}^{\top}) = \hat{\mathbf{P}}\hat{\mathbf{R}}_{\mathbf{p}}^{\top} + \hat{\mathbf{R}}_{\mathbf{p}}\hat{\mathbf{P}}^{\top}$$

$$\min_{\hat{\mathbf{D}}_{\mathbf{q}}=\hat{\mathbf{D}}_{\mathbf{q}}^{\top}} \left\| \underbrace{\dot{\hat{\mathbf{P}}} + \hat{\mathbf{F}}_{\mathbf{p}}(\hat{\mathbf{Q}}, \hat{\mathbf{P}})}_{-\mathbf{R}_{\mathbf{q}}} + \hat{\mathbf{D}}_{\mathbf{q}}\hat{\mathbf{Q}} \right\|_F \rightarrow (\hat{\mathbf{Q}}\hat{\mathbf{Q}}^{\top})\hat{\mathbf{D}}_{\mathbf{q}} + \hat{\mathbf{D}}_{\mathbf{q}}(\hat{\mathbf{Q}}\hat{\mathbf{Q}}^{\top}) = \hat{\mathbf{Q}}\hat{\mathbf{R}}_{\mathbf{q}}^{\top} + \hat{\mathbf{R}}_{\mathbf{q}}\hat{\mathbf{Q}}^{\top}$$

  can be solved via Lyapunov equations.

[3] Based on Operator Inference: [Data-driven operator inference for nonintrusive projection-based model reduction, Peherstorfer & Willcox, CMAME, 306, 196-215 351 2016]

Error bound on the learned operators

1. Time stepping scheme for the FOM is convergent

$$\max_{i\in\{1,\cdots,T/\Delta t\}} \left|\left|\mathbf{y}_i - \mathbf{y}(t_i)\right|\right|_2 \to 0 \quad as \quad \Delta t \to 0$$

2. Derivatives approximated from projected states converge to $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{y}}(t_k)$

$$\max_{i\in\{1,\cdots,T/\Delta t\}} \left|\left|\dot{\hat{\mathbf{y}}}_i - \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{y}}(t_i)\right|\right|_2 \to 0 \quad as \quad \Delta t \to 0$$

Theorem 1 ([Sharma/Wang/K, 2022][4])

Let $\tilde{\mathbf{D}}_{\mathbf{q}}$ and $\tilde{\mathbf{D}}_{\mathbf{p}}$ be the intrusively projected ROM operators. If the snapshot data matrix has full column rank, then for every $\epsilon > 0$, there exists $2r \leq 2n$ and $\Delta t > 0$ such that for the difference between the learned operators $\hat{\mathbf{D}}_{\mathbf{q}}, \hat{\mathbf{D}}_{\mathbf{p}}$ and the projection-based $\tilde{\mathbf{D}}_{\mathbf{q}}, \tilde{\mathbf{D}}_{\mathbf{p}}$, we have

$$||\hat{\mathbf{D}}_{\mathbf{q}} - \tilde{\mathbf{D}}_{\mathbf{q}}||_F \leq \epsilon, \qquad ||\hat{\mathbf{D}}_{\mathbf{p}} - \tilde{\mathbf{D}}_{\mathbf{p}}||_F \leq \epsilon.$$

[4]Sharma H, Wang Z, Kramer B. Hamiltonian operator inference: Physics-preserving learning of reduced-order models for canonical Hamiltonian systems. Physica D: Nonlinear Phenomena, 431, 133122, 2022.

Error bound on the solutions

Theorem 1 ([Geng/Singh/Ju/K./Wang, 2024][5])

Let $\mathbf{y}(t)$ be the solution of the FOM on $[0,T]$ and $\mathbf{y}_r(t)$ be the solution of the structure-preserving ROM on the same interval. Suppose $\nabla_{\mathbf{y}} H(\mathbf{y})$ is Lipschitz continuous, then the ROM approximation error satisfies

$$\int_0^T \|\mathbf{y} - \mathbf{\Phi}\mathbf{y}_r\|^2 \,\mathrm{d}t \leq$$

$$C(T)\left(\underbrace{\int_0^T \|\mathbf{y} - \mathbf{\Phi}\mathbf{\Phi}^\intercal\mathbf{y}\|^2 \mathrm{d}t}_{\textit{projection error}} + \underbrace{\int_0^T \|\dot{\mathbf{y}} - \mathcal{D}_t[\mathbf{y}]\|^2 \,\mathrm{d}t}_{\textit{data error}} + \underbrace{\int_0^T \|\mathbf{\Phi}^\intercal\mathcal{D}_t[\mathbf{y}] - \mathbf{D}_r\mathbf{\Phi}^\intercal\nabla_{\mathbf{y}} H(\mathbf{y})\|^2 \,\mathrm{d}t}_{\textit{optimization error}}\right),$$

where $C(T) = \max\{1 + C_2^2, 2\}T\alpha(T)$, $\alpha(T) = 2\int_0^T e^{2C_1(T-\tau)}\,\mathrm{d}\tau$, and the constants $C_1 = \mathcal{C}_{\mathrm{log-Lip}}[\mathbf{\Phi}\mathbf{D}_r\mathbf{\Phi}^\intercal\nabla_{\mathbf{y}} H]$ and $C_2 = \|\mathbf{\Phi}\mathbf{D}_r\mathbf{\Phi}^\intercal\|\,\mathcal{C}_{\mathrm{Lip}}[\nabla_{\mathbf{y}} H]$.

[5] Gradient preserving operator inference: data-driven reduced-order models for equations with gradient structure. *Geng/Singh/Ju/K./Wang*, CMAME 427, 117033, 2024.

Nonlinear Schrödinger equation

- Cubic Schrödinger equation

$$i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = 0$$

- Writing $\psi = p + iq$ with space-time continuous Hamiltonian $\mathcal{H}$

$$\mathcal{H}(q,p) = \int \frac{1}{2}\left[p_x^2 + q_x^2 - \frac{\gamma}{2}[q^2 + p^2]^2\right]\,dx$$

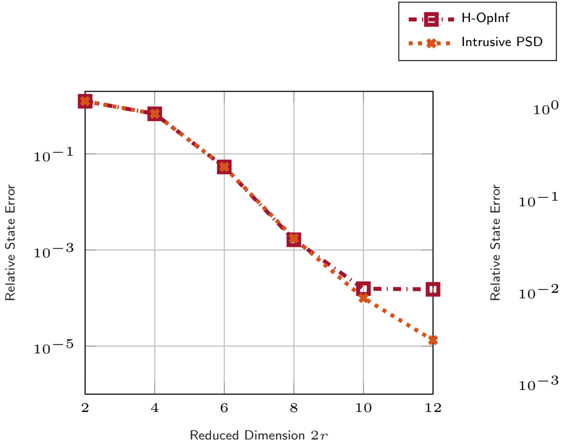
leads to canonical Hamiltonian PDE form $y_t = J\frac{\partial \mathcal{H}}{\partial y}$ for $y = [p,q]^\top$

$$y_t = \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta \mathcal{H}}{\delta p} \\ \frac{\delta \mathcal{H}}{\delta q} \end{bmatrix} = \begin{bmatrix} -q_{xx} - \gamma(q^2 + p^2)q \\ p_{xx} + \gamma(q^2 + p^2)p \end{bmatrix}$$

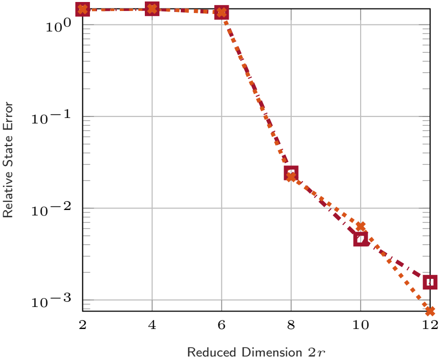
- Mass and momentum invariants of motion

$$\mathcal{M}_1(q,p) = \int \left[p^2 + q^2\right]\,dx \qquad \mathcal{M}_2(q,p) = \int \left[p_x q - q_x p\right]\,dx$$

Nonlinear Schrödinger equation ($2n = 128$): state error

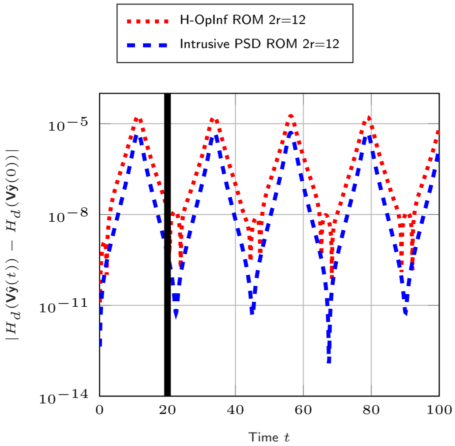


(a) Training data

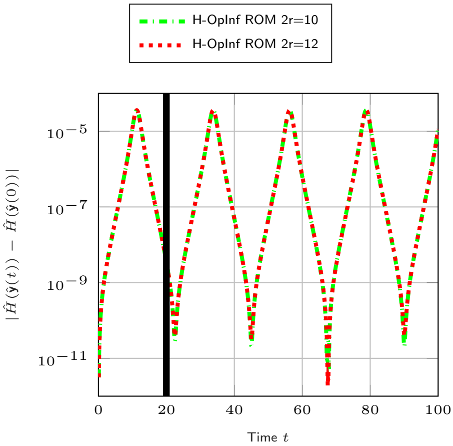


(b) Test data (400% outside)

Nonlinear Schrödinger equation: energy error

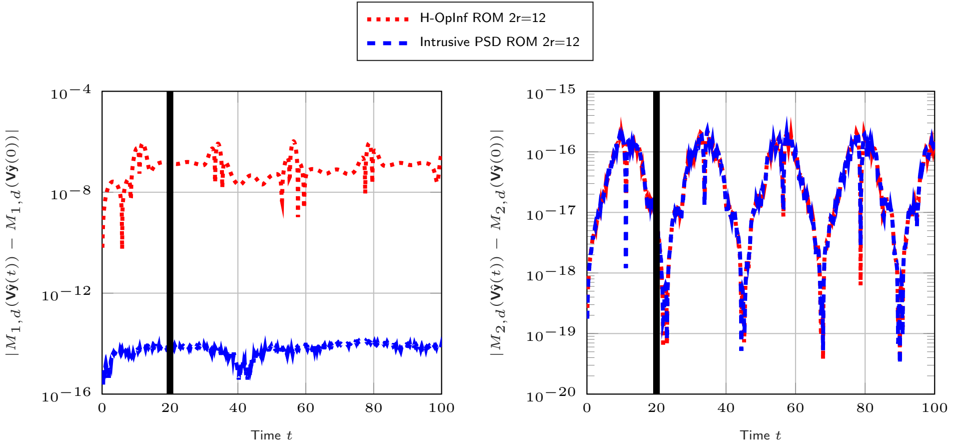


(a) FOM energy error



(b) ROM energy error

Nonlinear Schrödinger equation: invariants of motion



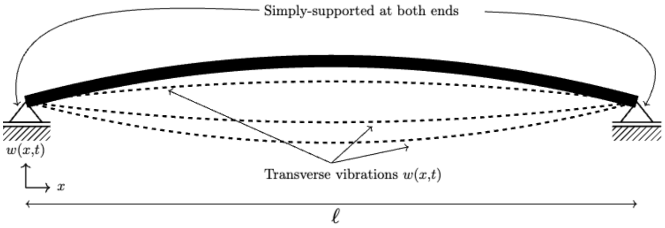
(a) Mass conservation **(b)** Momentum conservation

H-OpInf conserves mass and momentum invariants of motion

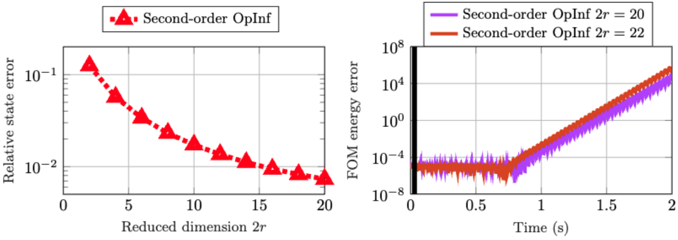
Part II: Learning Lagrangian reduced models from high-dimensional data

- *Preserving Lagrangian structure in data-driven reduced-order modeling of large-scale mechanical systems*, Sharma, H. & Kramer, B., Physica D: Nonlinear Phenomena, Vol 462, 134128, 2024.

Why do we need to preserve the Lagrangian structure?



Euler-Bernoulli beam with transverse vibrations in response to a nonzero initial condition.



(a) State error (training data)

(b) FOM energy error

- Unconstrained second-order model learning provides monotonically decaying state error in training
- BUT: predictive capabilties limited due to energy growth and finite-time blowup.

Lagrangian mechanics: simple mechanical systems

Consider a Lagrangian system with a finite-dimensional configuration manifold Q, state space TQ and a Lagrangian $L : \mathsf{TQ} \to \mathbb{R}$. The **forced Euler-Lagrange equations** define the dynamics:

$$\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}.$$

For simple mechanical systems with configuration manifold $\mathsf{Q} = \mathbb{R}^n$:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \underbrace{T(\dot{\mathbf{q}})}_{\text{kinetic energy}} - \underbrace{U(\mathbf{q})}_{\text{potential energy}} = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^\top \mathbf{K}\mathbf{q}, \qquad \mathbf{M} = \mathbf{M}^\top \succ 0, \quad \mathbf{K} = \mathbf{K}^\top.$$

The force is often modeled via a dissipative force and an external time-dependent input as

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = -\mathbf{C}\dot{\mathbf{q}} + \mathbf{B}\mathbf{u}(t),$$

with $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times n}$ the damping and input matrix, $\mathbf{u}(t) \in \mathbb{R}^m$ the time-dependent inputs. The resulting linear equations of motion are

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{B}\mathbf{u}(t).$$

Lagrangian mechanics: nonlinear wave equations

Consider for illustration the class of 1d nonlinear wave equations

$$\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} + \frac{\mathsf{d}U_{\mathsf{nl}}(q)}{\mathsf{d}q} = 0, \tag{1}$$

where $U_{\mathsf{nl}}(q)$ is the nonlinear component of the potential energy. Discretization of the space-time continuous Lagrangian (w. symmetric FDs or pseudo-spectral methods) at n equally spaced points:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{\partial q_i}{\partial t}\right)^2 - \left(\sum_{k=1}^{n} D_{ik} q_k\right)^2\right) - \sum_{i=1}^{n} U_{\mathsf{nl}}(q_i), \qquad \mathbf{q} = [q_1, q_2, \cdots, q_n]^\top$$

where $q_i := q(t, x_i)$, and $\frac{\partial q}{\partial x}(x_i) \approx \sum_{k=1}^{n} D_{ik} q_k$. The Euler-Lagrange equations are

$$\ddot{\mathbf{q}} = \mathbf{K}\mathbf{q} + \frac{\mathsf{d}U_{\mathsf{nl}}(\mathbf{q})}{\mathsf{d}\mathbf{q}}, \quad \mathbf{K} = \mathbf{K}^\top. \tag{2}$$

The nonlinear FOM described by (2) conserves the total energy

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top\mathbf{K}\mathbf{q} + \sum_{i=1}^{n} U_{\mathsf{nl}}(q_i). \tag{3}$$

High-dimensional Lagrangian systems and (data-driven) model reduction

1. Intrusive structure-preserving model reduction for Lagrangian systems ([Lall et al., 2004], [Carlberg et al., 2015])
   Drawback: Requires access to FOM operators
2. Structure-preserving neural networks ([Cranmer et al., 2019], [Lutter et al., 2019], [Gupta et al., 2020])
   Drawback: Ill-suited for high-dimensional systems
3. Nonintrusive model reduction via operator inference (OpInf)
   - Operator inference for nonlinear systems ([Peherstorfer and Willcox, 2016], [Benner et al., 2020])
   - Lift & Learn ([Qian et al., 2020], [Swischuk et al., 2020])

   Drawback: Does not preserve the Lagrangian structure
4. Operator Inference for linear mechanical systems [Filanova et al., MSSP 200 (2023): 110620]. $\Rightarrow$ Similar to our approach, independently derived.

Our contribution: Embed Lagrangian structure into a Operator Inference learning framework

The data we need for learning

Given: Solutions from Lagrangian FOM simulation with inputs and outputs stored in matrices

$$\mathbf{Q} = [\mathbf{q}_1, \cdots, \mathbf{q}_K] \in \mathbb{R}^{n \times K}, \quad \mathbf{Y} = [\mathbf{y}_1, \cdots, \mathbf{y}_K] \in \mathbb{R}^{p \times K}, \quad \mathbf{U} = [\mathbf{u}(t_1), \cdots, \mathbf{u}(t_K)] \in \mathbb{R}^{m \times K}$$

- Compute proper orthogonal decomposition basis via SVD

$$\mathbf{Q} = \mathbf{V}\Xi\mathbf{W}^\top, \qquad \mathbf{V} \in \mathbb{R}^{n \times n}, \Xi \in \mathbb{R}^{n \times n}, \mathbf{W} \in \mathbb{R}^{K \times n}$$

- Project FOM data to obtain reduced snapshot data

$$\hat{\mathbf{Q}} = \mathbf{V}_r^\top \mathbf{Q} = [\hat{\mathbf{q}}_1, \cdots, \hat{\mathbf{q}}_K] \in \mathbb{R}^{r \times K}$$

- Generate or collect reduced time-derivative data

$$\dot{\hat{\mathbf{Q}}} = [\dot{\hat{\mathbf{q}}}_1, \cdots, \dot{\hat{\mathbf{q}}}_K] \in \mathbb{R}^{r \times K}, \qquad \ddot{\hat{\mathbf{Q}}} = [\ddot{\hat{\mathbf{q}}}_1, \cdots, \ddot{\hat{\mathbf{q}}}_K] \in \mathbb{R}^{r \times K}$$

Next step: Fit reduced operators to the projected trajectories in a structure-preserving way

Problem formulation: model form for learning ROM

- Reduced Lagrangian with reduced mass matrix $\hat{\mathbf{M}} = \mathbb{I}_r$

$$\hat{L}_r(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}}) = \frac{1}{2}\dot{\hat{\mathbf{q}}}^\top \dot{\hat{\mathbf{q}}} - \frac{1}{2}\hat{\mathbf{q}}^\top \hat{\mathbf{K}}\hat{\mathbf{q}},$$

- Reduced forcing

$$\hat{\mathbf{f}}(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}}, t) = \hat{\mathbf{C}}\dot{\hat{\mathbf{q}}} - \hat{\mathbf{B}}\mathbf{u}(t)$$

- Model form for learning Lagrangian ROMs based on $\hat{L}_r(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}})$

$$\ddot{\hat{\mathbf{q}}}(t) + \hat{\mathbf{C}}\dot{\hat{\mathbf{q}}}(t) + \hat{\mathbf{K}}\hat{\mathbf{q}}(t) = \hat{\mathbf{B}}\mathbf{u}(t)$$

along with the reduced output equation

$$\mathbf{y}(t) = \hat{\mathbf{E}}\hat{\mathbf{q}}(t)$$

Model form ensures that the reduced models are Lagrangian

Lagrangian Operator Inference for simple mechanical systems

- Constrained optimization problem to compute $\hat{\mathbf{C}} \in \mathbb{R}^{r\times r}, \hat{\mathbf{K}} \in \mathbb{R}^{r\times r}$, and $\hat{\mathbf{B}} \in \mathbb{R}^{r\times m}$

$$\min_{\substack{\hat{\mathbf{K}}=\hat{\mathbf{K}}^\top\succ 0, \hat{\mathbf{C}}=\hat{\mathbf{C}}^\top\succ 0,\\ \hat{\mathbf{B}}}} ||\ddot{\hat{\mathbf{Q}}} + \hat{\mathbf{C}}\dot{\hat{\mathbf{Q}}} + \hat{\mathbf{K}}\hat{\mathbf{Q}} - \hat{\mathbf{B}}\mathbf{U}||_F$$

  where the specific choice of $\hat{\mathbf{M}}$ simplifies the constrained inference problem ([Gosea, Gugercin, and Werner, 2023])
- Separate linear least-squares problem to compute $\hat{\mathbf{E}} \in \mathbb{R}^{p\times r}$

$$\min_{\hat{\mathbf{E}}} ||\mathbf{Y} - \hat{\mathbf{E}}\hat{\mathbf{Q}}||_F$$

- Constrained optimization problem solved using the semidefinite programming mode in CVX[6]

[6] M. Grant and S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.1, 2014

Lagrangian Operator Inference for nonlinear wave equations

$$\mathcal{L}(x, q, q_x, q_t) = \frac{1}{2}\left(\left(\frac{\partial q}{\partial t}\right)^2 - \left(\frac{\partial q}{\partial x}\right)^2\right) - U_{\mathsf{nl}}(q)$$

- Use knowledge about the nonlinear potential energy U_{nl} at the PDE level to build the nonlinear forcing snapshot data matrix

$$\mathbf{F}_{\mathsf{nl}} = [\mathbf{f}_{\mathsf{nl}}(\mathbf{q}_1), \cdots, \mathbf{f}_{\mathsf{nl}}(\mathbf{q}_K)] \in \mathbb{R}^{n \times K}$$

- Projecting FOM snapshot data $\mathbf{Q}$ and forcing snapshot data $\mathbf{F}_{\mathsf{nl}}$

$$\hat{\mathbf{Q}} = \mathbf{V}_r^\top \mathbf{Q} \in \mathbb{R}^{r \times K}, \qquad \hat{\mathbf{F}}_{\mathsf{nl}} = \mathbf{V}_r^\top \mathbf{F}_{\mathsf{nl}} \in \mathbb{R}^{r \times K}$$

- Constrained optimization problem to compute $\hat{\mathbf{K}} \in \mathbb{R}^{r \times r}$

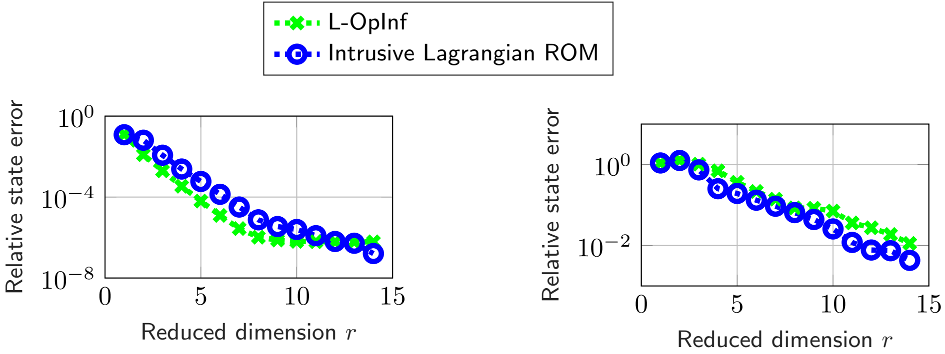
$$\min_{\hat{\mathbf{K}} = \hat{\mathbf{K}}^\top} \|\ddot{\hat{\mathbf{Q}}} - \hat{\mathbf{F}}_{\mathsf{nl}} - \hat{\mathbf{K}}\hat{\mathbf{Q}}\|_F$$

Learned ROM operator $\hat{\mathrm{K}}$ respects the symmetric property introduced during the structure-preserving spatial discretization

Sine-Gordon equation (n=2000): state error

Nonlinear hyperbolic PDE with a nonpolynomial nonlinearity

$$\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} - \sin(q)$$

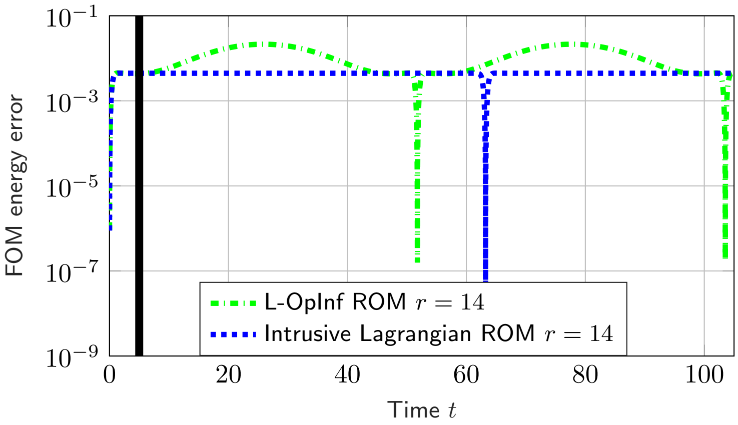


(a) Training regime $[0, 5]$s

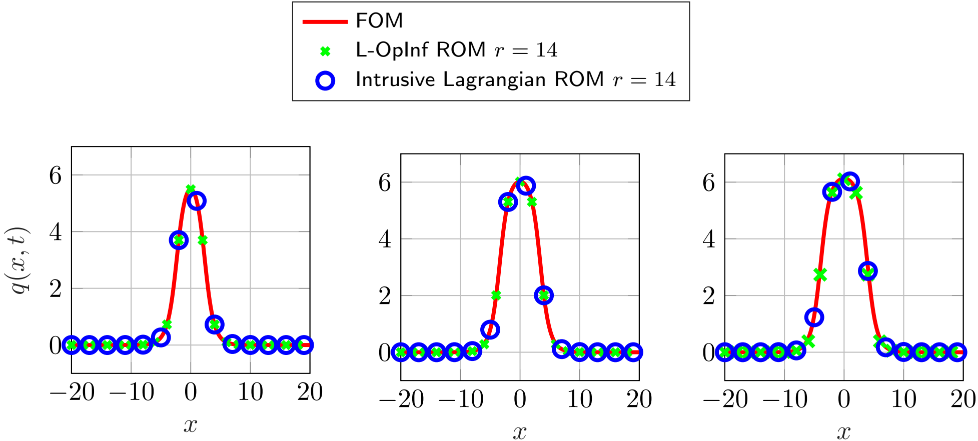
(b) Testing regime $[5, 25]$s

Sine-Gordon equation ($n = 2000$): bounded energy error

Preserving Lagrangian structure yields stable ROMs with bounded energy error far outside the training data regime (here: over 20x past training interval)



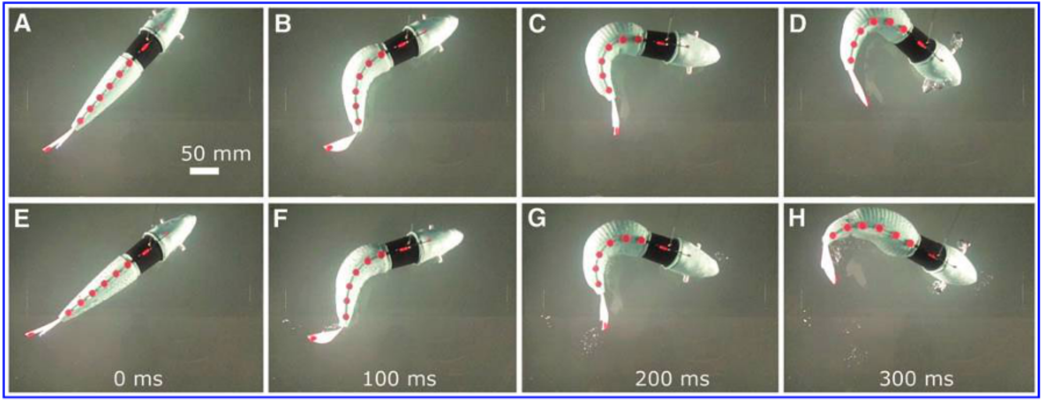
Sine-Gordon equation ($n = 2000$): extrapolation in time



(a) $t = 5$ **(b)** $t = 15$ **(c)** $t = 25$

Accurate predictions 400% outside training time interval

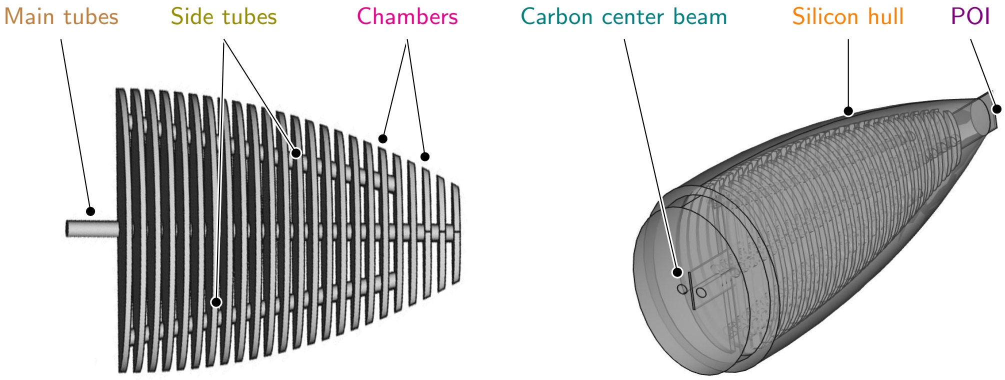
Benchmark soft-robotic fishtail



- Soft robotic fish[7] designed to emulate escape responses in addition to forward swimming because such maneuvers require rapid body accelerations and continuum-body motion
- Fish's soft body is an array of fluidic elastomer actuators

[7] A. D. Marchese, C. D. Onal, and D. Rus, Autonomous soft robotic fish capable of escape maneuvers using fluidic elastomer actuators, Soft Robotics, 1 (2014), pp. 75–87.

Benchmark fishtail CAD model[8]

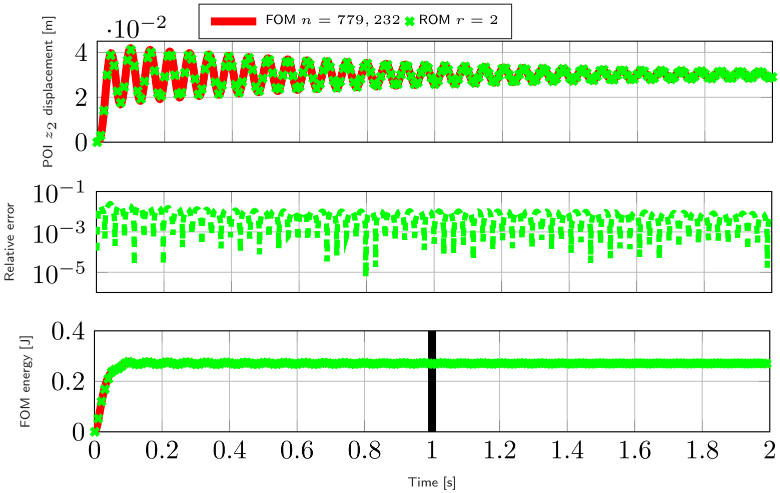


(a) Fluid chamber system **(b)** Complete fishtail model

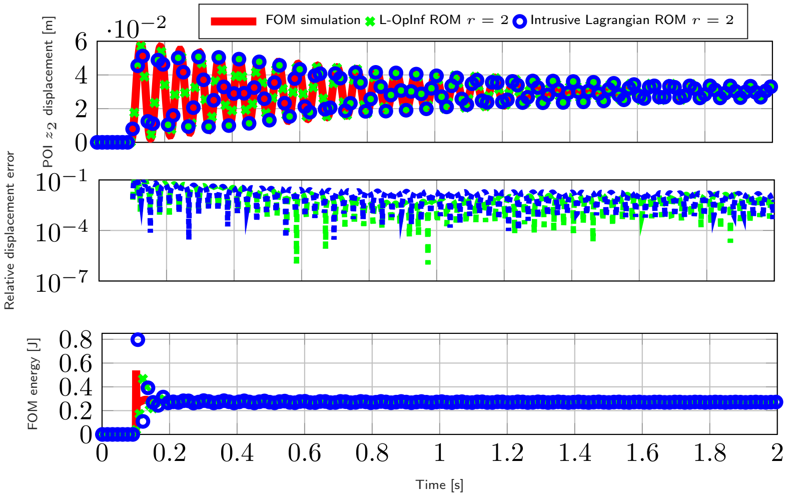
$$\rho \frac{\partial^2 \underline{q}(t, \mathbf{z})}{\partial t^2} = \nabla_{\mathbf{z}} \cdot \underline{\sigma}(t, \mathbf{z})$$

[8] D. Siebelts, A. Kater, and T. Meurer, Modeling and motion planning for an artificial fishtail, IFAC-PapersOnLine, 51 (2018), pp. 319–324. Available at `https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Artificial_Fishtail`

Soft-robotic fishtail ($n = 779,232$): sigmoid input



Soft-robotic fishtail ($n = 779,232$): step input



L-OpInf works well even for unknown control inputs

Full-Body Optimal Control of a Swimming Soft Robot Enabled by Data-Driven Model Reduction

Iman Adibnazari, Harsh Sharma, Jacobo Cervera Torralba, Boris Krämer , Michael T. Tolley
UC San Diego

Collaboration with the Bioinspired Robotics and Design Lab, UCSD (Prof. Mike Tolley)

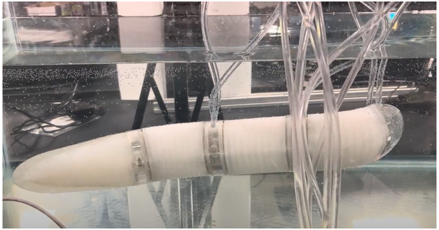
A swimming trout....

Taken from [Beal, D. N. et al (2006)]
What's the catch in the left video?

Video from the Tolley Lab

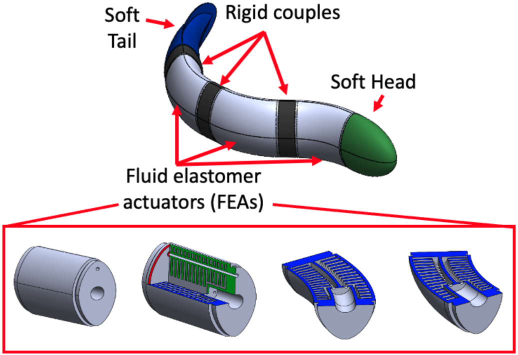
The SERPENT V1 aquatic soft robot

- Autonomous underwater vehicles (AUVs) are mechanically safe and provide silent operation
- Problem: Soft robots are (infinite)-dimensional, so hard to control and simulate.



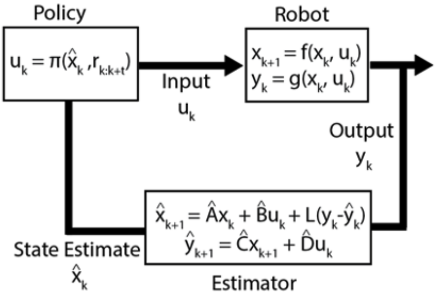
Built "SERPENT V1" from Bioinspired Robotics and Design Lab, UCSD (Prof. Mike Tolley)

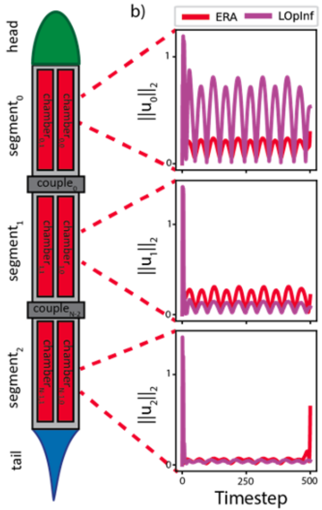
- Segmentation and computational discretization in SOFA ($n = 251,000$)
- $m = 6$ controls: fluid-elastomer actuators
- $p = 40$ outputs: centerline trajectories



Model predictive control w LOPINF for anguilliform swimming

- Control loop requires fast online state estimation
- Discretized model not available ($\rightarrow$ SOFA), need data-driven ROMs
- Lagrangian OPINF due to the second-order nature. Linear for faster online estimation.





Actuation effort in each segment

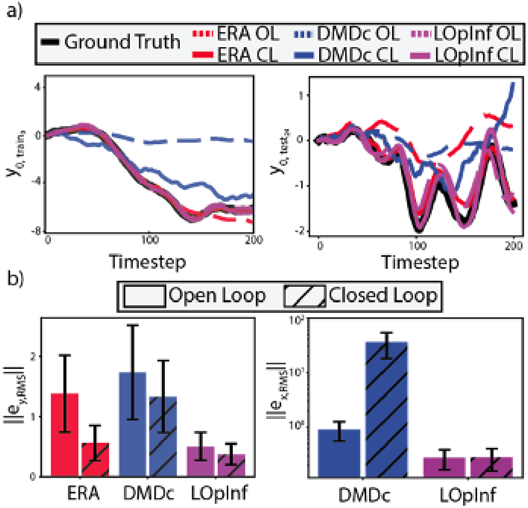
Preliminary results

Compared prevalent linear system identification methods

1. Dynamic Mode Decomposition w/ control (DMDc)
2. Eigensystem Realization Algorithm/Observer/Kalman Identification Algorithm (ERA/OKID)
3. Structure-preserving LOPINF

Current takeaways:

- LOPINF ROM most accurate in RMS error over many trajectories (from random ICs)
- Works well in open and closed-loop
- Certain regimes are too nonlinear ⇒ need to embed nonlinear terms in LOPINF



Nonlinear Lagrangian ROMs

H. Sharma, D. Najera, M. Todd, B. Kramer
UC San Diego

Lagrangian operator inference enhanced with structure-preserving machine learning for nonintrusive model reduction of mechanical systems, *Sharma/Najera/Todd/K.*, CMAME, Vol. 423, 116865, 2024.

Nonlinear Lagrangian mechanical systems

Consider Lagrangian system with a finite-dimensional configuration manifold Q, state space TQ and a Lagrangian $L : \mathsf{TQ} \to \mathbb{R}$. The forced Euler-Lagrange equations define the dynamics:

$$\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}.$$

For simple mechanical systems with configuration manifold $\mathsf{Q} = \mathbb{R}^n$:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \underbrace{T(\dot{\mathbf{q}})}_{\text{kinetic energy}} - \underbrace{U(\mathbf{q})}_{\text{potential energy}} = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^\top \mathbf{K}\mathbf{q} - U_{\mathsf{nl}}(\mathbf{q}), \qquad \mathbf{M} = \mathbf{M}^\top \succ 0$$

We model viscous damping

$$\mathbf{f}(\dot{\mathbf{q}}) = -\mathbf{C}\dot{\mathbf{q}} - \frac{\partial \mathcal{F}_{\mathsf{nl}}(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}},$$

with $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times n}$ the damping and input matrix, $\mathbf{u}(t) \in \mathbb{R}^m$ the time-dependent inputs. The resulting equations of motion are

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \frac{\partial \mathcal{F}_{\mathsf{nl}}(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} + \mathbf{K}\mathbf{q}(t) + U_{\mathsf{nl}}(\mathbf{q}) = \mathbf{B}\mathbf{u}(t).$$

LOPINF enhanced with structure-preserving machine learning ('LOPINF-SpML')

We now have a nonlinear constrained optimization problem

$$\min_{\hat{\mathbf{M}}=\hat{\mathbf{M}}^\top\succ\mathbf{0},\hat{\mathbf{C}}=\hat{\mathbf{C}}^\top\succ\mathbf{0},\hat{\mathbf{K}}=\hat{\mathbf{K}}^\top\succ\mathbf{0},\widehat{\mathcal{F}}_{\mathsf{nl}},\widehat{U}_{\mathsf{nl}}} \left\| (\hat{\mathbf{I}}+\hat{\mathbf{M}})\ddot{\hat{\mathbf{Q}}} + \hat{\mathbf{C}}\dot{\hat{\mathbf{Q}}} + \frac{\partial \widehat{\mathcal{F}}_{\mathsf{nl}}(\dot{\hat{\mathbf{Q}}})}{\partial \dot{\hat{\mathbf{Q}}}} + \hat{\mathbf{K}}\hat{\mathbf{Q}} + \frac{\partial \widehat{U}_{\mathsf{nl}}(\hat{\mathbf{Q}})}{\partial \hat{\mathbf{Q}}} = \mathbf{0}, \right\|_F$$

Our approach: a two-step approach

- Step 1: LOPINF to learn the linear reduced stiffness matrix $\hat{\mathbf{K}}$ and the linear reduced damping matrix $\hat{\mathbf{C}}$ ✓

$$\min_{\hat{\mathbf{K}}=\hat{\mathbf{K}}^\top\succ\mathbf{0},\hat{\mathbf{C}}=\hat{\mathbf{C}}^\top\succ\mathbf{0}} \left\| \ddot{\hat{\mathbf{Q}}} + \hat{\mathbf{C}}\dot{\hat{\mathbf{Q}}} + \hat{\mathbf{K}}\hat{\mathbf{Q}} \right\|_F .$$

- Step 2: Structure-preserving machine learning to learn the reduced mass matrix $\hat{\mathbf{M}}$, the nonlinear components of the reduced potential energy function $\widehat{U}_{\mathsf{nl}}(\hat{\mathbf{q}})$, and the nonlinear components of the reduced dissipation function $\widehat{\mathcal{F}}_{\mathsf{nl}}(\dot{\hat{\mathbf{q}}})$.

Structure-preserving neural networks

Parametrization of the nonlinear terms via polynomial-augmented multilayer perceptrons (MLPs); we choose $(P_1 = P_2 = 4)$

$$\widehat{U}_{\mathsf{NN}}(\hat{\mathbf{q}};\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta}_{\widehat{U}_{\mathsf{NN}}}) = \sum_{i_1,i_2,\cdots,i_r}^{i_1+i_2+\cdots+i_r=P_1} \alpha_{i_1,i_2,\cdots,i_r}\hat{q}_1^{i_1}\hat{q}_2^{i_2}\cdots\hat{q}_r^{i_r} + \sum_{i}^{N}\lambda_i\mathcal{U}^{(i)}(\hat{\mathbf{q}}),$$

$$\widehat{\mathcal{F}}_{\mathsf{NN}}(\dot{\hat{\mathbf{q}}};\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\theta}_{\widehat{\mathcal{F}}_{\mathsf{NN}}}) = \sum_{i_1,i_2,\cdots,i_r}^{i_1+i_2+\cdots+i_r=P_2} \beta_{i_1,i_2,\cdots,i_r}\dot{\hat{q}}_1^{i_1}\dot{\hat{q}}_2^{i_2}\cdots\dot{\hat{q}}_r^{i_r} + \sum_{i}^{N}\gamma_i\mathcal{F}^{(i)}(\dot{\hat{\mathbf{q}}}),$$

We parametrize the reduced kinetic energy term as

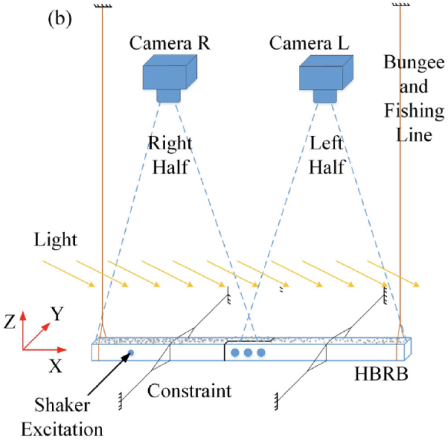
$$\widehat{T}_{\mathsf{NN}}(\dot{\hat{\mathbf{q}}};\boldsymbol{\zeta}) = \sum_{i_1,i_2,\cdots,i_r}^{i_1+i_2+\cdots+i_r=2} \zeta_{i_1,i_2,\cdots,i_r}\dot{\hat{q}}_1^{i_1}\dot{\hat{q}}_2^{i_2}\cdots\dot{\hat{q}}_r^{i_r} \quad\Rightarrow\quad [\hat{\mathbf{M}}_{\mathsf{NN}}]_{ij} = [\hat{\mathbf{M}}_{\mathsf{NN}}]_{ji} = \frac{\partial^2\widehat{T}_{\mathsf{NN}}\left(\dot{\hat{\mathbf{q}}}\right)}{\partial\dot{\hat{q}}_i\partial\dot{\hat{q}}_j}$$

We then minimize the squared loss function under structure-preserving constraints

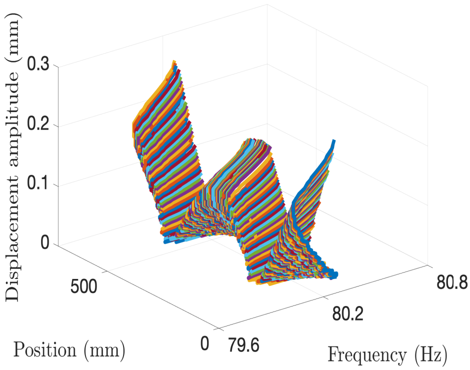
$$\min_{\boldsymbol{\zeta},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\gamma},\boldsymbol{\theta}_{\widehat{U}_{\mathsf{NN}}},\boldsymbol{\theta}_{\widehat{\mathcal{F}}_{\mathsf{NN}}}} \mathcal{J}(\boldsymbol{\zeta},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\gamma},\boldsymbol{\theta}_{\widehat{U}_{\mathsf{NN}}},\boldsymbol{\theta}_{\widehat{\mathcal{F}}_{\mathsf{NN}}}) \quad \text{such that} \quad \frac{1}{2}\dot{\hat{\mathbf{q}}}^\top(\mathbf{I}_r + \hat{\mathbf{M}}_{\mathsf{NN}})\dot{\hat{q}} > 0,\ \widehat{\mathcal{F}}(\dot{\hat{\mathbf{q}}}) \geq 0.$$

Experimental data from the half Brake-Reuss beam

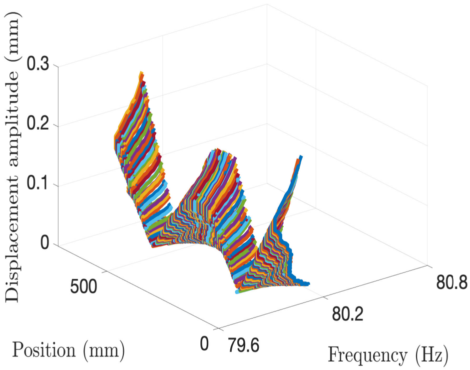
- Modeling of jointed structures remains a challenging problem due to the strong nonlinearities at the frictional interfaces found in joints
- Experimental dataset from [Chen et al., Measurement and identification of the nonlinear dynamics of a jointed structure using full-field data, Part I: Measurement of nonlinear dynamics. MSSP, 2022;166:108401]; high-speed cameras combined with digital image correlation provide the full-field response of the structure
- Repository: `https://github.com/mattiacenedese/BRBtesting`.



Backbone curves: comparison of FOM and LOPINF-SpML



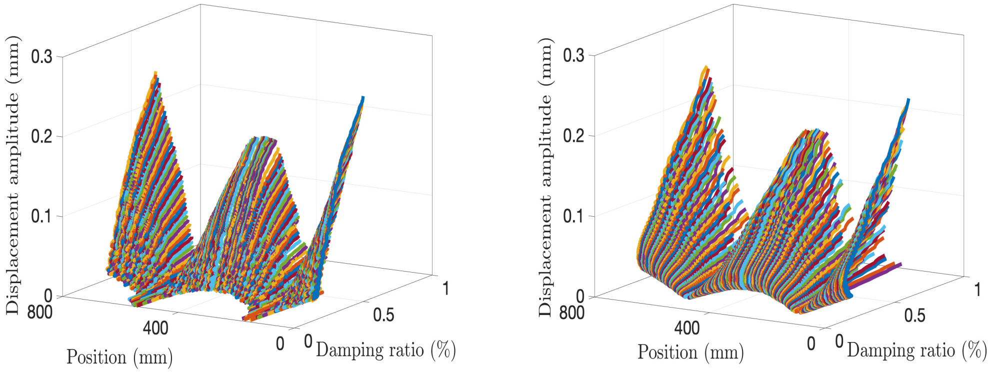
(a) Experimental data



(b) LOpInf-SpML ROM $r = 3$

Figure: Half Brake-Reuß beam. The LOpInf-SpML ROM of size $r = 3$ accurately predicts the amplitude-dependent frequency characteristics and yields backbone curves that appear to agree with the backbone curves obtained directly from the experimental data.

Damping plots: comparison of FOM and LOPINF-SpML



(a) Experimental data

(b) LOpInf-SpML ROM $r = 3$

Figure: Half Brake-Reuß beam. The amplitude-dependent damping plots based on the LOpInf-SpML ROM of dimension $r = 3$ are reasonably similar to the plots obtained from the experimental data.

Today's talk: review and outlook

Review: Embedding geometric/mechanical structure (Hamiltonian, Lagrangian) in model learning creates

- Physically interpretable and analyzable models that engineers are familiar with
- Stable ROMs with bounded energy error
- Accurate long-time predictions far outside the training data regime
- Reduces the need for large training data

Looking ahead:

- Neural networks are expressive, but not as nicely interpretable as symbolic expressions. Good methods for incorporating nonlinearities (polynomial or non-polynomial) in an interpretable manner required (e.g, SINDy, Higher-order OPINF, etc).
- Better methods to extrapolate in parameter space

Papers on Hamiltonian/Gradient systems

1. Hamiltonian operator inference: physics-preserving learning of reduced-order models for Hamiltonian systems. *Sharma/Wang/K.*, Physica D: Nonlinear Phenomena, Vol. 431, 133122, 2022.
2. Gradient preserving operator inference: data-driven reduced-order models for equations with gradient structure. *Geng/Singh/Ju/K./Wang*, CMAME 427, 117033, 2024.
3. Symplectic model reduction of Hamiltonian systems using data-driven quadratic manifolds. *Sharma/Mu/Buchfink/Geelen/Glas/K*. CMAME, 417, 116402, 2023.
4. Bayesian identification of nonseparable Hamiltonians with multiplicative noise using deep learning and reduced-order modeling, *Galioto/Sharma/K./Gorodetsky*, CMAME, 430, 117194, 2024.
5. Bayesian Identification of nonseparable Hamiltonian systems using stochastic dynamic models, *Sharma/Galioto/Gorodetsky/K.* 2022 IEEE 61st Conference on Decision and Control (CDC), 2022, pp. 6742-6749.

Papers on Lagrangian systems

1. Full-Body Optimal Control of a Swimming Soft Robot Enabled by Data-Driven Model Reduction. *Adibnazari/Sharma/Torralba/Kramer/Tolley*, 2023 Southern California Robotics (SCR) Symposium, September 14-15, 2023.
2. Lagrangian operator inference enhanced with structure-preserving machine learning for nonintrusive model reduction of mechanical systems, *Sharma/Najera/Todd/K.*, CMAME, Vol. 423, 116865, 2024.
3. Preserving Lagrangian structure in data-driven reduced-order modeling of large-scale mechanical systems, *Sharma/K.*, Physica D: Nonlinear Phenomena, Vol 462, 134128, 2024.