IN SEARCH OF MARKOV SOLUTIONS

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Two part talk:

1. **Mimicking one dimensional marginals by Markov processes**

Athreya, S., Borkar, V. S. and Gadhiwala, N., 2023. '*Controlled martingale problems and their Markov mimics*', SIAM Journal of Control and Optimization, to appear,

and its many precursors (shall mostly talk about the latter).

2. **A selection procedure for a Markov solution for degenerate diffusions**

1. Borkar, V. S. and Suresh Kumar, K., 2010. ‘*A new Markov selection procedure for degenerate diffusions*’, Journal of Theoretical Probability, 23, 729-747.

2. Anugu, S. R. and Borkar, V. S., 2023. ‘*A Selection Procedure for Extracting the Unique Feller Weak Solution of Degenerate Diffusions*’, Applied Mathematics and Optimization, 87(3), online.

I. Mimicking one dimensional marginals by Markov processes

Consider an Ito process in $\mathcal{R}^d$ given by

$$dZ(t) = \xi(t)dt + \psi(t)dW(t) \tag{1}$$

where $W(\cdot)$ is a standard brownian motion and $\xi(\cdot), \psi(\cdot)$ are square-integrable processes that are non-anticipative with respect to $W(\cdot)$,

i.e., for all $t > s$, $W(t) - W(s)$ is independent of $\sigma(W(y), \xi(y), \psi(y), y \leq s)$ for all $s \geq 0$.

Consider another process in $\mathcal{R}^d$ given by

$$dX(t) = m(X(t))dt + \sigma(X(t))dW(t).$$

$X(\cdot)$ is a Markov control mimic (C-mimic for short) of $Z(\cdot)$ if the one dimensional marginals of $X(\cdot), Z(\cdot)$ agree in law (i.e., the law of $X(t)$= the law of $Z(t)$ $\forall t$).

It is a Markov mimic (M-mimic for short) of $Z(\cdot)$ if the above holds and in addition, $X(\cdot)$ is a Markov process.

Under suitable conditions, (Gÿongy, '86) showed the existence of a C-mimic assuming non-degeneracy for ψ, i.e.,

$$\exists\lambda > 0 \text{ such that } x^T\psi\psi^T x \geq \lambda\|x\|^2 \ \forall x \in \mathcal{R}^d.$$

Around the same time, motivated by control problems, for $\xi(t) = m(X(t), u(t))$ and $\psi(t) = \sigma(x(t))$, (B., '86) showed the existence of an M-mimic when $\sigma(\cdot)$ above is Lipschitz and non-degenerate.

Subsequent works by (Brunick and Shreve '13), (Mikami, '95), etc.

Extended to controlled martingale problems as a consequence of a result about stationary distributions in (Bhatt and B., '96).

A more direct proof closer to that of (B., '86) appears in (Athreya, B. and Gadhiwala, 2023).

A different take from (B., '91) is to consider equivalence classes under the equivalence relation $(X(\cdot), u(\cdot)) \equiv (X'(\cdot), u'(\cdot))$ when their marginals match a.s. For fixed initial law, they form a convex compact set.

Then the extremal equivalence classes are singletons containing a Markov process.

The proof uses the idea of 'markovianization at a time point t'.

This means replacing the original $X(\cdot)$ by an $X'(\cdot)$ by setting the laws of $X(t), X'(t)$ identical and rendering $X'(t+\cdot), X'(t-\cdot)$ conditionally independent given $X'(t)$ with the same conditional laws as for $X(t\pm\cdot)$ given $X(t)$.

This retains the marginals.

(Bhatt and B., '96) extends this to controlled martingale problems.

A special situation that has drawn considerable interest, motivated by optimal transport, is as follows.

Assume that the processes under consideration have laws absolutely continuous with a base measure Q on the path space under which the process is Markov, with the Radon-Nikodym derivative(s) denoted by Λ.

Suppose also that the set of laws under consideration is closed under markovianization at any time point t.

Then if the relative entropy $E_Q[\Lambda \log \Lambda]$ attains its minimum, the minimum is unique and corresponds to a Markov process. Sufficient condition for attaining the minimum can be given in terms uniform integrability of $\Lambda \log \Lambda$. (Athreya, B., Gadhiwala, '23)

This improves upon prior results in (Baradat and Léonard, '20). Related work in (Chen, Georgiou and Pavon, '21,'22), (Mikami, '21).

(Athreya, B., Gadhiwala, '23) also shows that under the absolute continuity condition above, every marginal class has a Markov representative.

II. A selection procedure for a Markov solution for degenerate diffusions

A 'half-open' problem?

The martingale problem (Stroock and Varadhan):

Given a probability measure μ on $\mathcal{R}^d$, $X(\cdot)$ solves the martingale problem for $(\mathcal{L}, \mu)$ if $X(0)$ has law μ and for $f \in C_b^2(\mathcal{R}^d)$,

$$f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds, \ t \geq 0,$$

is a martingale w.r.t. $\{\mathcal{F}_t\}$, $\mathcal{F}_t :=$ the completion of $\cap_{t'>t}\sigma(X(s), s \leq t)$.

It is the unique solution to this martingale problem if two such solutions agree in law.

Martingale problem for $\mathcal{L}$ (now called the **'extended generator'**) is well-posed if the martingale problem for $(\mathcal{L}, \delta_x)$ is $\forall x \in \mathcal{R}^d$. The corresponding unique measures then satisfy the Chapman-Kolmogorov equations to yield a Markov process.

True, e.g., if σ is non-degenerate and Lipschitz and m bounded measurable, or if both m, σ are Lipschitz.

m, σ continuous $\Longrightarrow$ existence of a solution is guaranteed, but no uniqueness.
However, a characterization of 'all solutions' is possible (Chapter 12, Stroock & Varadhan).

The solution set $\mathcal{A}_x$ (resp., $\mathcal{A}_\nu$) for a given initial condition x (resp., law ν) is nonempty convex and compact.

The problem of Markov selection:

Given $\mathcal{A}_x,\ x \in \mathcal{R}^d$, find $P_x^* \in \mathcal{A}_x,\ x \in \mathcal{R}^d$, such that $\{P_x^*,\ x \in \mathcal{R}^d\}$ is a Markov family,

i.e., the Chapman-Kolmogorov equation holds.

Krylov selection procedure:

Let $\{r_i\} \subset (0, \infty)$ be an enumeration of positive rationals and $\{f_i\} \subset C_b(\mathcal{R}^d)$ a countable separating class, i.e.,

$$\int f_i d\mu = \int f_i d\mu' \ \forall i \implies \mu = \mu'.$$

Let

$$G_{ij}(\Gamma) = E\left[\int_0^\infty e^{-r_i t} f_j(X(t))dt\right]$$

$$= \int \left(\int_0^\infty e^{-r_i t} f_j(x(t))dt\right) d\Gamma(x(\cdot)),$$

where $\Gamma =$ the law of $X(\cdot)$.

Let $F_i(\cdot), i \geq 0$, be an enumeration of $\{G_{ij}(\cdot)\}$. Let $\mathcal{A}_0(x) = \mathcal{A}_x$ and for $m \geq 0$, define

$$\mathcal{A}_{m+1}(x) = Argmin_{\{\Gamma \in \mathcal{A}_m(x)\}} F_m(\Gamma).$$

Then $\{\mathcal{A}_m(x)\}$ is a nested decreasing sequence of compact convex nonempty subsets of $\mathcal{A}(x)$.

Fact: For a stopping time τ w.r.t. the natural σ-fields of $X(\cdot)$, the law of $X(\cdot) \in \mathcal{A}_m(x) \Longrightarrow$ the law of $X(\tau + \cdot) \in \mathcal{A}_m(X(\tau))$ a.s. on $\{\tau < \infty\}$.

This follows by a 'dynamic programming' like argument.

By finite intersection property of compact sets, $A_\infty(x) := \cap_{m\geq 0}\mathcal{A}_m(x)$ is nonempty.
By our choice of $\{F_i\}$, it is a singleton $\{P_x^*\}$ and by the above observation, $\{P_x^*\}$ form a (strong) Markov process.

For the general case of 'martingale problems', see Ethier and Kurtz.

Problems with this construction: Not unique, it can depend on the choice of F_i's, r_i's and the order of minimization.

Is a 'principled' alternative possible?

Yes, based on the Kolmogorov philosophy* of selecting a 'physical solution(s)' of an ill-posed problem by adding non-degenerate noise to 'regularize' it and then let noise tend to zero to get a limit solution(s).

Examples: stochastically stable equilibria in evolutionary games, viscosity solutions in control, 'hysteresis' in electric circuits

This is the approach I shall outline next.

*as reported by Eckmann and Ruelle

Assume bounded continuous $m(\cdot), \sigma(\cdot)$.

Preliminaries:

Consider the backward equation with $\sigma(\cdot)$ degenerate:

$$(\dagger) \qquad \frac{\partial u}{\partial t} + \mathcal{L}u = 0,\ t \in [0,T];\ u(x,T) = f(x).$$

Classical solution may not exist, but a viscosity solution does 'under suitable conditions'.

Consider a non-degenerate bounded continuous approximation $\sigma^\epsilon(\cdot), \epsilon > 0$, to $\sigma(\cdot)$ such that it converges to $\sigma(\cdot)$ uniformly on compacts (e.g., $\sigma^\epsilon(\cdot) = \sqrt{\sigma(\cdot)\sigma^T(\cdot) + \epsilon I}$).

Let $\mathcal{L}^\epsilon$ denote the corresponding extended generator and $u_f^\epsilon(\cdot, \cdot)$ a solution to

$$\frac{\partial u_f^\epsilon}{\partial t} + \mathcal{L}^\epsilon u_f^\epsilon = 0, \; t \in [0, T]; \; u_f^\epsilon(x, T) = f(x),$$

the corresponding backward equation.

This has a unique classical solution in $C_b^{2,1}(\mathcal{R}^d \times [0, T])$ which is also its unique viscosity solution. Furthermore, $u_f^\epsilon(x, t) = E[f(X^\epsilon(T))|X^\epsilon(t) = x] \; \forall t \in [0, T]$.

As $\epsilon \downarrow 0$, it converges to the unique viscosity solution $u_f(\cdot,\cdot)$ of (†) uniformly on compacts. (The comparison principle plays a key role in this.)

Let $X^\epsilon(\cdot)$ be the process corresponding to $\mathcal{L}^\epsilon$ with a fixed initial law, and $X(\cdot)$ one of its subsequential limits as $\epsilon \downarrow \infty$.

(VB-KSK) shows that there exists a Feller solution with the same one dimensional marginals as $X(\cdot)$, in particular all subsequential limits have common one dimensional marginals.

The latter follows from the convergence of $u_f^\epsilon(\cdot, \cdot)$ to the unique viscosity solution $u_f(\cdot, \cdot)$ of (†), the degenerate backward equation for $f \in$ a countable convergence determining class.

The Feller process comes from a diagonal argument to claim simultaneous convergence of u_f^ϵ defined over f as above on $[s, t], t > s$ rational in $[0, T]$, $f \in$ a countable convergence determining class.

Using Riesz theorem, we identify $u_f(x, t)$ as $\int p(dy|x, t)f(y)$.

These transition kernels satisfy the Chapman-Kolmogorov equation because those for u_f^ϵ do, and the C-K equation is preserved under the limiting operation.
This implies Markov property. Since $u_f(\cdot,\cdot)$, hence $p(dy|\cdot,\cdot)$ is continuous, the process is Feller.

(ASR-VB) improves this to the claim that the Feller solution is in fact the unique limit in law.

This follows from the convergence of arbitrary finite dimensional marginals, proved using uniform convergence of u_f^ϵ to u.

Additional contribution of (ASR-VB): The theory of existence and uniqueness of viscosity solutions in this context is not off-the-shelf, but is established in (ASR-VB) for the backward Kolmogorov equation using some very recent developments and under significantly more general conditions than in (VB-KSK).

The key hypotheses involve certain Hölder continuity conditions on the coefficients.

The anti-climax: Conditions on $m(\cdot), \sigma(\cdot)$:

There exist $\alpha, \beta > 0$ satisfying $1 + \alpha - 2\beta > 0$ and $\beta > \frac{1}{2}$, such that

$$\|m(x) - m(y)\| \leq C_1 \|x - y\|^{\alpha}, \ \|\sigma(x) - \sigma(x)\| \leq \|x - y\|^{\beta}$$

and, at points x where either of them is not Lipschitz, they both vanish and satisfy: for some $\epsilon > 0$,

$$c\|x - y\|^{2\beta}\|v\|^2 \leq v^T \sigma(y) \sigma^T(y) v \leq c^{-1} \|x - y\|^{2\beta} \|v\|^2,$$

for all y in the open ϵ-ball centred at x where $m(\cdot), \sigma(\cdot)$ are Lipschitz. (Examples in the aticle.)

Future directions:

Relaxing the conditions on coefficients

Extensions to ‘controlled’ diffusions

A purely probabilistic proof?

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