Obliquely reflecting diffusions in curved, nonsmooth domains

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joint work with Thomas G. Kurtz

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Some examples from stochastic networks

W.N. Kang and R. J. Williams (2012): Diffusion approximation for an input-queued switch operating under a maximum weight matching policy, Stochastic Systems, 2, 277-321

For some values of the parameters, the conjectured diffusion approximation for the workload process is an obliquely reflecting Brownian motion in a piecewise smooth, nonpolyhedral cone

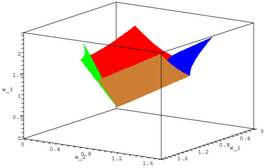


FIG 4. *A portion of the workload cone* $\mathcal{W}_2$ *is shown for a* 2×2 *input-queued switch with* $\alpha = 2$.

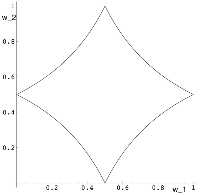


FIG 5. *A cross-section of the workload cone* $\mathcal{W}_2$ *depicted in Figure 4 taken at* $w_3 = 1$.

Some examples from stochastic networks

W.N. Kang, F.P. Kelly, N.H. Lee and R.J. Williams (2009): State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy, The Annals of Applied Probability, 19, 1719-1780

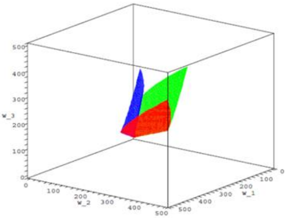


FIG. 8. *A portion of the workload cone $\mathcal{W}_{0.5}$ is shown for a linear network with three resources and four routes with $\alpha = 0.5$ and $\nu_i = \mu_i = \kappa_i = 1$ for all $i \in \mathbb{I}$.*

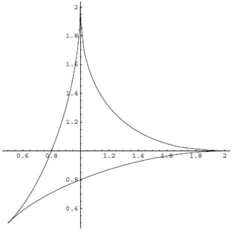
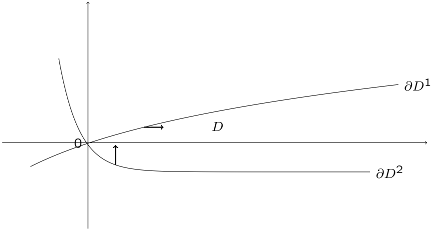


FIG. 9. *A cross-section of the workload cone $\mathcal{W}_{0.5}$ depicted in Figure 8 taken at $w_3 = 1$.*

An example from singular stochastic control

S.A. Williams, P-L. Chow and J-L. Menaldi (1994): Regularity of the free boundary in singular stochastic control, Journal of Differential Equations 111, 175-201

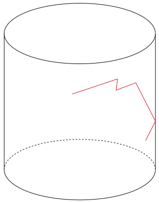
The candidate for the optimally controlled process is an obliquely reflecting Brownian motion in a curved, piecewise smooth domain



$$D = D^1 \cap D^2, \quad \partial D^1 = \{x : x_1 = \psi_1(x_2)\}. \quad \partial D^2 = \{x : x_2 = \psi_2(x_1)\}.$$

Examples from diffusion approximation of transport processes

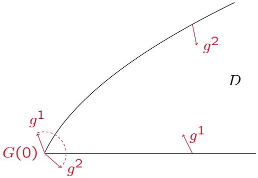
Transport processes describe particle behaviour in many areas of physics and chemistry



A. Bensoussan, J.L. Lions and G.C. Papanicolaou (1979): Boundary layers and homogenization of transport processes, Publ. RIMS, Kyoto University, 15, 53-157

C. - T.G. Kurtz (2006): Diffusion approximation for transport processes with general reflection boundary conditions, Math. Models Methods Appl. Sci., 5, 717-762

Semimartingale obliquely reflecting diffusions



Stochastic Differential Equation with Reflection (SDER)

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)d\lambda(s),$$

$$X(s) \in \overline{D}, \quad \gamma(s) \in G(X(s)),\ G(x) \text{ a cone}, \quad |\gamma(s)| = 1,\ d\lambda - a.e.,$$

$$\lambda \text{ nondecreasing, continuous}, \quad d\lambda\big(\{s \le t : X(s) \in \partial D\}\big) = \lambda(t),$$

X is a solution if there exist W, γ, λ s.t. SDER is satisfied

For non semimartingale reflecting diffusions: Varadhan-Williams (1984), Williams (1985), Kwon-Williams (1981), Ramanan (2006), Ramanan-Reiman (2008), Kang-Ramanan (2010), Lakner-Reed-Zwart (2017), Atar-Budhiraja (2024), etc.

For normal reflection: Tanaka (1979), Saisho (1987), Z.-Q. Chen (1993), Bass-Hsu (2000), Bass-Burdzy (2006, 2008), etc.

Strong existence and pathwise uniqueness via the Skorohod problem

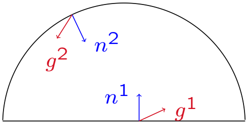
- **Harrison and Reiman (1981)**: for Brownian motion in an orthant with constant, oblique direction of reflection on each face pointing towards the origin, under the condition that the spectral radius of the identity minus the reflection matrix is strictly less than 1
- Lions and Sznitman (1984): in a domain that can be approximated by smooth domains, with varying oblique directions of reflection, for a very specific class of directions of reflection
- C. (1992): in a piecewise $\mathcal{C}^1$ domain with varying, oblique directions of reflection, some existence and compactness results but no uniqueness
- **Dupuis and Ishii (1993)**: in a piecewise $\mathcal{C}^1$ domain with varying, oblique directions of reflection, under the the condition that, for each point x on the boundary there exists a certain compact, convex set defined in terms of the cone of normal directions and the cone of directions of reflection at x. In the orthant, strictly more general than the Harrison-Reiman condition

Lipschitz continuity for the Skorohod problem

- **Dupuis and Ishii (1991)**: in a convex polyhedron with constant directions of reflection on each face, a very similar condition is sufficient for the Skorohod map to be Lipschitz continuous
- **Dupuis and Ramanan (1999, part I)**: reformulates the Dupuis and Ishii (1991) condition in a way that sheds light on its meaning and makes it easier to verify
- **Dupuis and Ramanan (1999, part II)**: shows how to use the new formulation, establishes Lipschitz continuity for the Skorohod problem that arises from a generalized processor sharing model

Dupuis and Ishii (1993) is very general, but still leaves out:

- situations where the directions of reflection are very oblique, but "compensating":



$$|\text{angle}(g^1, n^1)| = |\text{angle}(g^2, n^2)| = \text{constant} \geq \frac{\pi}{4}$$

- piecewise smooth cones
- domains with cusps

Weak existence and uniqueness in distribution

- **Varadhan and Williams (1984)**: for Brownian motion in a wedge, with oblique, constant directions of reflection, characterized as the solution of a submartingale problem; Williams (1985) gives a necessary and sufficient condition for the solution to be a semimartingale
- **Kwon and Williams (1991)**: for Brownian motion in a smooth cone in $\mathbb{R}^d$, with radially constant direction of reflection, characterized as a solution of a submartingale problem; C. and Kurtz (2024a) give a sufficient condition for the solution to be a semimartingale.
- **Dai and Williams (1995)**: for Brownian motion in a convex polyhedron in $\mathbb{R}^d$, with constant direction of reflection on each face, under a generalization of the completely-$\mathcal{S}$ condition used in Taylor and Williams (1993) for an orthant. This condition allows to deal with very oblique but "compensating" directions of reflection. Moreover, it is necessary for existence.

Question:

In a non polyhedral domain, can one obtain weak existence and uniqueness under some "generalized completely-$\mathcal{S}$ condition"?

- Yes for a d-dimensional domain with only one singular point (C.-Kurtz 2024a)
- Yes for a 2-dimensional, piecewise $\mathcal{C}^{1,1}$ domain (C.-Kurtz 2024b), allowing cusps (C.-Kurtz 2018)
- Yes for a piecewise smooth cone, under some assumptions (C. 2024)

Key tools:

- constrained martingale problems for existence
- a new reverse ergodic theorem for inhomogeneous killed Markov chains for uniqueness

Piecewise smooth domains in $\mathbb{R}^2$

D bounded, connected, open set in $\mathbb{R}^2$

Define
$$N(x) := \left\{ n : \liminf_{y \in \overline{D},\, y \to x} \frac{(y-x)}{|y-x|} \cdot n \geq 0 \right\}. \quad x \in \partial D$$

Assume D admits the following representation:
$$D = \bigcap_{i=1}^{m} D^i, \quad \partial D^i \in \mathcal{C}^{1,1},$$

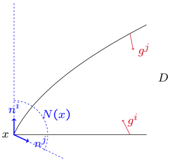
and, defining
$$I(x) := \{ i : \, x \in \partial D^i \}, \quad x \in \partial D,$$

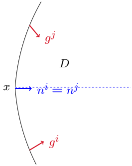
$$|I(x)| \leq 2 \quad \text{and the set of "corners" } \{x \in \partial D : \, |I(x)| = 2\} \text{ is finite.}$$

$$g^i(x) \text{ direction of reflection at } x \in \partial D^i, \qquad \inf_{x \in \partial D^i} g^i(x) \cdot n^i(x) > 0.$$

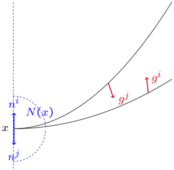
$$G(x) \text{ the closed, convex cone generated by } \{ g^i(x),\, i \in I(x) \}$$

Cone points and cusp points





$N(x)$ does not contain any full straight line: cone case.



$N(x)$ contains a full straight line: cusp case.

In the cusp case we assume that the contact that ∂D^i and ∂D^j have between themselves is of order not higher than each of them has with their common tangent.

Uniqueness for SDER in a piecewise smooth domain in $\mathbb{R}^2$

g^i, b, σ Lischitz continuous, $\sigma(x)$ nonsingular at every corner $x \in \partial D$

Theorem (C. - Kurtz 2024b)

Assume for every $x \in \partial D$, there exists $e \in N(x)$ such that

$$e \cdot g > 0, \quad \forall g \in G(x) - \{0\}.$$

Then, for every initial condition $X_0 \in \overline{D}$, the solution of SDER is unique in distribution.

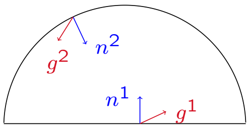
Remark

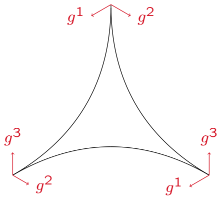
In the case of obliquely reflecting BM in a convex polygon with constant directions of reflection, our condition coincides with that of Dai and Williams (1995) which is necessary for existence: in this sense it is optimal.

Remark

In the case of obliquely reflecting BM in a cusp with directions of reflection as in De Blassie and Toby (1993a, 1993b), our condition coincides with theirs.

Piecewise smooth domains in $\mathbb{R}^2$: examples





Piecewise smooth domains in $\mathbb{R}^2$: reductions

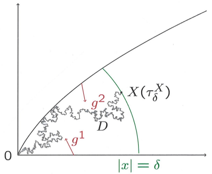
- by the localization results of C. - Kurtz (2024b), we can reduce to consider a domain that is smooth except at one point (taken to be 0); then the proof follows the approach of Kwon and Williams (1991)
- by C. - Kurtz (2019), there exist strong Markov solutions and it is enough to prove uniqueness among them
- by Dupuis and Ishii (1993), starting at $x \neq 0$ the distribution of any solution is the same up to the first time 0 is hit $\implies$ consider only solutions starting at 0
- for any solution X

$$\mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{0\}}(X(t))\, dt\right] = 0,$$

(the assumption on the vector e enters here implicitely).

Uniqueness follows from uniqueness of the exit distribution

- for every bounded neighborhood U_δ of 0,



$$\tau_\delta^X := \inf\{t \geq 0 : X(t) \in \partial U_\delta\},$$

the assumption on the vector e ensures that, for any solution X,

$$\mathbb{E}[\tau_\delta^X] < \infty.$$

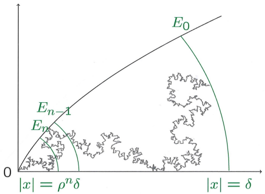
- for two strong Markov solutions starting at 0, X and $\widetilde{X}$, if, for some family $\{U_\delta\}$ of neighborhoods of 0, shrinking to $\{0\}$ as $\delta \to 0$,

$$\mathcal{L}\big(X(\tau_\delta)\big) = \mathcal{L}\big(\widetilde{X}(\widetilde{\tau}_\delta)\big) \quad \forall \delta \qquad (\tau_\delta := \tau_\delta^X,\ \widetilde{\tau}_\delta := \tau_\delta^{\widetilde{X}})$$

then

$$\mathcal{L}(X) = \mathcal{L}(\widetilde{X}).$$

Uniqueness of the exit distribution: intuition



$E_0 := \partial U_\delta, \quad \tau_k := \inf\{t \geq 0 : X(t) \in E_k\}, \quad \tau_n < \tau_{n-1} < \ldots < \tau_0 = \tau_\delta$

- consider the Markov chain

$$\xi_0 := X(\tau_n), \quad \xi_1 := X(\tau_{n-1}), \quad \ldots, \quad \xi_n := X(\tau_0) = X(\tau_\delta)$$

  killed if X reaches the origin before the next layer
- for any two solutions X and $\widetilde{X}$, the two killed Markov chains $\{\xi_h\}$ and $\{\widetilde{\xi}_h\}$ have the same transition kernels, the difference between them is only in their initial distributions on E_n
- if the family of the transition kernels is "ergodic", then as $n \to \infty$ the initial distributions will be "forgotten" and

$$\mathcal{L}\big(X(\tau_\delta)\big) = \lim_{n\to\infty} \mathcal{L}\big(\xi_n\big) = \lim_{n\to\infty} \mathcal{L}\big(\widetilde{\xi}_n\big) = \mathcal{L}\big(\widetilde{X}(\widetilde{\tau}_\delta)\big)$$

A reverse ergodic theorem for inhomogeneous killed MCs

$X, \widetilde{X}$ two strong Markov solutions starting at 0,

$$Q_k(x, C) := \mathbb{P}\big(\tau^x_{k-1} < \vartheta^x,\, X^x(\tau^x_{k-1}) \in C\big) = \mathbb{P}\big(\widetilde{\tau}^x_{k-1} < \widetilde{\vartheta}^x,\, \widetilde{X}^x(\widetilde{\tau}^x_{k-1}) \in C\big),$$

$$x \in E_k, \quad C \subseteq E_{k-1}, \quad X^x, \widetilde{X}^x \text{ starting at } x,$$

$$\vartheta^x := \inf\{t \geq 0 : X^x(t) = 0\}, \quad \widetilde{\vartheta}^x := \inf\{t \geq 0 : \widetilde{X}^x(t) = 0\}$$

One can prove

$$\mathbb{E}\big[f(X(\tau_\delta))\big] = \mathbb{E}\big[f(\xi_n)\big] = \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\mu_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\mu_n(dx)}, \; \mu_n(C) = \mathbb{P}(X(\tau_n) \in C)$$

$$\mathbb{E}\big[f(\widetilde{X}(\widetilde{\tau}_\delta))\big] = \mathbb{E}\big[f(\widetilde{\xi}_n)\big] = \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\widetilde{\mu}_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\widetilde{\mu}_n(dx)}, \; \widetilde{\mu}_n(C) = \mathbb{P}(\widetilde{X}(\widetilde{\tau}_n) \in C)$$

Goal:

$$\lim_{n\to\infty} \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\ \mu_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\ \mu_n(dx)} \text{ is independent of } \{\mu_n\}$$

A reverse ergodic theorem for inhomogeneous killed MCs

Theorem (C. - Kurtz 2024a)

$E_0, \dots E_n, \dots$ a sequence of compact metric spaces, Q_k a subprobability transition kernel from E_k to E_{k-1}

$f_{k,\tilde{x}}(x,\cdot)$ the Radon-Nykodim derivative of $Q_k(x,\cdot)$ w.r.t. $\big(Q_k(x,\cdot)+Q_k(\tilde{x},\cdot)\big)$

$$\epsilon_k(x,\tilde{x}) := \int \big(f_{k,\tilde{x}}(x,y) \wedge f_{k,x}(\tilde{x},y)\big)\big(Q_k(x,dy)+Q_k(\tilde{x},dy)\big), \quad x,\tilde{x} \in E_k.$$

Assume Q_k is not identically zero and there exist $c_0 > 0$ and $\epsilon_0 > 0$ such that

(i) $\inf_{x,\tilde{x}\in E_k} \epsilon_k(x,\tilde{x}) \geq \epsilon_0$, $\forall k$,

(ii) $\inf_{x,\tilde{x}\in E_n} \big(Q_n \cdots Q_1\big)(x,E_0)/\big(Q_n \cdots Q_1\big)(\tilde{x},E_0) \geq c_0$, $\forall n$.

Then $\inf_{x\in E_n} Q_n \cdots Q_1 \mathbf{1}(x) > 0$ and, for every $f \in \mathcal{C}(E_0)$, $\{\mu_n\}$, $\mu_n \in \mathcal{P}(E_n)$, the limit

$$\lim_{n\to\infty} \frac{\int Q_n \cdots Q_1 f(x)\mu_n(dx)}{\int Q_n \cdots Q_1 \mathbf{1}(x)\mu_n(dx)}$$

exists and is independent of $\{\mu_n\}$.

Ergodic theorem: interpretation of the assumptions

- (i) is a condition on the one-step transition kernel Q_k. For $x, \widetilde{x} \in E_k$,

$$\epsilon_k(x, \widetilde{x}) \geq Q_k(x, E_{k-1}) \vee Q_k(\widetilde{x}, E_{k-1}) - \|Q_k(x, \cdot) - Q_k(\widetilde{x}, \cdot)\|_{TV},$$

  hence the condition is satisfied if

$$\|Q_k(x, \cdot) - Q_k(\widetilde{x}, \cdot)\|_{TV} \leq Q_k(x, E_{k-1}) \vee Q_k(\widetilde{x}, E_{k-1}) - \epsilon_0, \quad \forall x, \widetilde{x} \in E_k, \ \forall k.$$

  We obtain this inequality by a scaling result and a coupling lemma (C. - Kurtz 2018)

- (ii) is a condition on the n-step transition kernel $Q_n \cdots Q_1$. In our case

$$\inf_{x, \widetilde{x} \in E_n} \frac{\left(Q_n \cdots Q_1\right)(x, E_0)}{\left(Q_n \cdots Q_1\right)(\widetilde{x}, E_0)} = \inf_{x, \widetilde{x} \in E_n} \frac{\mathbb{P}\left(\tau_\delta^x < \vartheta^x\right)}{\mathbb{P}\left(\tau_\delta^{\widetilde{x}} < \vartheta^{\widetilde{x}}\right)} \geq c_0, \quad \forall n.$$

  Note that, when 0 can be reached,

$$\mathbb{P}\left(\tau_\delta^x < \vartheta^x\right) \rightarrow_{n \to \infty} 0, \quad \forall x \in E_n.$$

One-step transition kernel: scaling for the cone case

$$U_\delta := \{x \in \overline{D} : |x| < \delta\}, \quad E_k := \{x \in \overline{D} : |x| = \rho^k \delta\}, \quad 0 < \rho < 1.$$

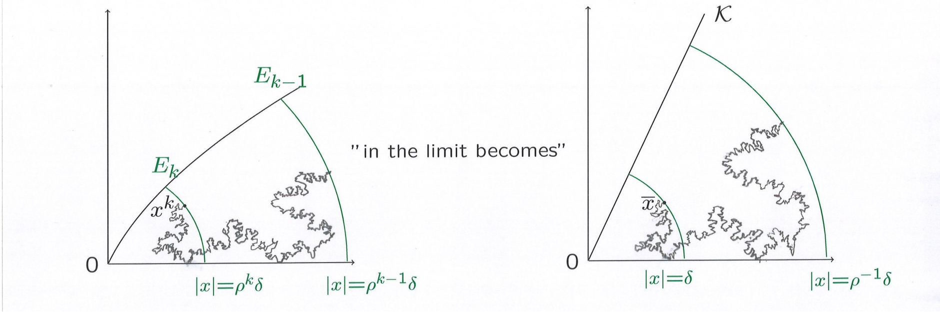
Let $x^k \in E_k$ be s.t. $\rho^{-k} x^k \to_{k\to\infty} \bar{x}$. Then

$$\rho^{-k} X^{x^k}(\rho^{2k} \cdot) \xrightarrow{\mathcal{L}}_{k\to\infty} \overline{X}^{\bar{x}},$$

where $\overline{X}$ is the Reflecting Brownian Motion in the "tangent cone"

$$\mathcal{K} := \{x \in \mathbb{R}^2 : x \cdot n^1(0) > 0, \, x \cdot n^2(0) > 0\},$$

with directions of reflection $g^1(0)$, $g^2(0)$ and coefficients $b = 0$, $\sigma = \sigma(0)$.

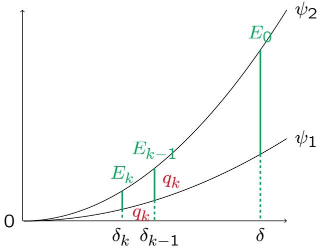


One-step transition kernel: scaling for the cusp case

Same argument as in the cone case, but different choice of U_δ and $\{E_k\}$:

$$U_\delta := \{x \in \overline{D} : x_1 < \delta\}, \qquad E_0 := \{x \in \overline{D} : x_1 = \delta\}$$

$$E_k := \{x \in \overline{D} : x_1 = \delta_k\}$$



$$\delta_1 := \delta - q_1, \quad q_1 := \psi_2(\delta) - \psi_1(\delta),$$

$$\delta_k := \delta_{k-1} - q_k, \quad q_k := \psi_2(\delta_{k-1}) - \psi_1(\delta_{k-1}).$$

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One-step transition kernel: scaling for the cusp case

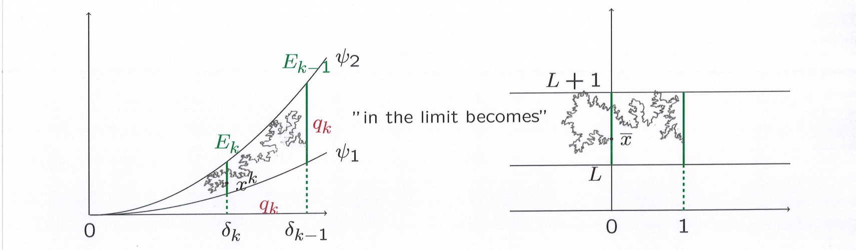
Let $x^k \in E_k$ be s.t. $q_k^{-1} x_2^k \to_{k\to\infty} \bar{x}_2$ ($q_k^{-1} x_1^k = q_k^{-1}\delta_k \to_{k\to\infty} \infty$). Then

$$q_k^{-1}\left(X_1^{x^k}(q_k^2\cdot) - \delta_k, X_2^{x^k}(q_k^2\cdot)\right) \xrightarrow{\mathcal{L}}_{k\to\infty} \overline{X}^{\bar{x}},$$

where $\overline{X}$ is the Reflecting Brownian Motion in the infinite strip

$$\{x \in \mathbb{R}^2 : L < x_2 < L+1\}, \quad L := \lim_{x_1\to 0^+} \frac{\psi_1(x_1)}{\psi_2(x_1) - \psi_1(x_1)},$$

with directions of reflection $g^1(0)$, $g^2(0)$ and coefficients $b = 0$, $\sigma = \sigma(0)$.



n-step transition kernel: Lyapunov functions

$$Af(x) := \nabla f(x) \cdot b(x) + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma(x)^T D^2 f(x)\big)$$

$$\inf_{x,\widetilde{x}\in E_n} \frac{\mathbb{P}\big(\tau_\delta^x < \vartheta^x\big)}{\mathbb{P}\big(\tau_\delta^{\widetilde{x}} < \vartheta^{\widetilde{x}}\big)} \geq c_0, \quad \forall n,$$

if $\inf_{x\in U_\delta - \{0\}} \mathbb{P}\big(\tau_\delta^x < \vartheta^x\big) < 1$. In this case, if v is a solution of

$$\begin{aligned} Av(x) &= 0, & x &\in D \cap U_\delta, \\ \nabla v(x) \cdot g^i(x) &= 0, & x &\in \partial D^i \cap U_\delta - \{0\}, \end{aligned}$$

continuous on $\overline{D \cap U_\delta}$ and such that $v(0) = 0$, $v(x) = 1$ for $x \in \partial U_\delta$, then

$$\mathbb{P}\big(\tau_\delta^x < \vartheta^x\big) = v(x).$$

Key observation: we only need to bound $\mathbb{P}\big(\tau_\delta^x < \vartheta^x\big)$ from above and from below, hence it is sufficient to find V_- and V_+ such that

$$\begin{aligned} &AV_-(x) \leq 0, & &AV_+(x) \geq 0, & &x \in D \cap U_\delta, \\ &\nabla V_-(x) \cdot g^i(x) \leq 0, & &\nabla V_+(x) \cdot g^i(x) \geq 0, & &x \in \partial D^i \cap U_\delta - \{0\}. \end{aligned}$$

This is always possible for a domain with one singular point under our assumptions (in arbitrary dimension).

Piecewise smooth cones in $\mathbb{R}^d$

$$D = \bigcap_{i=1}^{m} D^i, \quad D^i := \{x = rz,\, z \in \mathcal{S}^i,\, r > 0\},$$

$$\mathcal{S}^i \quad \text{a domain in the unit sphere } S^{d-1}, \quad \partial\mathcal{S}^i \in \mathcal{C}^2,$$

$$n^i(x) = n^i(z) = n^{\overline{\mathcal{S}^i}}(z), \quad z := x/|x| \in \partial\mathcal{S}^i.$$

Defining

$$I(x) := \{i : \; x \in \partial D^i\},$$

assume the vectors $\{n^i(x)\}_{i\in I(x)}$ are linearly independent, $x \in \partial D - \{0\}$.

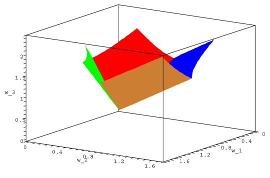


FIG 4. *A portion of the workload cone $\mathcal{W}_2$ is shown for a 2×2 input-queued switch with $\alpha = 2$.*

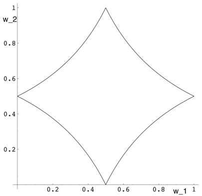


FIG 5. *A cross-section of the workload cone $\mathcal{W}_2$ depicted in Figure 4 taken at $w_3 = 1$.*

Lyapunov functions assumption

For some $\delta > 0$, either of the following is satisfied:

(i) there exists a function $V \in \mathcal{C}^2(\overline{D} - \{0\})$ such that

$$AV(x) \leq 0, x \in D - \{0\}, \ |x| < \delta,$$
$$\nabla V(x) \cdot g \leq 0, g \in G(x), \ x \in \partial D - \{0\}, \ |x| < \delta,$$
$$\lim_{x \in \overline{D}, x \to 0} V(x) = \infty;$$

(ii) there exist two functions $V_+, V_- \in \mathcal{C}^2(\overline{D} - \{0\})$ such that

$$\begin{array}{lll} AV_-(x) \leq 0, & AV_+(x) \geq 0, & x \in D - \{0\}, \ |x| < \delta, \\ \nabla V_-(x) \cdot g \leq 0, & \nabla V_+(x) \cdot g \geq 0, & g \in G(x), \ x \in \partial D - \{0\}, \ |x| < \delta, \end{array}$$

$$V_+(x) > 0, \quad V_-(x) > 0, \quad \text{for } x \in \overline{D} - \{0\}, \ |x| < \delta,$$
$$\lim_{x \in \overline{D}, x \to 0} V_+(x) = \lim_{x \in \overline{D}, x \to 0} V_-(x) = 0,$$
$$\inf_{0 < r \leq \delta} \frac{\inf_{|x|=r} V_+(x)}{\sup_{|x|=r} V_-(x)} > 0, \qquad \inf_{0 < r \leq \delta} \frac{\inf_{|x|=r} V_-(x)}{\sup_{|x|=r} V_+(x)} > 0.$$

Uniqueness for ORBM in a piecewise smooth cone in $\mathbb{R}^d$

$$g^i \text{ radially constant: } g^i(x) = g^i(z), \quad z := x/|x| \in \partial\mathcal{S}^i$$

$$g^i \text{ Lipschitz continuous,} \qquad \inf_{x\in\partial D^i} g^i(x)\cdot n^i(x) > 0,$$

$$b,\ \sigma \text{ constant,} \quad \sigma \text{ nonsingular}$$

Theorem (C. 2024)

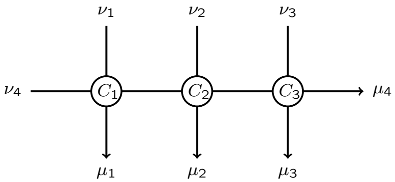
Suppose the Lyapunov functions assumption is verified and

a) *for $x \in \partial D - \{0\}$, the Dupuis and Ishii (1993) condition is satisfied*
b) *there exists $e \in N(0)$ such that*

$$e\cdot g > 0, \quad \forall g \in G(0) - \{0\}.$$

Then, for every initial condition $X_0 \in \overline{D}$, the solution of SDER is unique in distribution.

An example from bandwidth sharing networks



$$A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C := \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Poisson arrivals (ν_i^r), documents with i.i.d. exponential lengths (μ_i^r),

$\Lambda_i(n_1, n_2, n_3, n_4)$ fraction of the capacity of each resource allocated to route i

$\Lambda_i(\cdot)$ is determined by solving an optimization problem
α the exponent in the reward function of the optimization problem

heavy traffic:

$$\nu^r \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; \mu^r \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ \mu \end{bmatrix}, \quad \rho_i^r := \nu_i^r / \mu_i^r,$$

$$r\big(A\rho^r - C\big) \to b, \qquad \text{as } r \to \infty.$$

An example from bandwidth sharing networks

Kang, Kelly, Lee and Williams (2009) prove the state space collapse and conjecture that the diffusion approximation of the rescaled workload process is an obliquely reflecting Brownian motion in a piecewise smooth cone D:

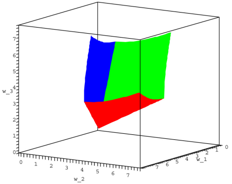


FIG. 6. *A portion of the workload cone $\mathcal{W}_2$ is shown for a linear network with three resources and four routes with $\alpha = 2$ and $\nu_i = \mu_i = \kappa_i = 1$ for all $i \in \mathbb{I}$.*

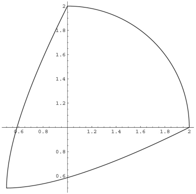


FIG. 7. *A cross-section of the workload cone $\mathcal{W}_2$ depicted in Figure 6 taken at $w_3 = 1$.*

$$\partial_1 D = \text{green face}, \quad \partial_2 D = \text{blue face}, \quad \partial_3 D = \text{red face}$$

D depends on α and μ

$$g^1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ on } \partial_1 D, \quad g^2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ on } \partial_2 D, \quad g^3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ on } \partial_3 D.$$

An example from bandwidth sharing networks

$$\sigma := 2 \begin{bmatrix} \left(1+\frac{1}{\mu^2}\right) & 1 & 1 \\ 1 & \left(1+\frac{1}{\mu^2}\right) & 1 \\ 1 & 1 & \left(\mu^2+1\right) \end{bmatrix}.$$

For $\mu > \sqrt{3(1+\sqrt{2})}$ the Lyapunov functions assumption (i) is satisfied by

$$V(x) := -\ln(|x|).$$

Theorem (C. 2024)

If $\alpha \geq 2$ and $\mu > \sqrt{3(1+\sqrt{2})}$, the obliquely reflecting Brownian motion in D with b, σ and g^i, $i = 1, 2, 3$ as above is uniquely determined in distribution.

Starting at 0, it immediately leaves 0, with probability one.

Starting at $x \neq 0$, it never reaches 0, with probability one.

Thank you for your attention!

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Existence for piecewise smooth domains in $\mathbb{R}^d$

D bounded, connected, open set in $\mathbb{R}^d$

Assume D admits a representation $D = \bigcap_{i=1}^m D^i$, $\partial D^i \in \mathcal{C}^1$. such that

$$N(x) = \{ \sum_{i \in I(x)} \eta_i n^i(x),\ \eta_i \geq 0 \}, \quad \forall x \in \partial D.$$

For $y \in \overline{D}^c$ let

$$I(y) := \{ i : y \notin D^i \},$$

For every $x \in \partial D$ there is $\delta(x) > 0$ such that $I(y) \subseteq I(x)$ for $y \in B_{\delta(x)}(x)$.

For $x \in \partial D$, define the following family of subsets of $I(x)$

$$\mathcal{I}(x) := \left\{ I \subseteq I(x) : \ I = I(y) \text{ for some } y \in \overline{D}^c \cap B_{\delta(x)}(x) \right\},$$

and the subcones

$$N^I(x) := \{ \sum_{i \in I} \eta_i n^i(x),\ \eta_i \geq 0 \}, \quad G^I(x) := \{ \sum_{i \in I} \eta_i g^i(x),\ \eta_i \geq 0 \}, \quad I \in \mathcal{I}(x).$$

Existence for piecewise smooth domains in $\mathbb{R}^d$

Theorem (C. - Kurtz 2019)

b, σ, g^i continuous, $\inf_{x \in \partial D^i} g^i(x) \cdot n^i(x) > 0$

a) *for every $x \in \partial D$, there exists $e \in N(x)$ such that*

$$e \cdot g > 0, \quad \forall g \in G(x) - \{0\}$$

b) *for every $x \in \partial D$, for every $I \in \mathcal{I}(x)$, $N^I(x)$ does not contain any full straight line and for every $n \in N^I(x) - \{0\}$ there is $v \in G^I(x)$ such that*

$$n \cdot v > 0$$

Then, for every initial condition $X_0 \in \overline{D}$, there exists a strong Markov solution of SDER.
If uniqueness in distribution holds among strong Markov solutions of SDER then it holds among all solutions.

Remark

In the case of a simple, convex polyhedron, our conditions are equivalent to those of Dai and Williams (1995)

Existence for piecewise smooth domains in $\mathbb{R}^d$

Keypoints of proof

- X is a solution of SDER if and only if X is a natural solution of the corresponding constrained martingale problem (introduced by Kurtz (1987) and (1989))
- One can construct a natural solution of the constrained martingale problem by a limiting procedure without proving oscillation estimates.

One can formulate constrained martingale problems for all sorts of boundary behaviour (Wentzell boundary conditions, jumps from the boundary, etc.)

Constrained martingale problem

$$Af(x) := \nabla f(x) \cdot b(x) + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma^T(x)D^2 f(x)\big)$$

$$\Xi := \{(x,u) \in \partial D \times \mathbb{R}^d : u \in G(x),\ |u| = 1\}, \quad Bf(x,u) := \nabla f(x) \cdot u$$

Constrained martingale problem (Kurtz 1987, 1989. C.- Kurtz 2019)

X is a solution of the *constrained martingale problem* for (A, D, B, Ξ) if there exists a random measure Λ on $[0,\infty) \times \Xi$ and a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_{[0,t]\times\Xi} Bf(x,u)\Lambda(ds \times dx \times du)$$

is a $\{\mathcal{F}_t\}$-local martingale. X is a *natural solution* if

$$X(t) = Y(\lambda_0^{-1}(t)), \ \ \Lambda([0,t]\times C) = \int_{[0,\lambda_0^{-1}(t)]\times\{u:\,|u|=1\}} \mathbf{1}_C(Y(s),u)\Lambda_1(ds\times du),$$

where $(Y, \lambda_0, \Lambda_1)$ is a solution of the *controlled martingale problem* for (A, D, B, Ξ) (a slowed down martingale problem).