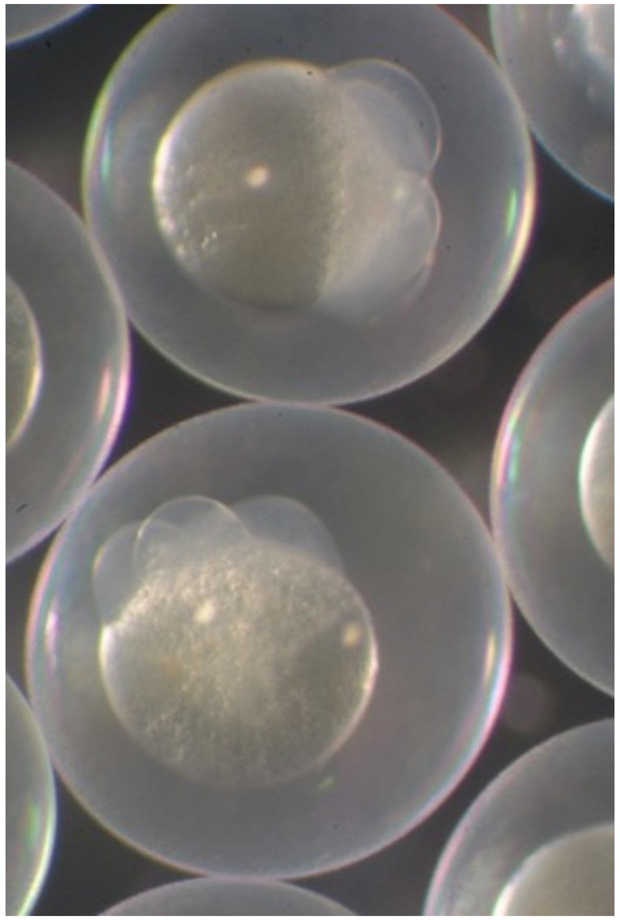

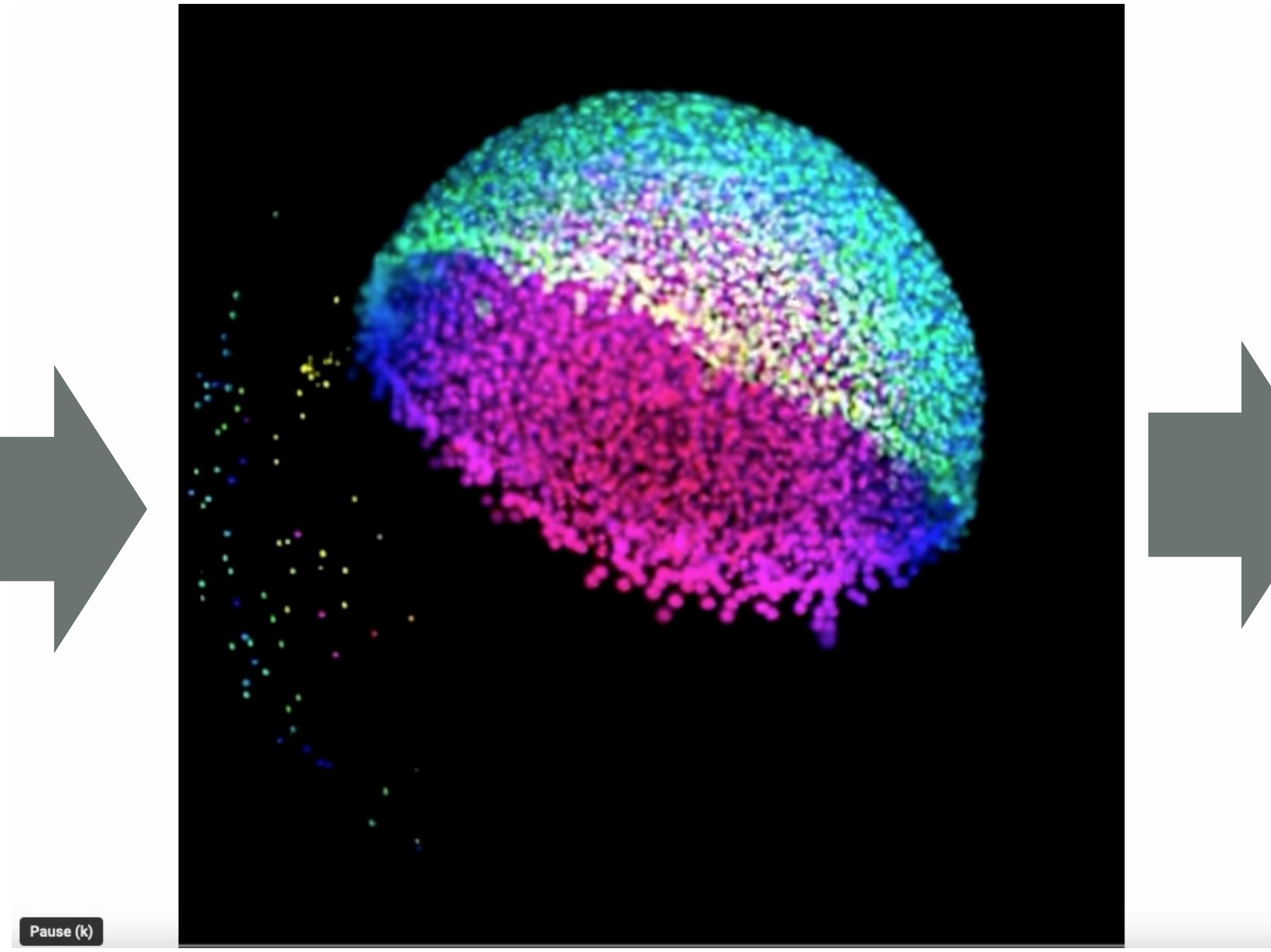

Gopi Shah, Max Planck Institute

Programming rigidity transitions and multifunctionality in disordered underconstrained spring networks

Geometry of Materials Workshop
ICERM, Providence, RI
8 April 2025

M. Lisa Manning
Syracuse University
BioInspired Institute
Department of Physics

Thanks to my group and collaborators!

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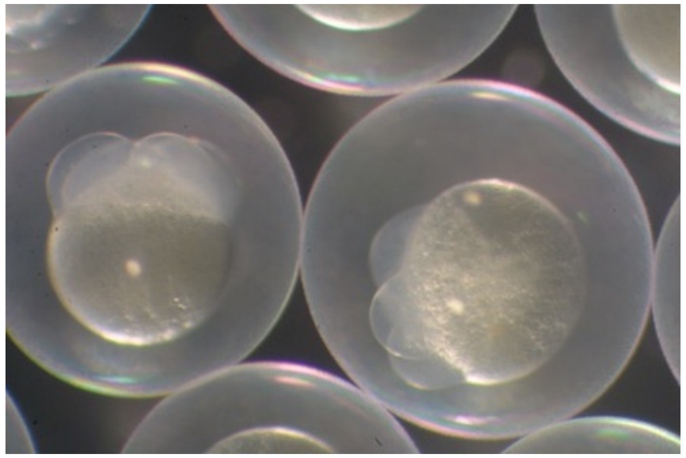
Margaret Gardel & John Devany (U Chicago)
Jeff Amack (SUNY Upstate Medical University)
Karen Kasza and Xun Wang (Columbia)
Ryan Hayward (CU Boulder)

How do you turn a blob of material into something that's the shape of a fish?

blob



zfin.org

development

Karlstrom et al, Development (1996)

fish

zfin.org

How do you turn a blob of material into something that's the shape of a fish?

glass blob

coursehorse.com

glass blower

Smithsonian Magazine

complex, stable, reproducible morphology

How do you turn a blob of material into something that's the shape of a fish?

Smithsonian Magazine

mandrel rod

applies localized forces:

pressures

shear stresses

heat

alters the local material properties:

elastic modulus

fluidity/viscosity

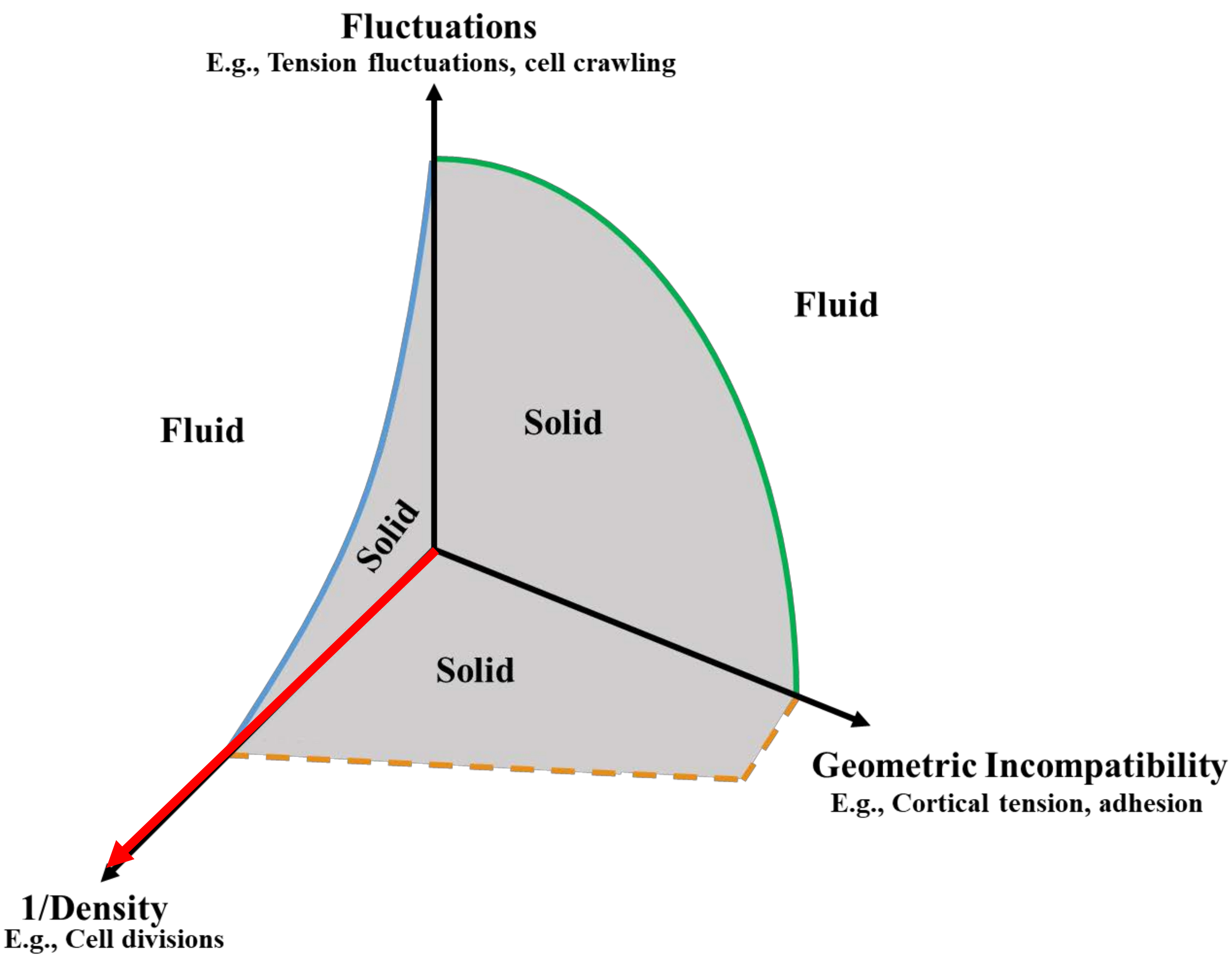
goes through a "glass transition"

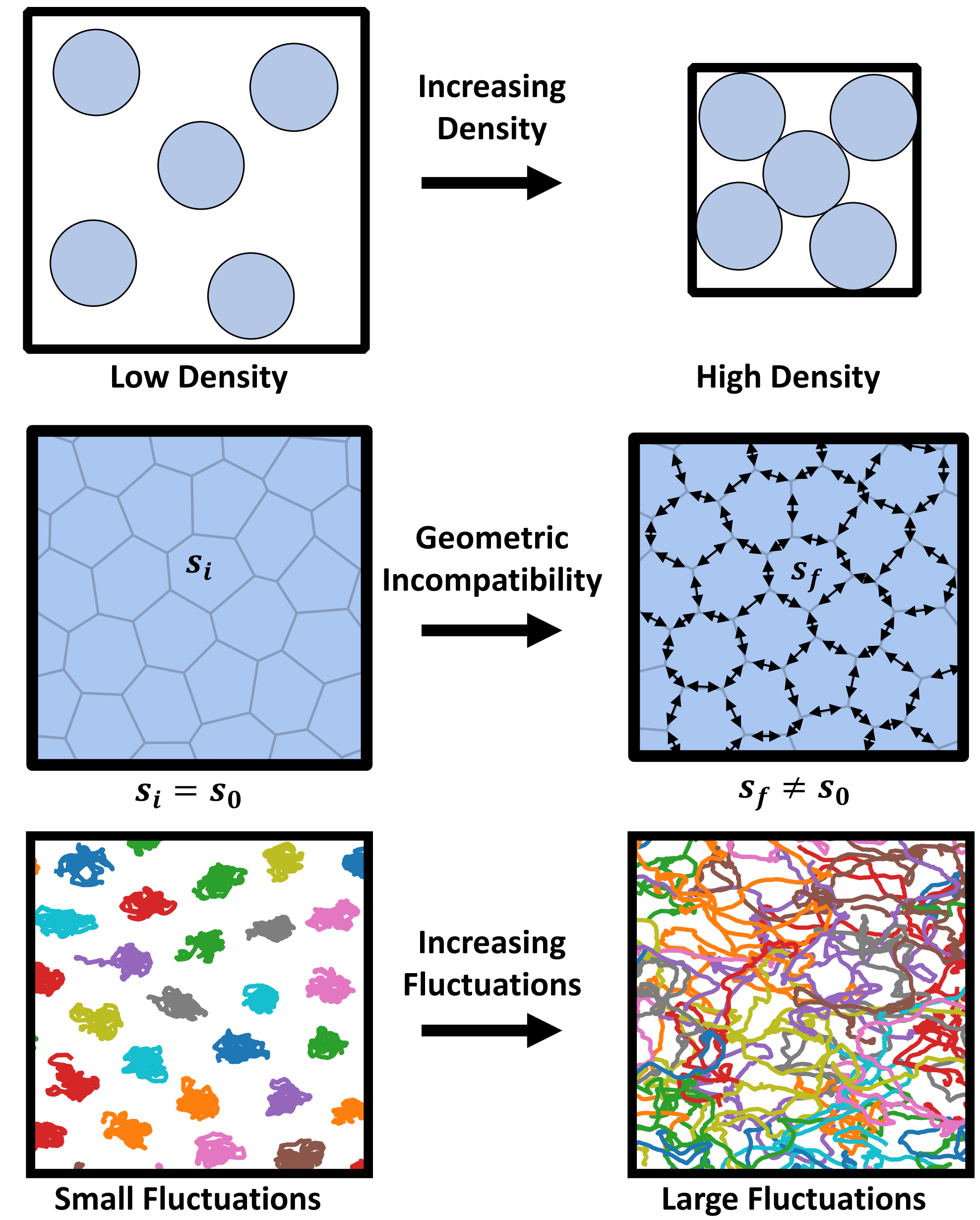
solid → fluid → solid

Research in my lab:

- predict the emergent **mechanical** behavior of disordered glassy/jammed materials or groups of cells (material properties and forces)
- predict how these mechanical properties help govern functional behaviors in biological tissues: morphogenesis, collective cell migration, tissue homeostasis
- design new types of adaptive materials based on these bioinspired principles

There are multiple physical mechanisms that can drive rigidity/fluidity in tissues (and mechanical networks):





Lawson-Keister++, Current Opinions in Cell Biology (2021)

adapted from Kim++ Nature Physics (2021) and Bi++ PRX 2016

Example: zero-temp granular materials

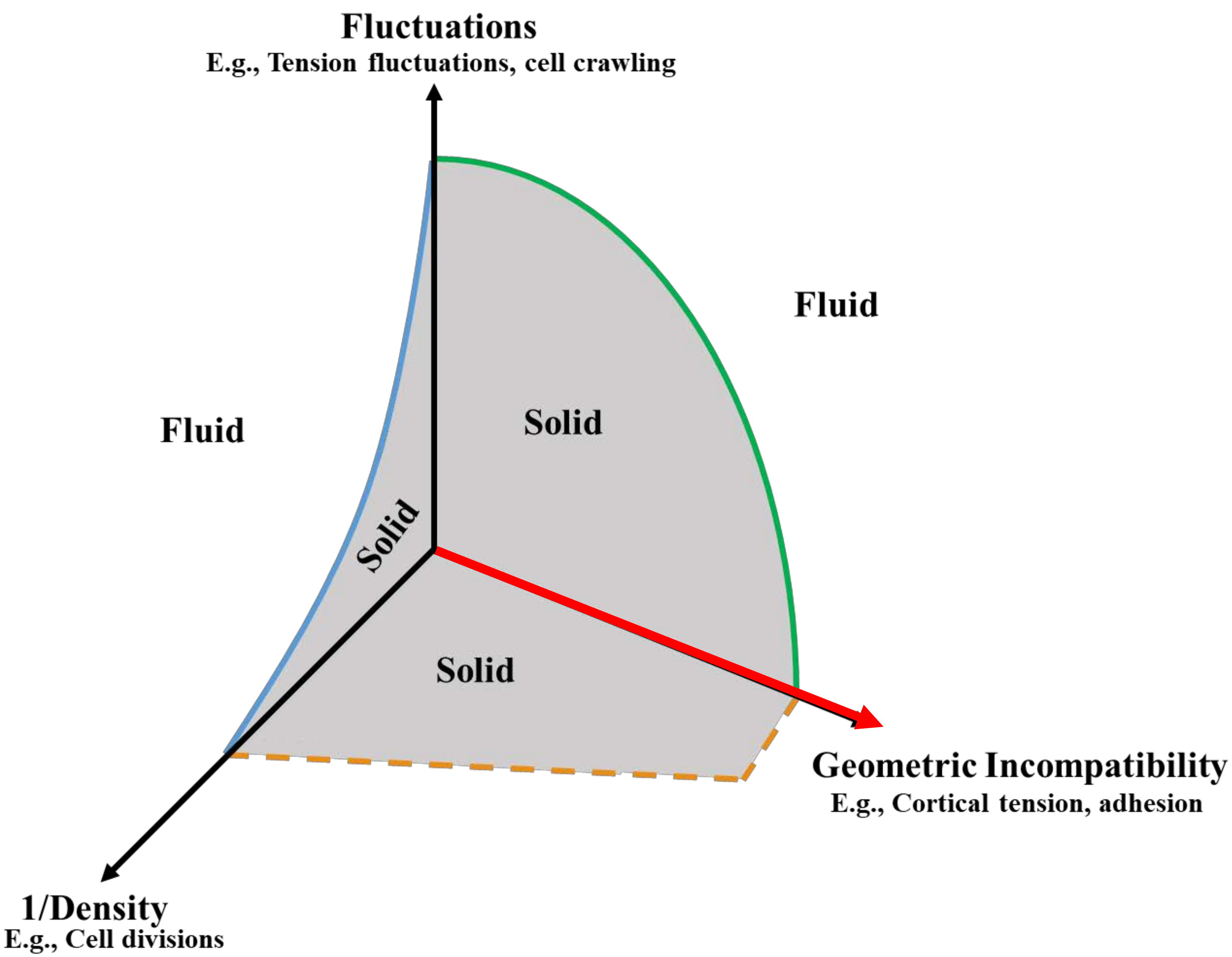
Intuition: Materials solidify as the packing fraction/number density increases (c.f. Eric Corwin's talk yesterday)

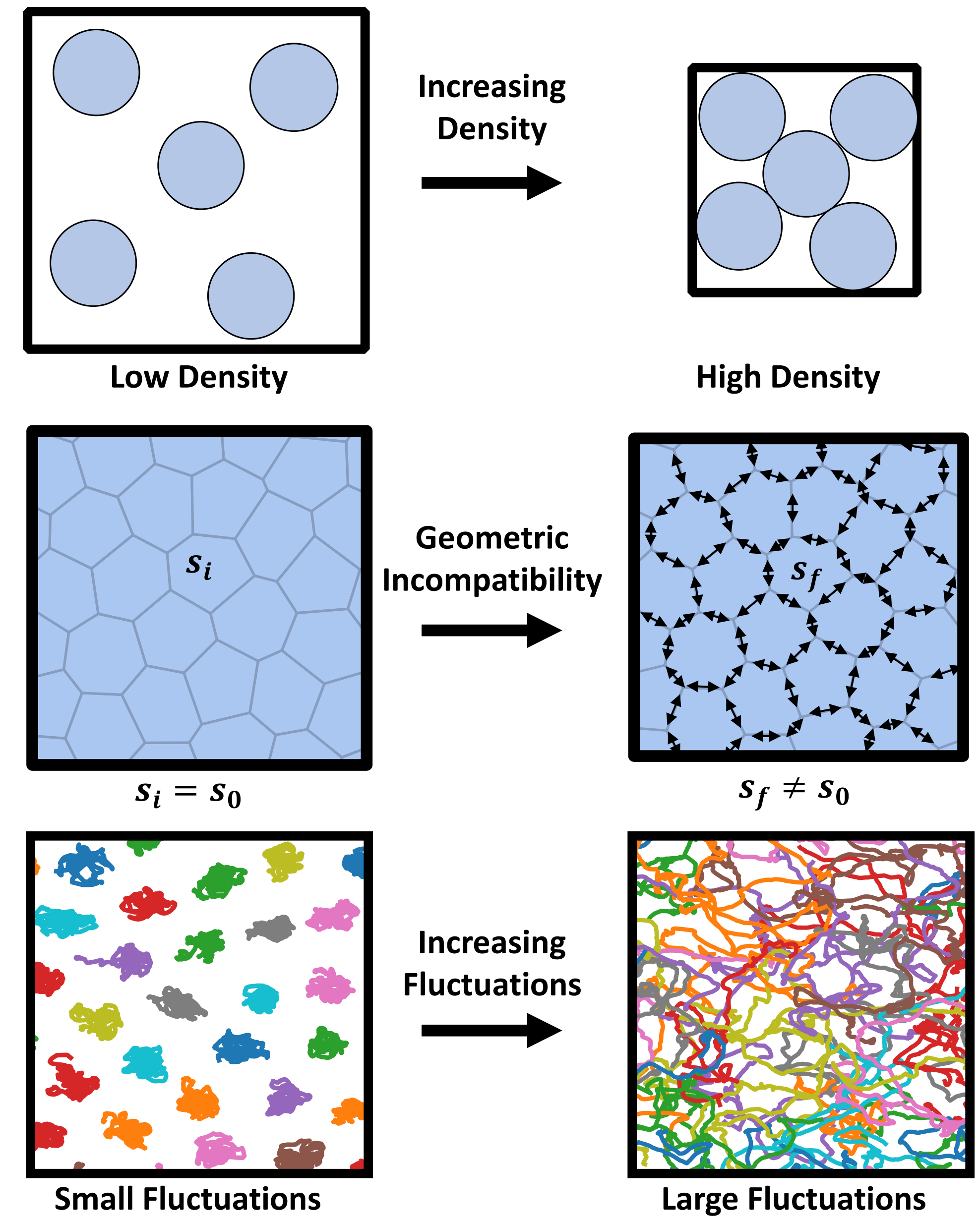
Why? In the simple model of frictionless disordered spheres at zero temperature (i.e. particulate jamming)

- there are $N_p d = N$ degrees of freedom and $N_p z/2 = M$ constraints, z is the number of contacts per particle.
- In mean field, expect rigidity to occur when when these two quantities are equal: $z=2d$.
- at low densities, $z < 2d$ and system is underconstrained.
- at high densities $z > 2d$ and system is overconstrained.

c.f. Gortler talk yesterday, for $z > 2d$ generally there is no (1,1) flex

There are multiple physical mechanisms that drive fluidity in mechanical networks:

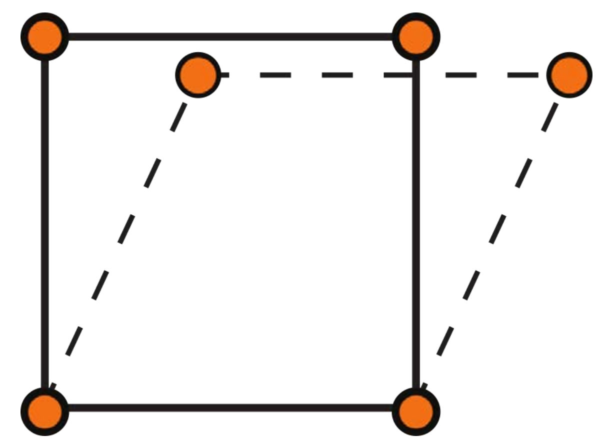




Lawson-Keister++, Current Opinions in Cell Biology (2021)
adapted from Kim++ Nature Physics (2021) and Bi++ PRX 2016

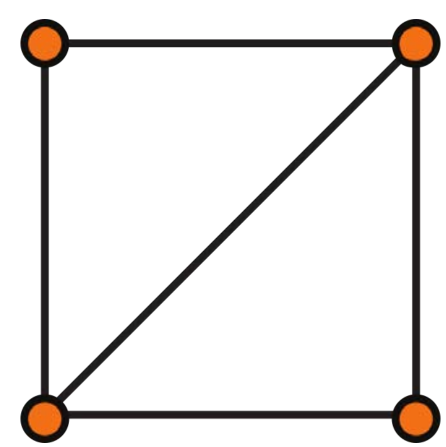
Let's revisit rigidity and Maxwell's constraint counting

J. C. Maxwell



8 degrees of freedom
-4 constraints
-3 rigid body motions
= 1 (nontrivial) floppy mode

→ **under-constrained**



8 degrees of freedom
-5 constraints
-3 rigid body motions
= 0 (nontrivial) floppy modes

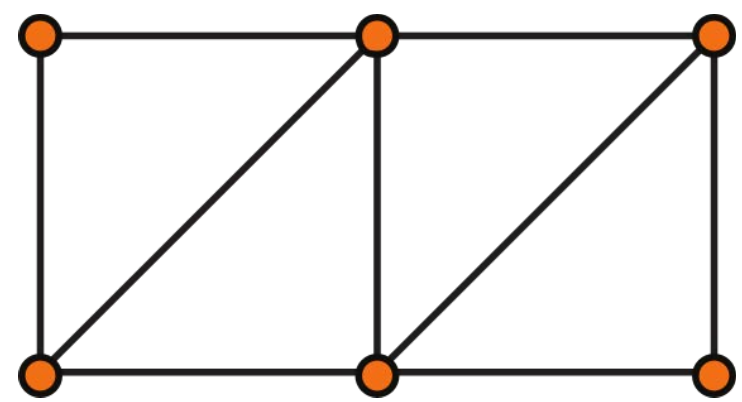
c.f. Gortler talk yesterday, there is no (1,1) flex

Maxwell, *Phil. Mag. Series* (1864).
Calladine, *Int. J. Solids Struct.* (1978).
Lubensky et al., *Rep. Prog. Phys.* (2015).

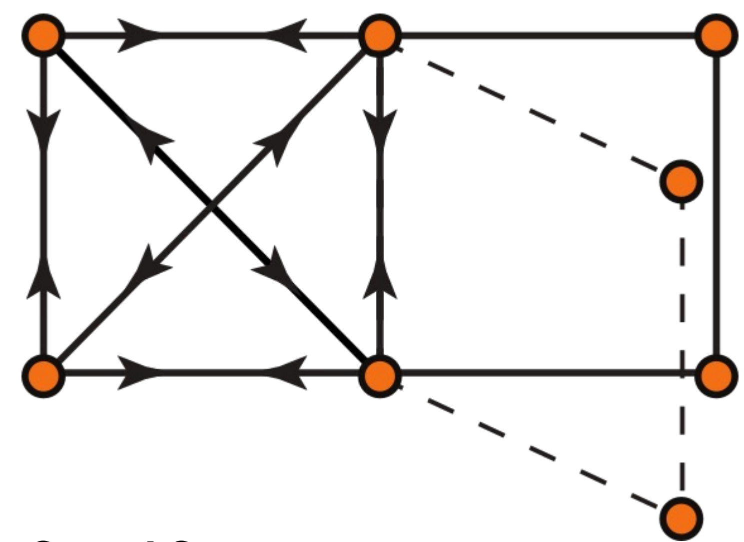
More linear theory: states of self-stress are generated by redundant constraints.

J. C. Maxwell

Rigid:



Floppy although same number of constraints:



→ because of **state of self-stress σ**

$$N_{dof} - M = \boxed{N_0} - \boxed{N_s}$$

Number of zero modes

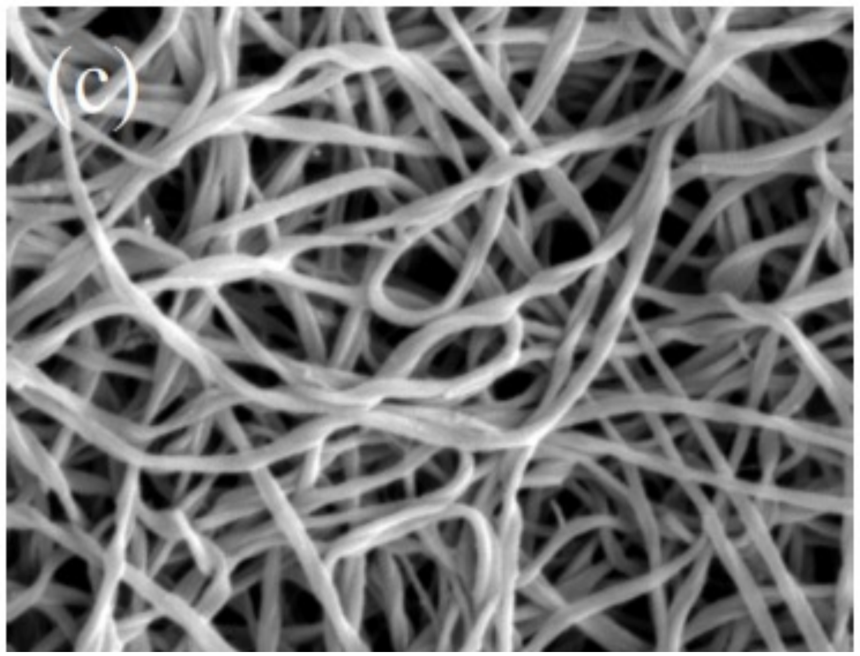
Number of states of self stress

Jamming happens when Ndof ~ M

Calladine, *Int. J. Solids Struct.* (1978).
Lubensky et al., *Rep. Prog. Phys.* (2015).

But, there is a whole class of materials that rigidify even though they are underconstrained.

Fiber networks in biology are often under-constrained



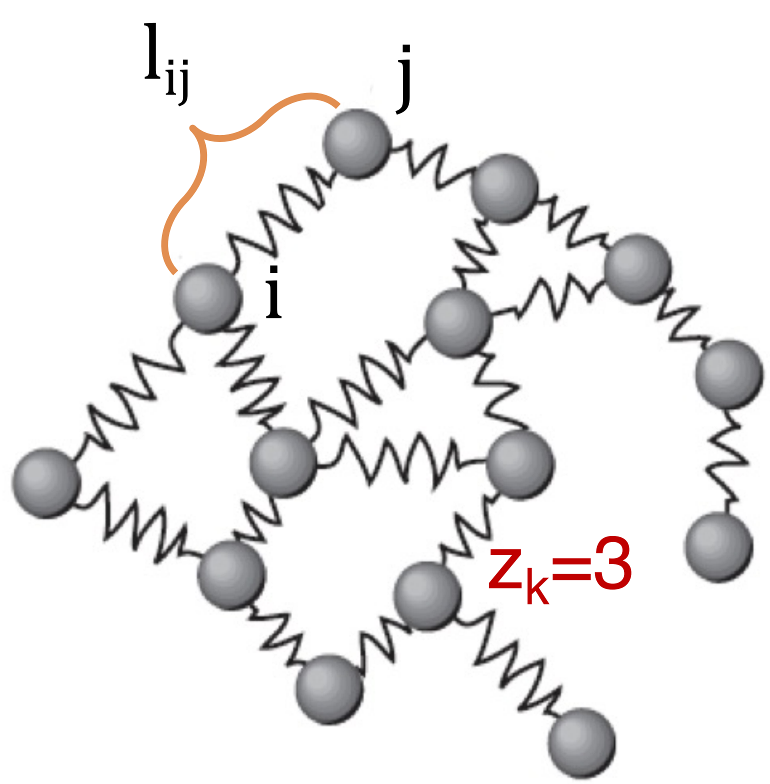
Sharma et al. Nature Phys. 2016.

can be approximated as a network of springs

$$e_{network} = k_{spring} \sum_{\langle ij \rangle} (l_{ij} - l_0)^2$$

actual length

rest length



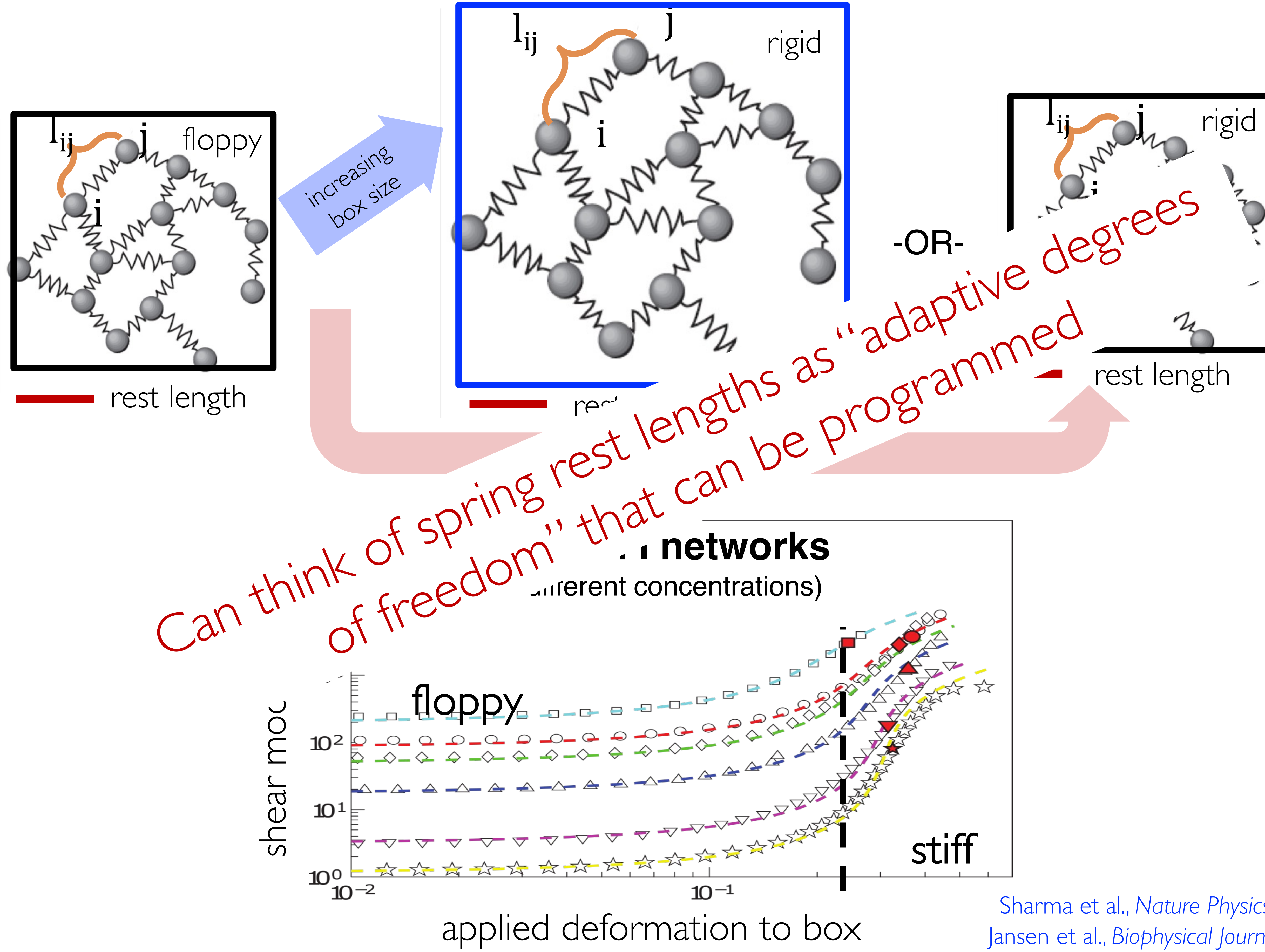
in biological tissues networks like collagen are almost always under-constrained:

network coordination

$z < z_c = 2d$

Sharma et al., *Nature Physics* (2016).
Jansen et al., *Biophysical Journal* (2018).

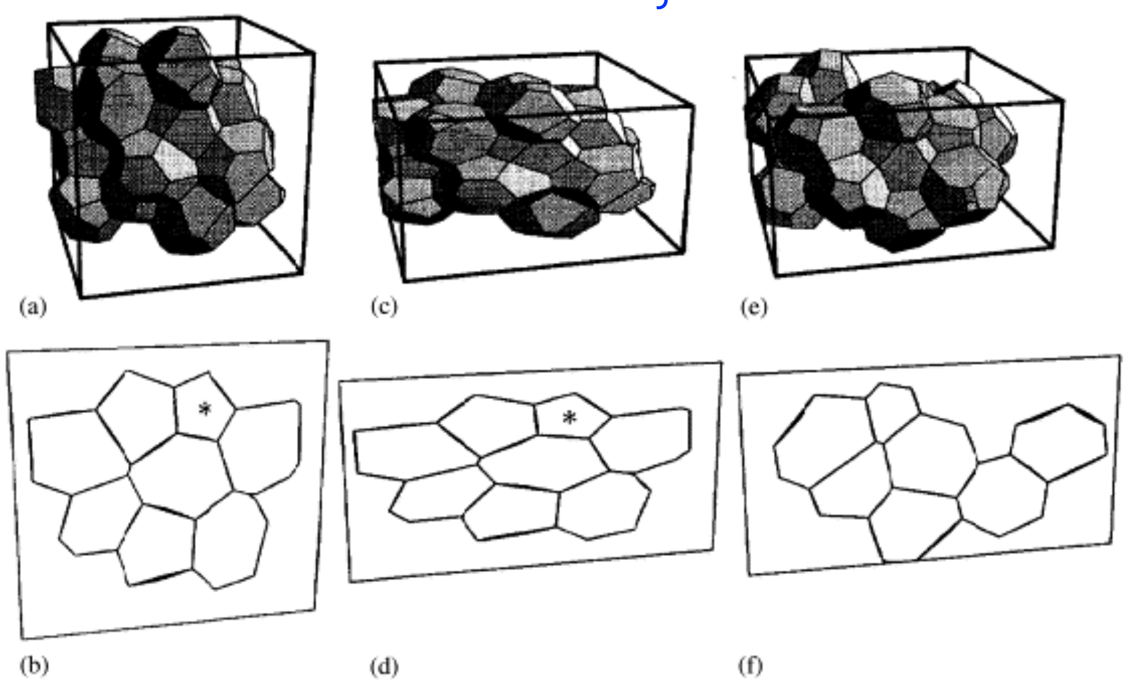
Fiber networks can rigidify via changing box size or spring rest length



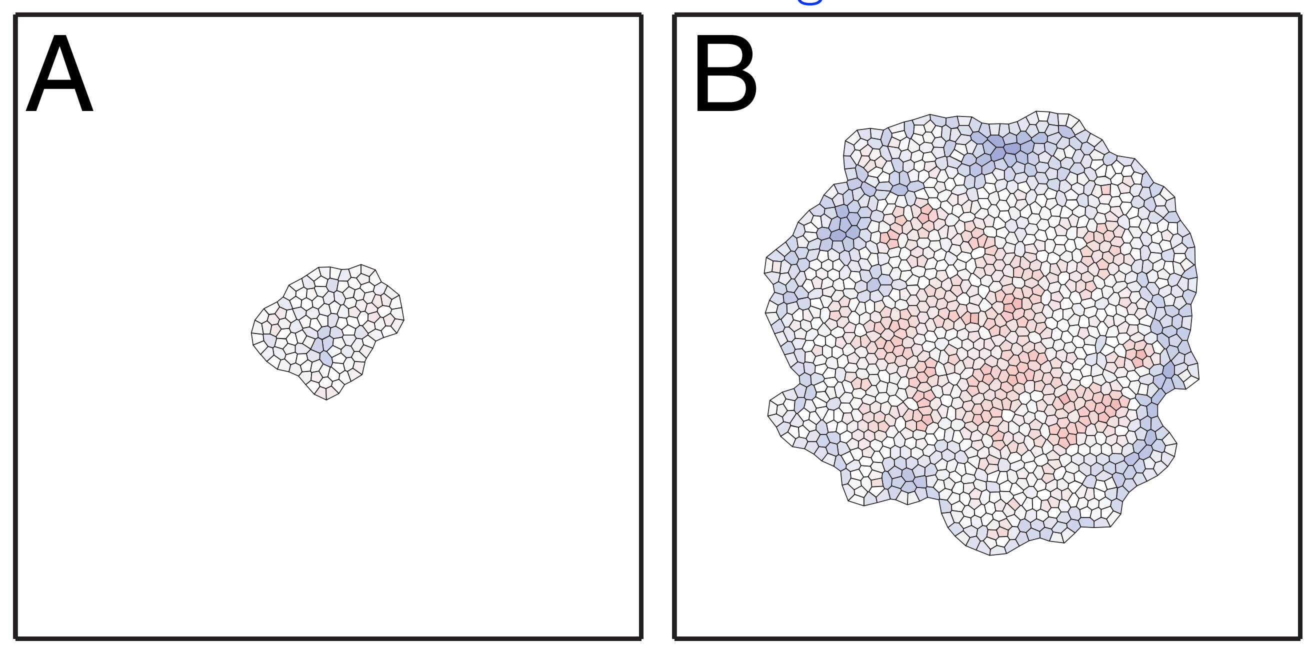
Sharma et al., *Nature Physics* (2016).
Jansen et al., *Biophysical Journal* (2018).

How to model confluent tissues composed of complicated cells?

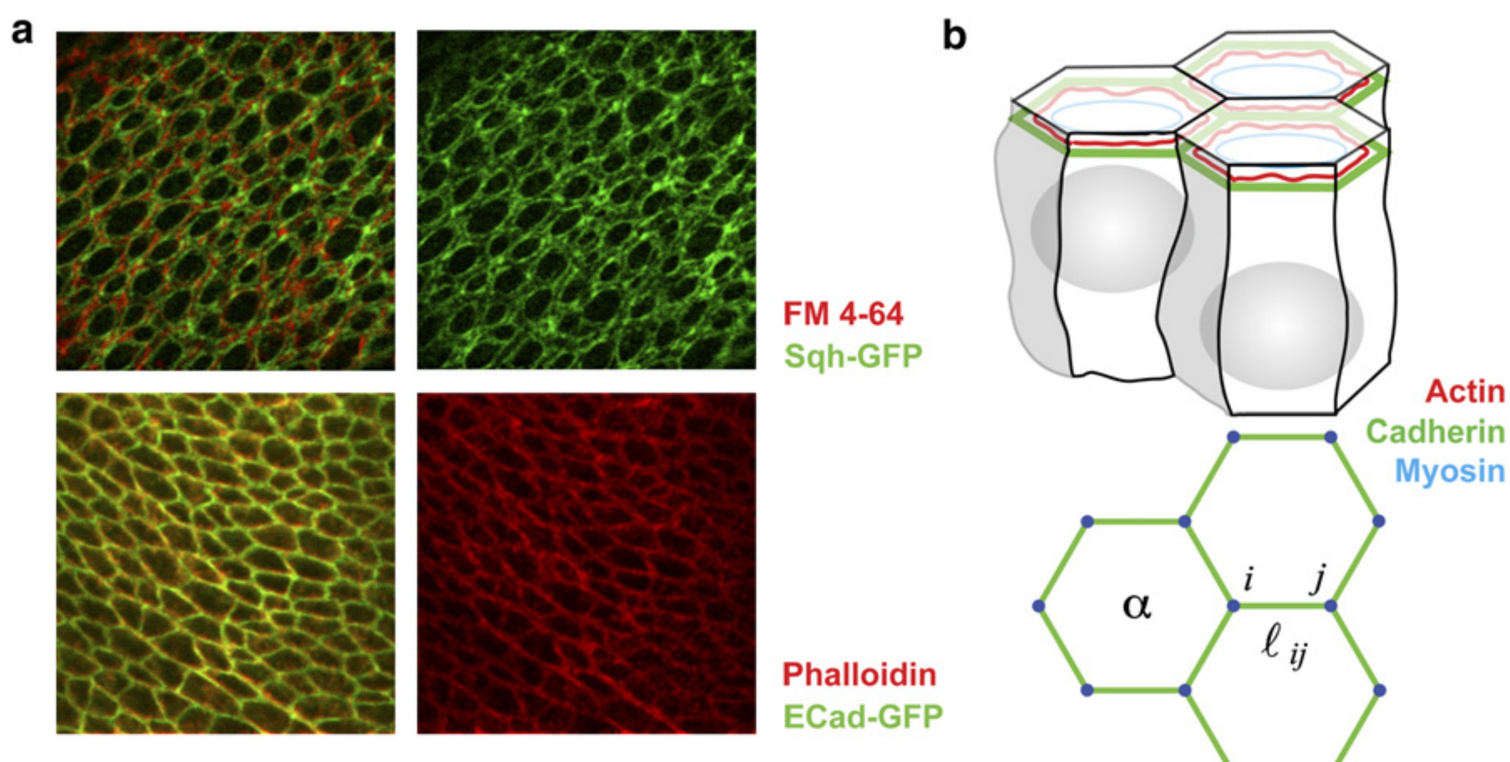
Honda++ J. Theo. Bio. 2004



Hufnagel++ PNAS 2007

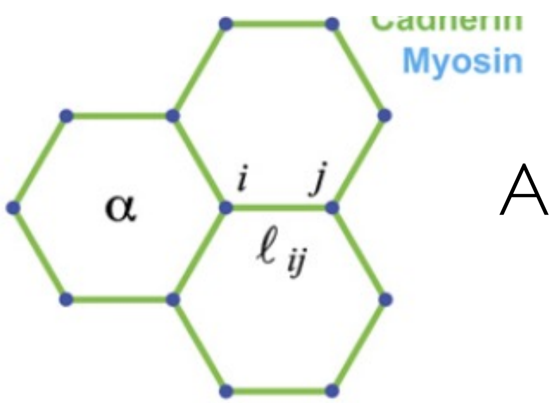


Vertex models



Farhadifar++ Curr. Bio. 2008

2D vertex models transition from fluid to solid as a function of cell shape

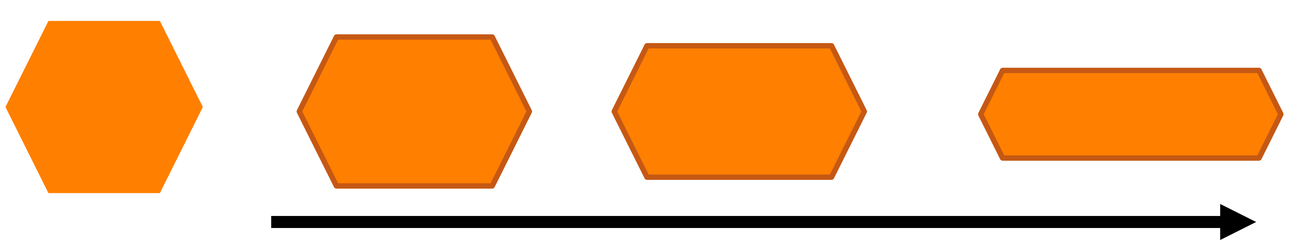


A = area, P = perimeter

$$E_{cell} = k_A(A - A_0)^2 + k\cdot\ \cdot$$

$$E_i = k_{a,i}(\cdot \quad\quad \cdot - p_{0,i})^2$$

Targe ... factor $= p_0 = P_0/\sqrt{A_0}$

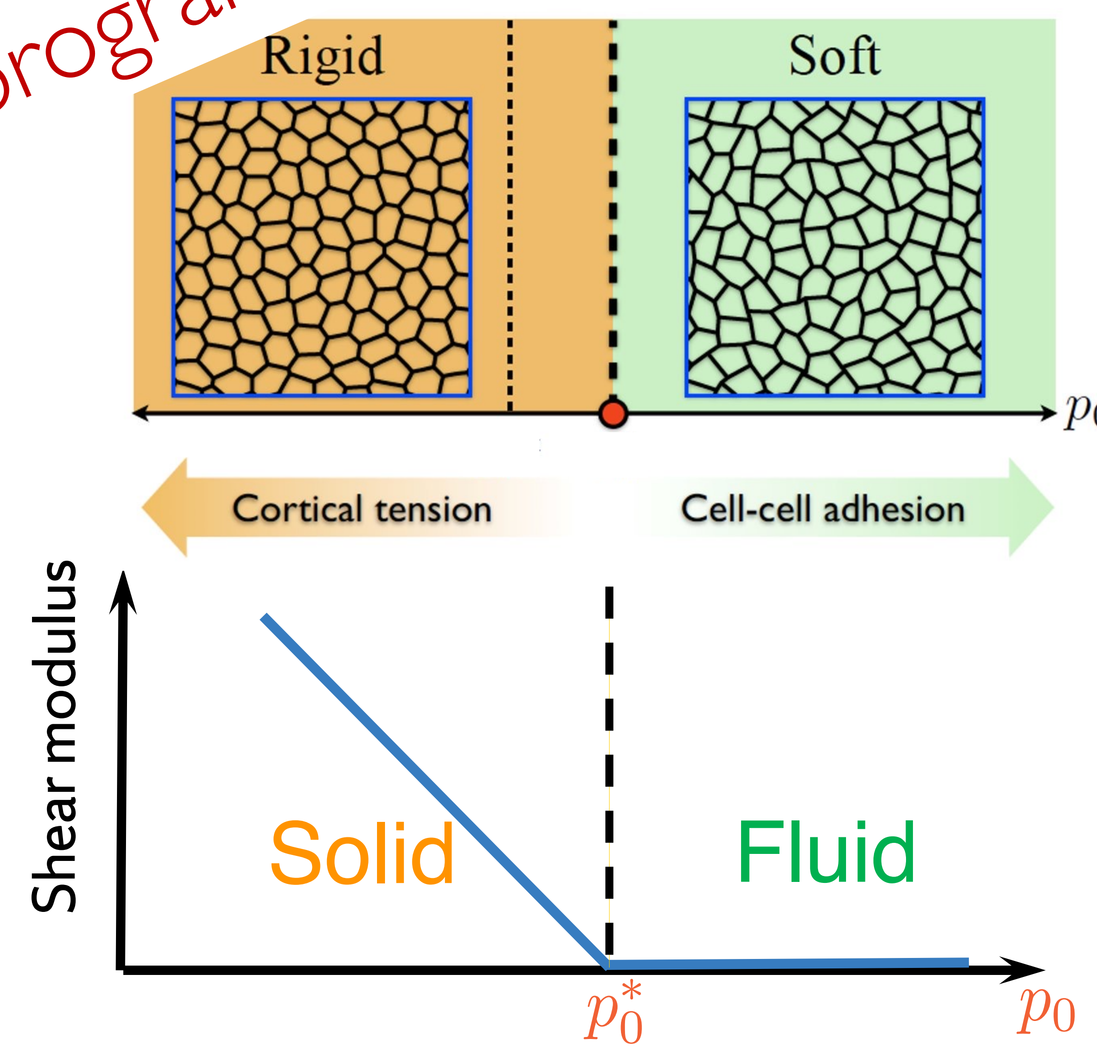


p0=3.72

increasing shape parameter

T. Nagai, H. Honda, Philo... 699 (2001)
Hufnagel et al, PN... p. 3835 (2007)
Farhadifar... 7)
Jülic...
...07–911 (2008)
...17 (2010)

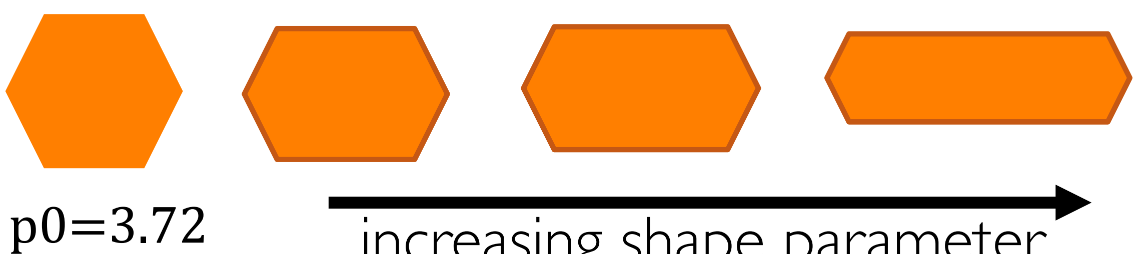
Can think of target cell shapes as "adaptive degrees of freedom" that can be programmed



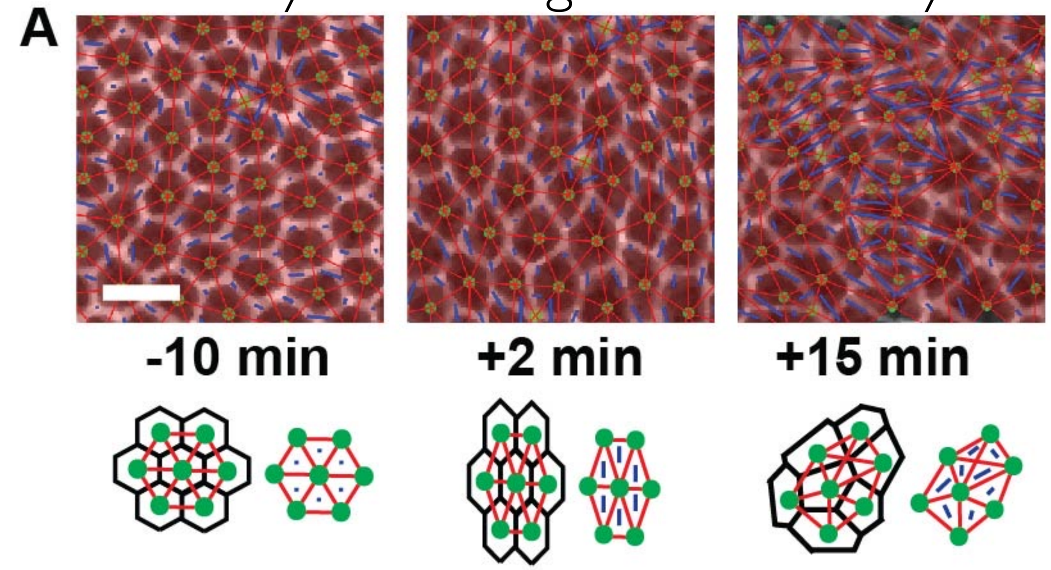
Bi ++., Nat. Phys. (2015)

experiments: confluent tissues rigidify by tuning cell shape

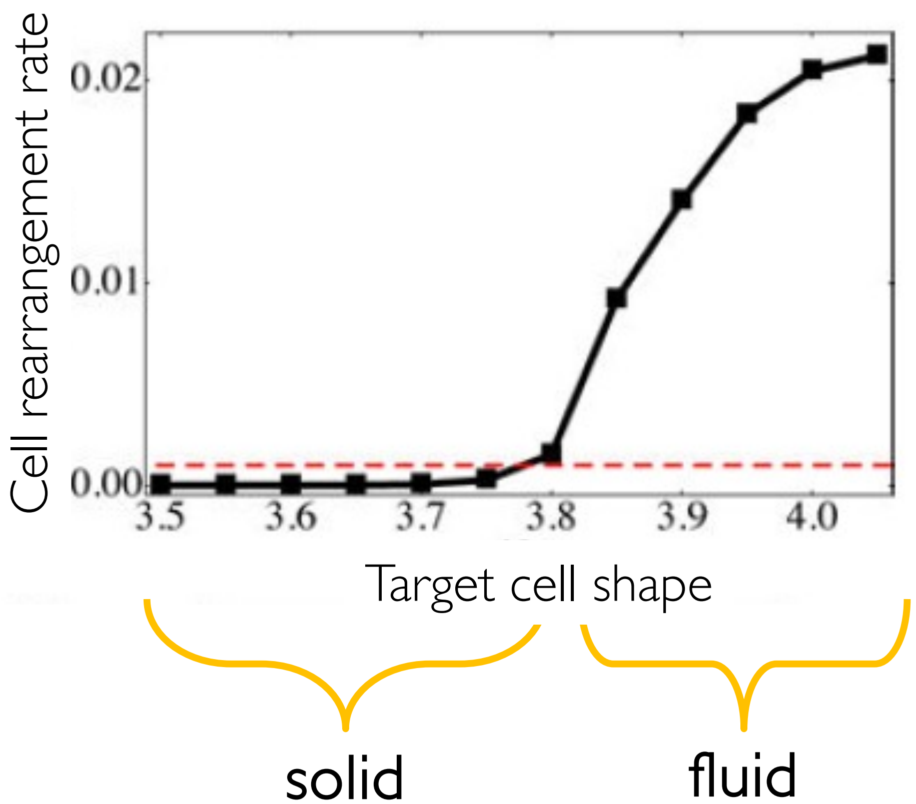
shape index: perimeter / $\sqrt{\text{area}}$



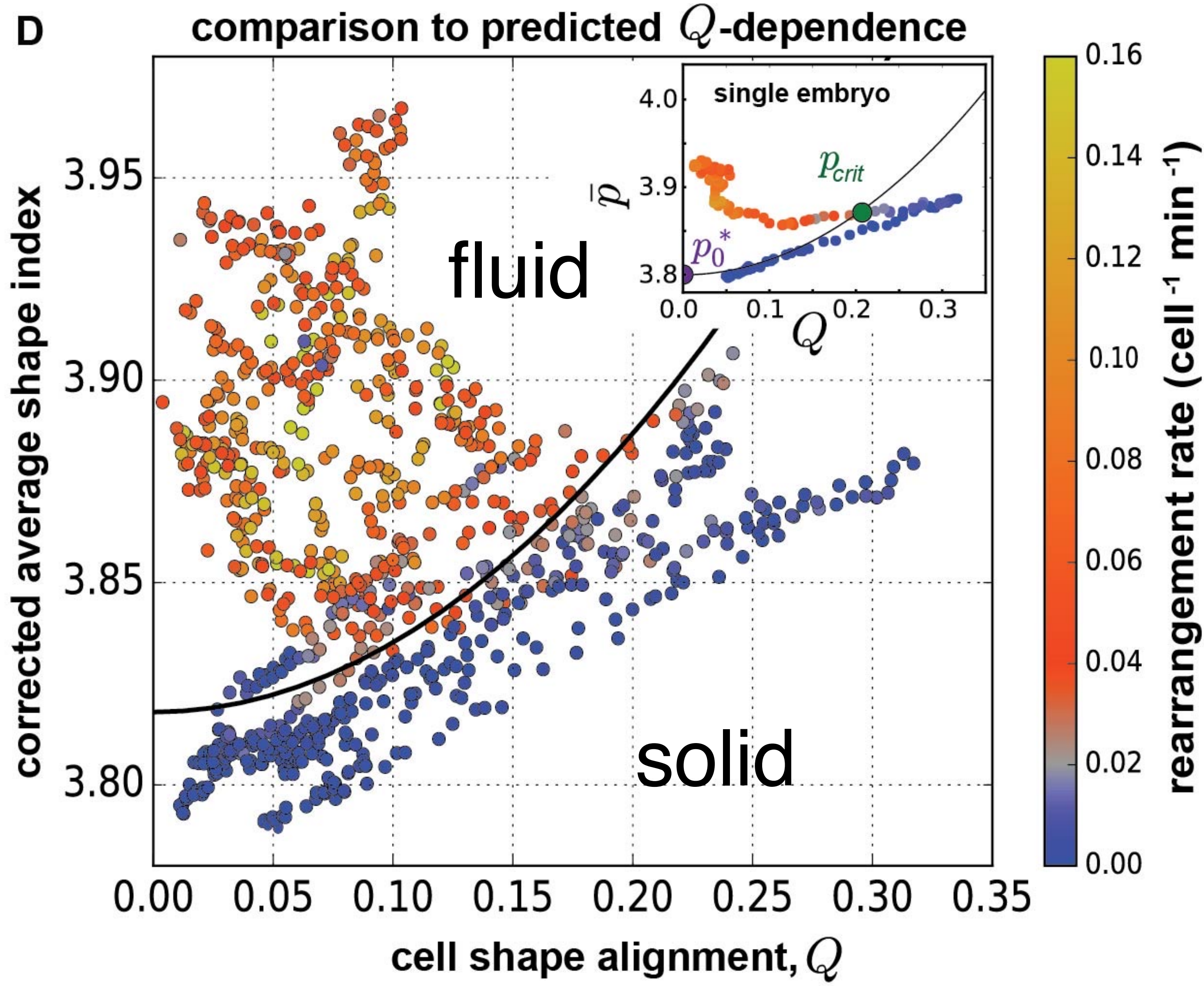
Body axis elongation in fruit fly



Model for confluent tissue



Bi++(2016)



Wang++ PNAS (2020)

Dr. Kim (Syracuse University, Schwarz group) is an expert, is here!

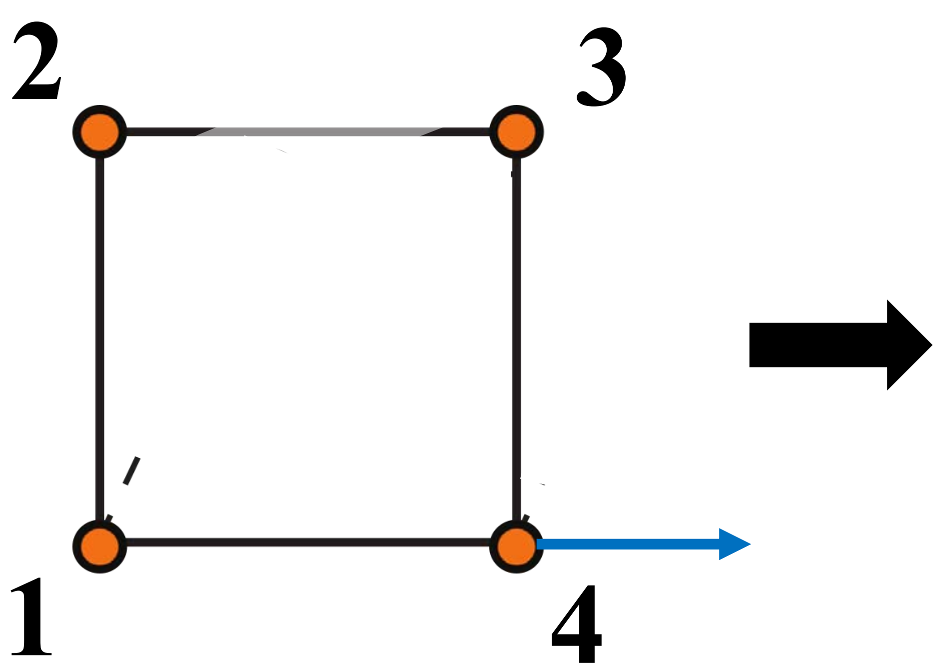
How are these materials becoming rigid?

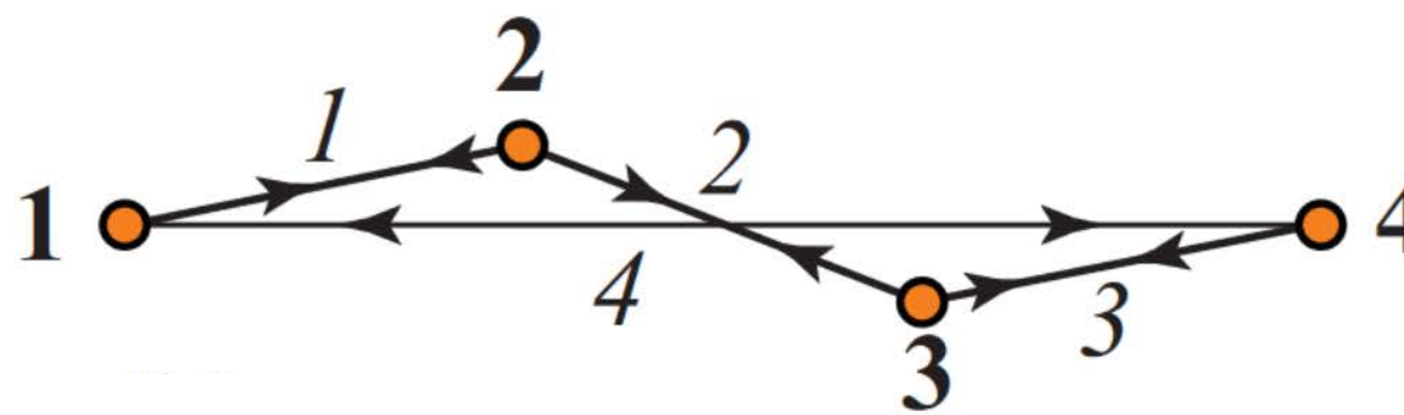
Linear theory: $N_{dof} - M = \boxed{N_0} - \boxed{N_s}$

Number of zero modes

Number of states of self stress

- BUT this isn't the only way to become rigid
- Underconstrained systems generally don't have any SSS: but in special singular configurations they can appear





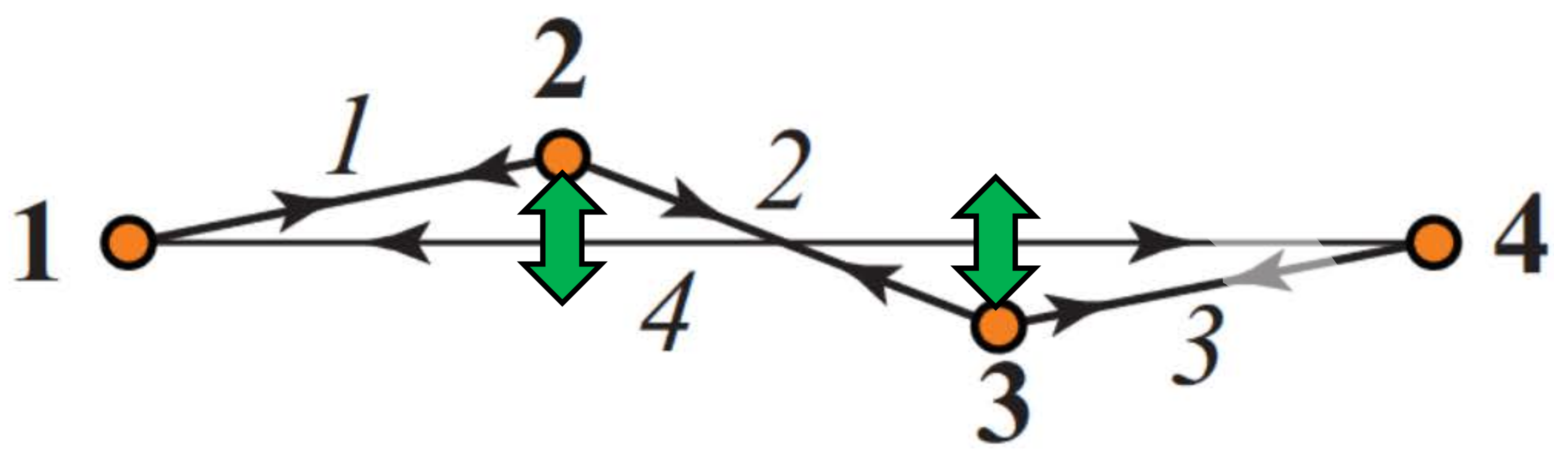
Becomes rigid when $L_{14} = L_{12} + L_{23} + L_{34}$

Connelly Advances in Mathematics (1980)

Lubensky et al., *Rep. Prog. Phys.* (2015).

c.f. Gorter talk yesterday, there is a (1,1) flex

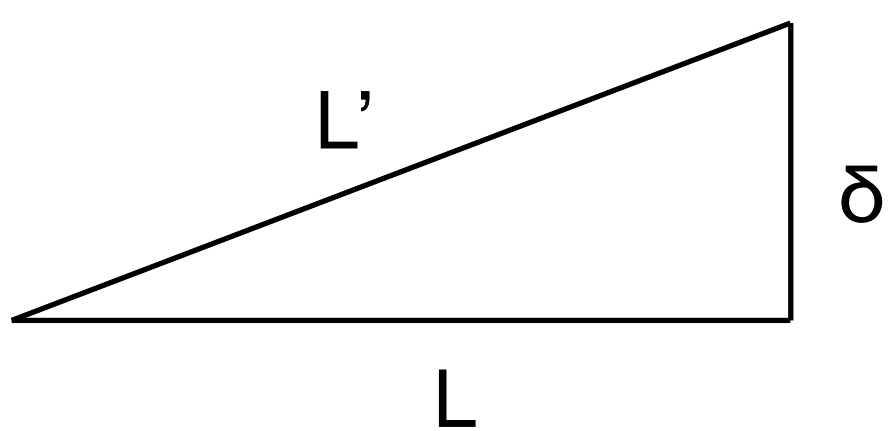
Revisit this example:



8 degrees of freedom (N_{dof})
4 constraints (M)
1 state of self stress (N_0)
3 rigid body motions

$$N_0 = N_{\rm dof} - M + N_s$$
$$= 8 - 4 + 1 = 5$$

$\Longrightarrow$ 2 non-trivial LZM's



$$L' = \sqrt{L^2 + \delta^2} = L + \frac{\delta^2}{2L} + \mathcal{O}\left(\frac{\delta}{L}\right)^4$$

Only changes constraints at 2nd order!

c.f. Gorter talk yesterday, there is no (1,2) flex

Connelly Advances in Mathematics (1980)

Lubensky et al., *Rep. Prog. Phys.* (2015).

Merkel et al PNAS 2019

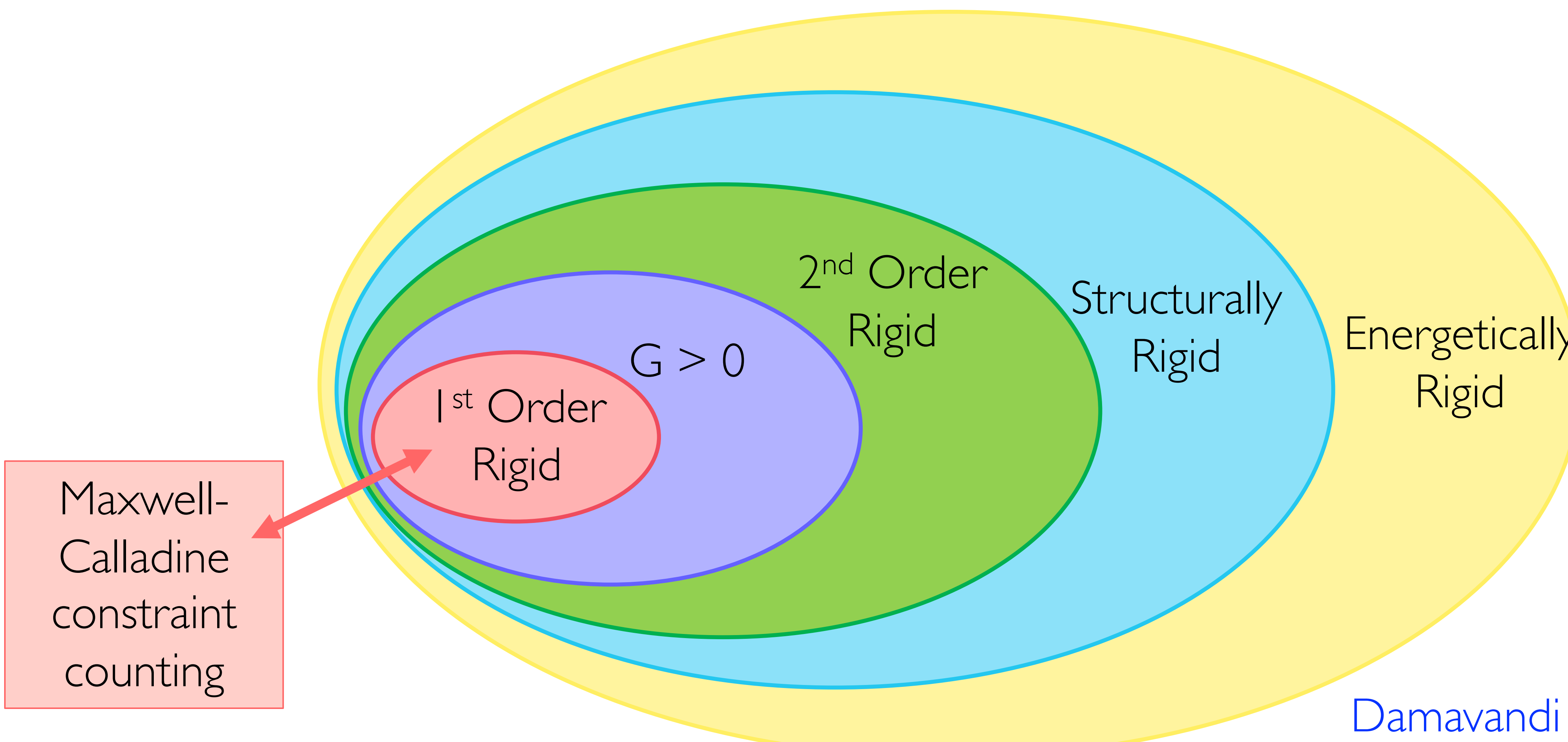
How do these more complicated 2D/3D disordered systems become rigid?


Damavandi


Hagh

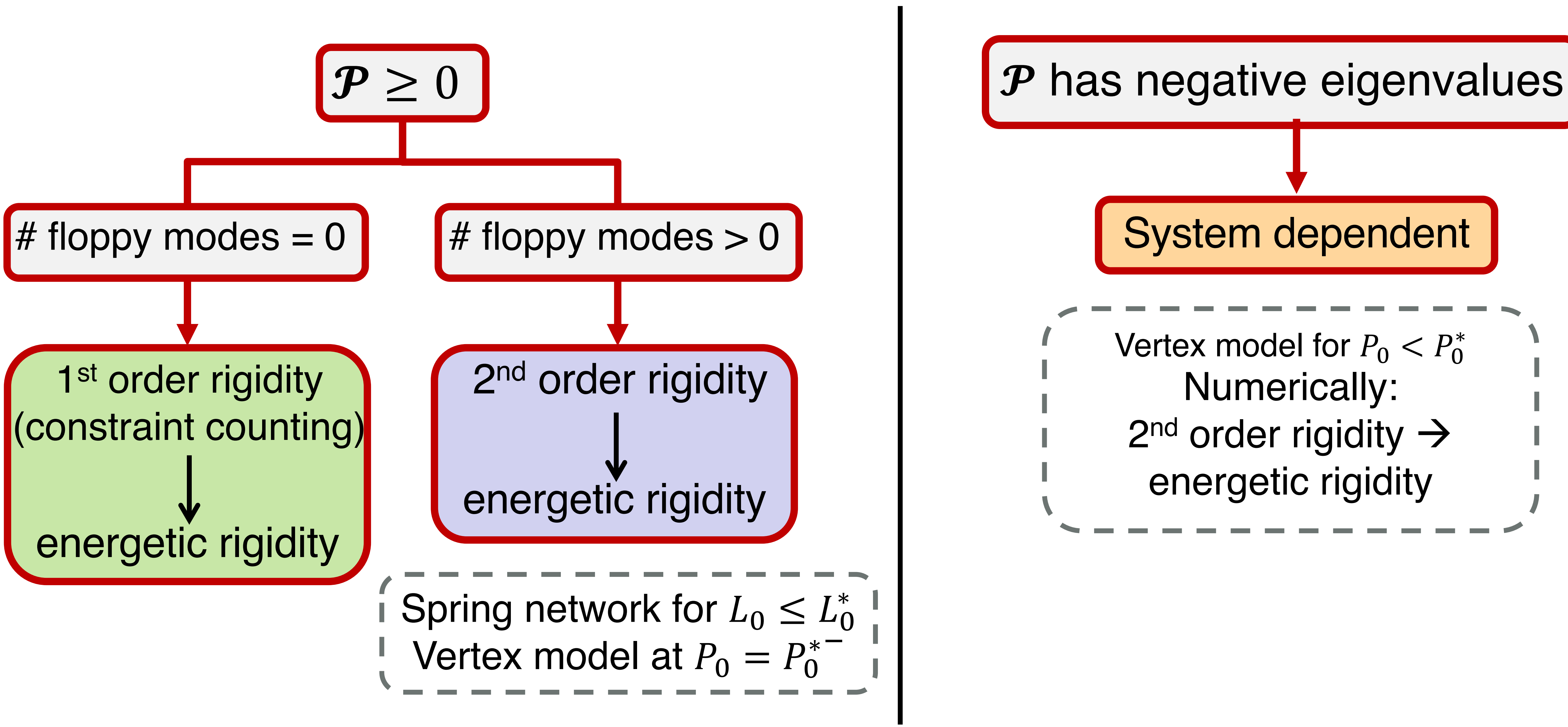

Santangelo

A system is …	when …
Energetically rigid	any nontrivial global motion increases the energy
Structurally rigid	no nontrivial global motion preserves the constraints f_α
First-order rigid	no nontrivial global motion preserves the constraints f_α to first order
Second-order rigid	no nontrivial global motion preserves the constraints f_α to second order



Damavandi et al *PRE* (2022)

Via second-order rigidity.

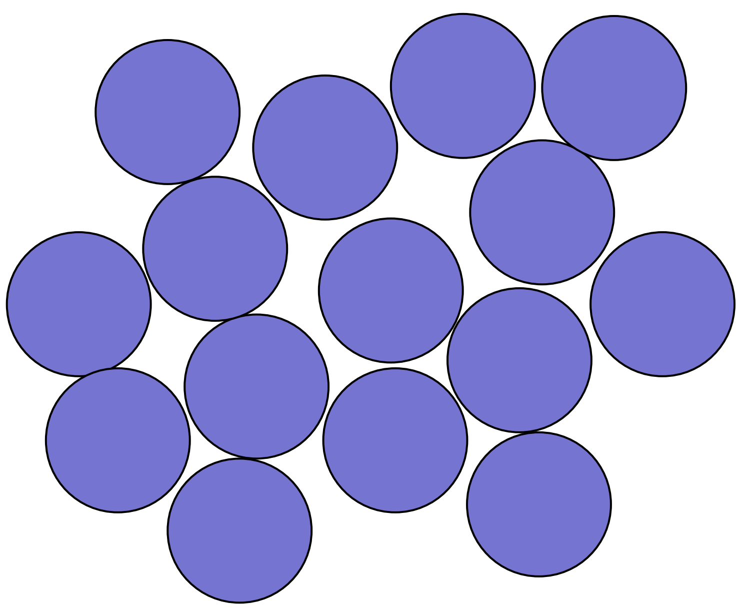


Damavandi et al, *PRE* (2022)

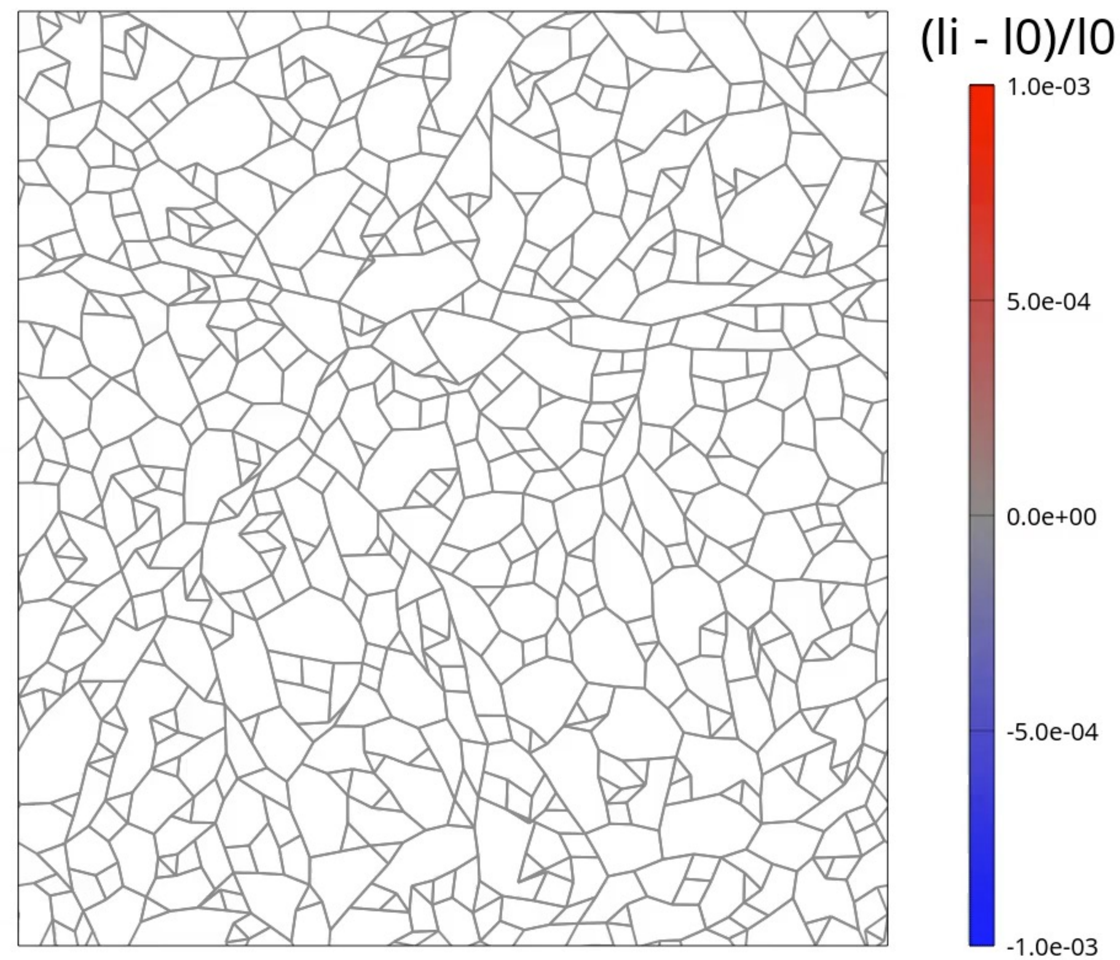
Goal: design materials that can be tuned to cross a rigidity transition via small changes to internal parameters

should we try do this via first-order or second-order rigidity?

First-order rigidity:
e.g. jamming transition
controlled by **topology** of contacts



- near the transition there are a very large number of "kissing contacts" or small gaps
- difficult to determine which ones become real contacts
- difficult to program
- Gardner transition physics



- network connectivity is fixed
- there are "adaptive degrees of freedom" (here the spring rest lengths) that are easy to program
- Can we enumerate all possible critically rigid configurations?
- Need to consider "floppy" vs. "fluid"

Chris Santangelo

Tyler Hain

Idea: Use second order rigidity to design mechanical metamaterials that can be tuned to cross a rigidity transition via small changes to internal parameters

focus on spring networks first.

Hain, Santangelo, Manning, PRE 2025

Description of a general spring network: mapping from vertex positions to bonds

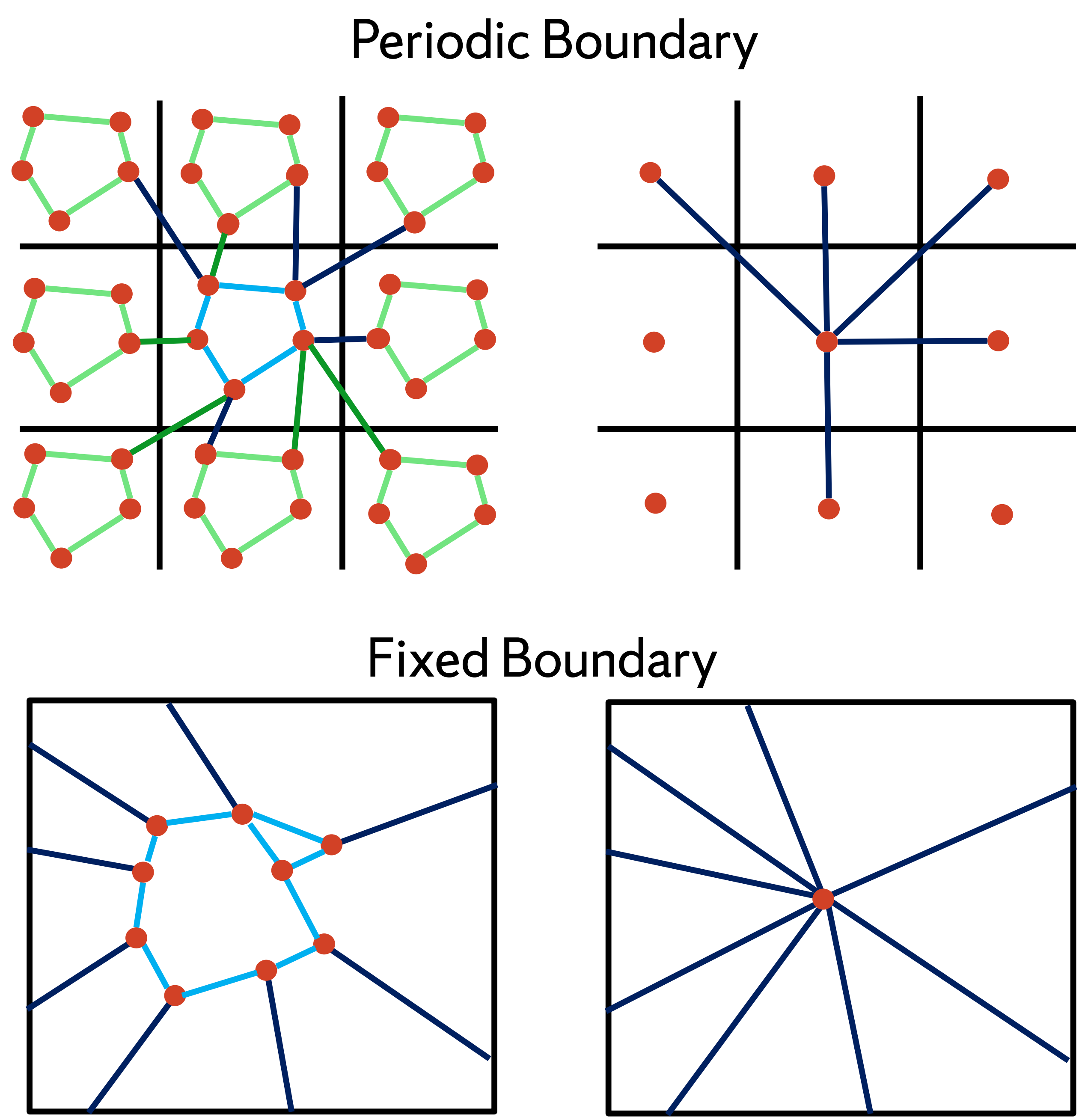
incidence matrix

$$L_{\alpha\mu} = g_{\alpha i} x_{i\mu} + b_{\alpha\mu}$$

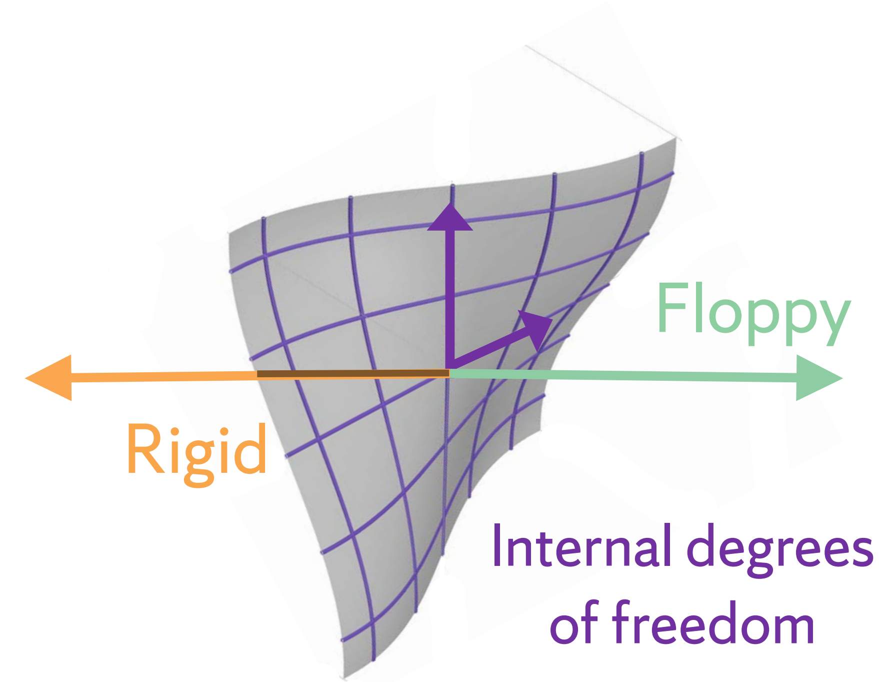
vertex positions

bonds/springs

Encodes boundary conditions: which edges don't go to zero when all vertices are collapsed?



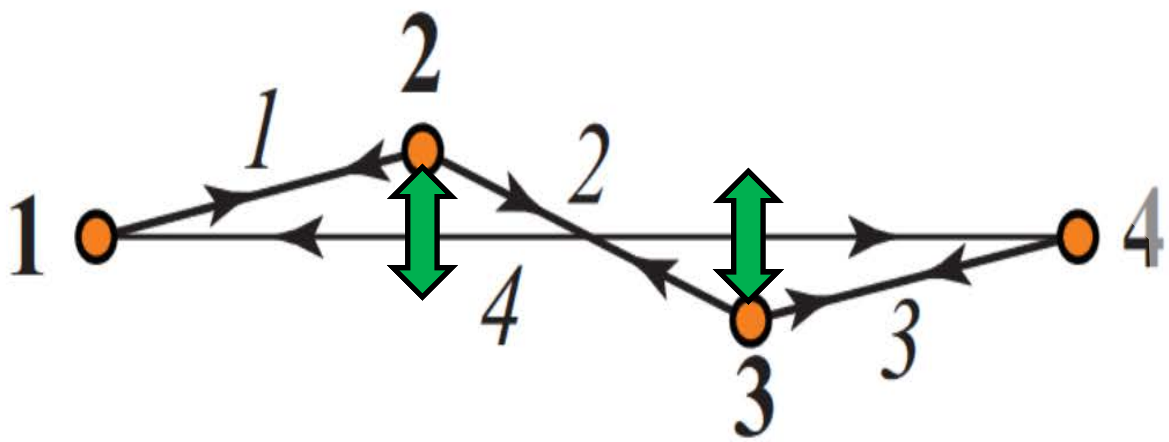
We need to figure out when a given configuration is precisely at the rigidity transition.

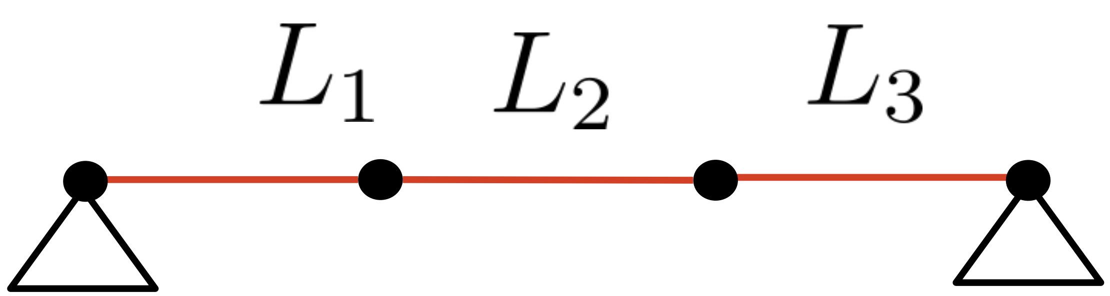


Our results (spoiler alert!):

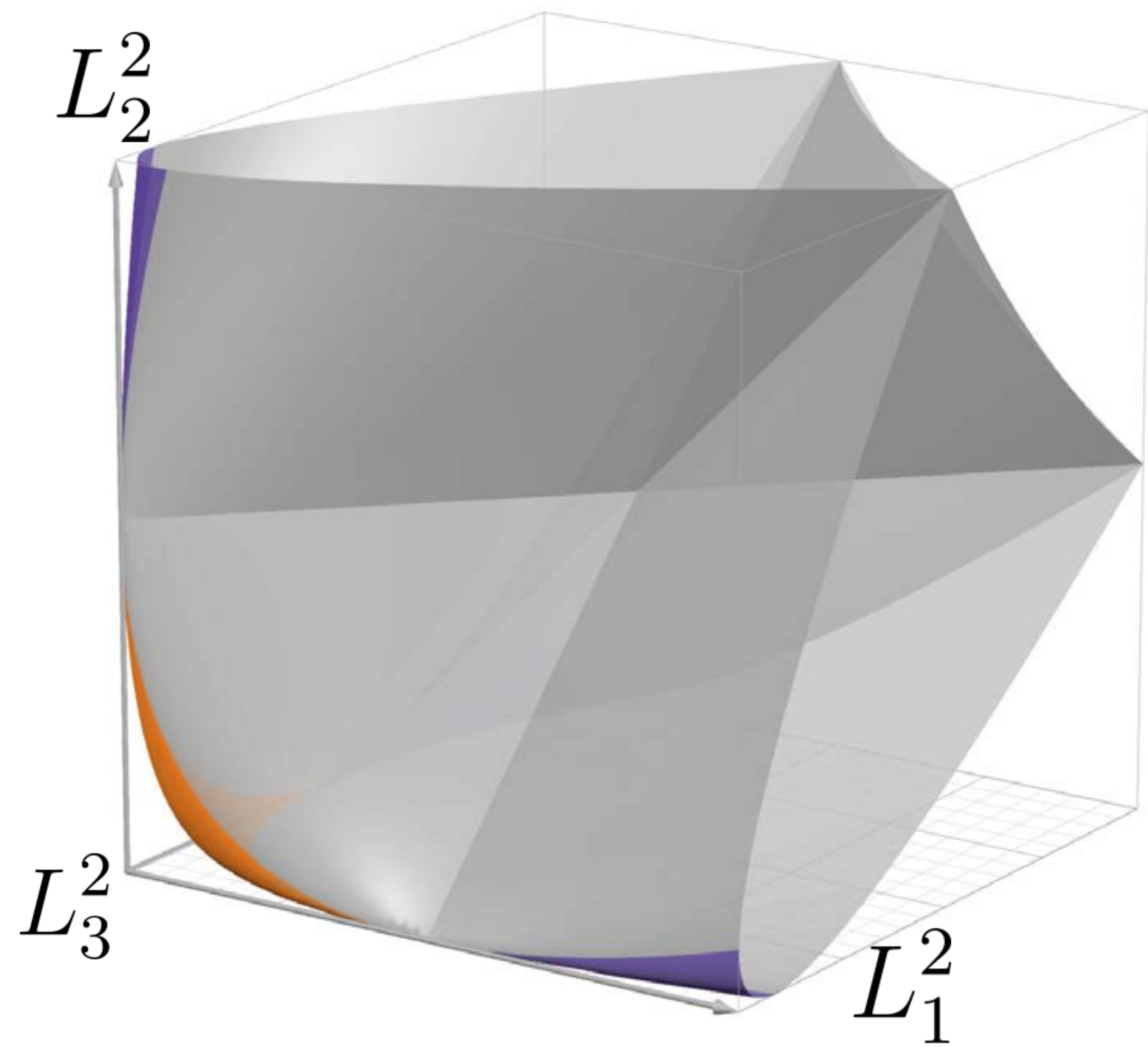
- There is a manifold of critically rigid states in the space of vertex configurations.
- This manifold is codimension one, which means you are very likely to “run into it”
- There is a set of natural coordinates that parameterize this manifold, so you can easily search on it.

to gain intuition, let's revisit this example (again):



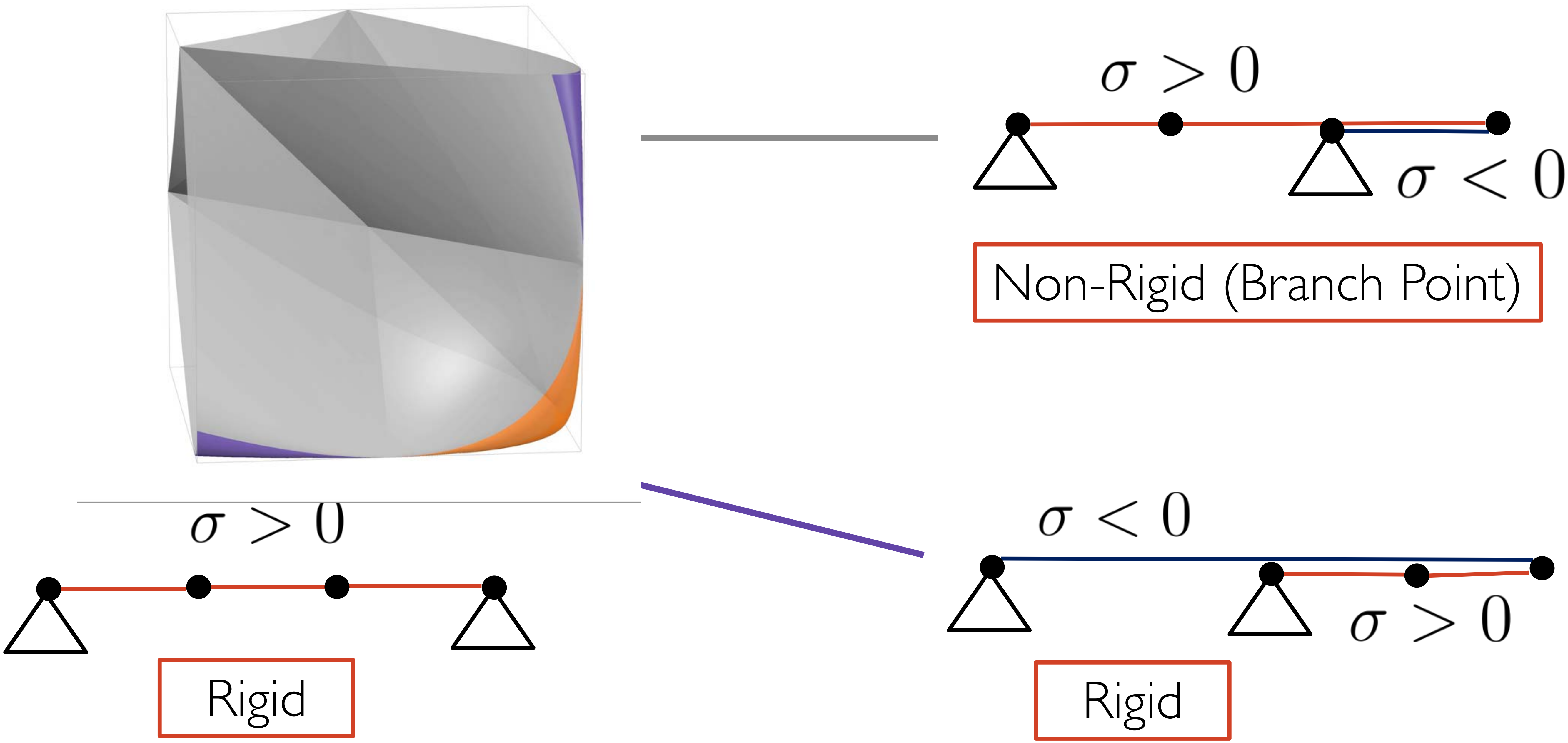


"3-Bar Linkage"

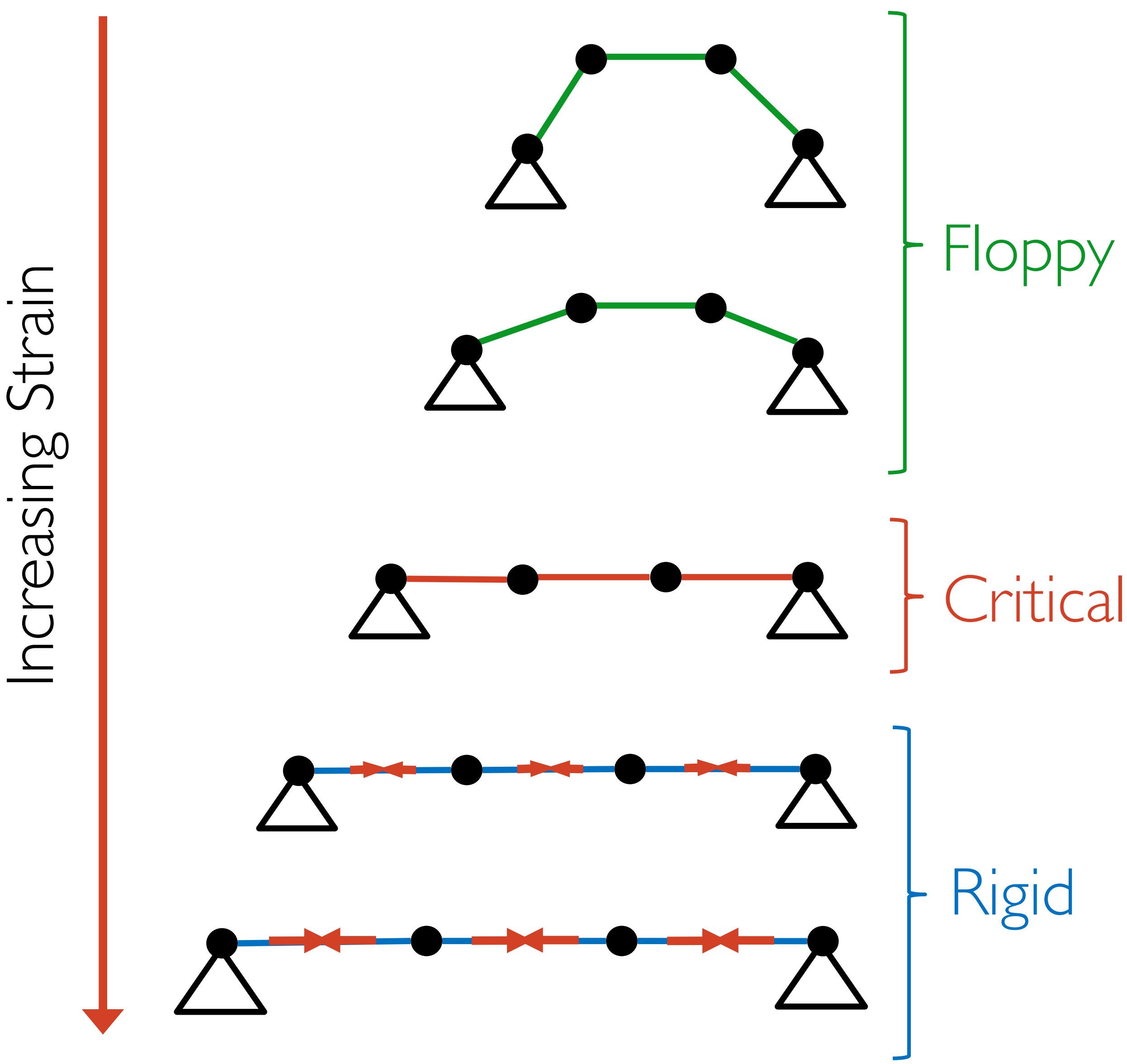


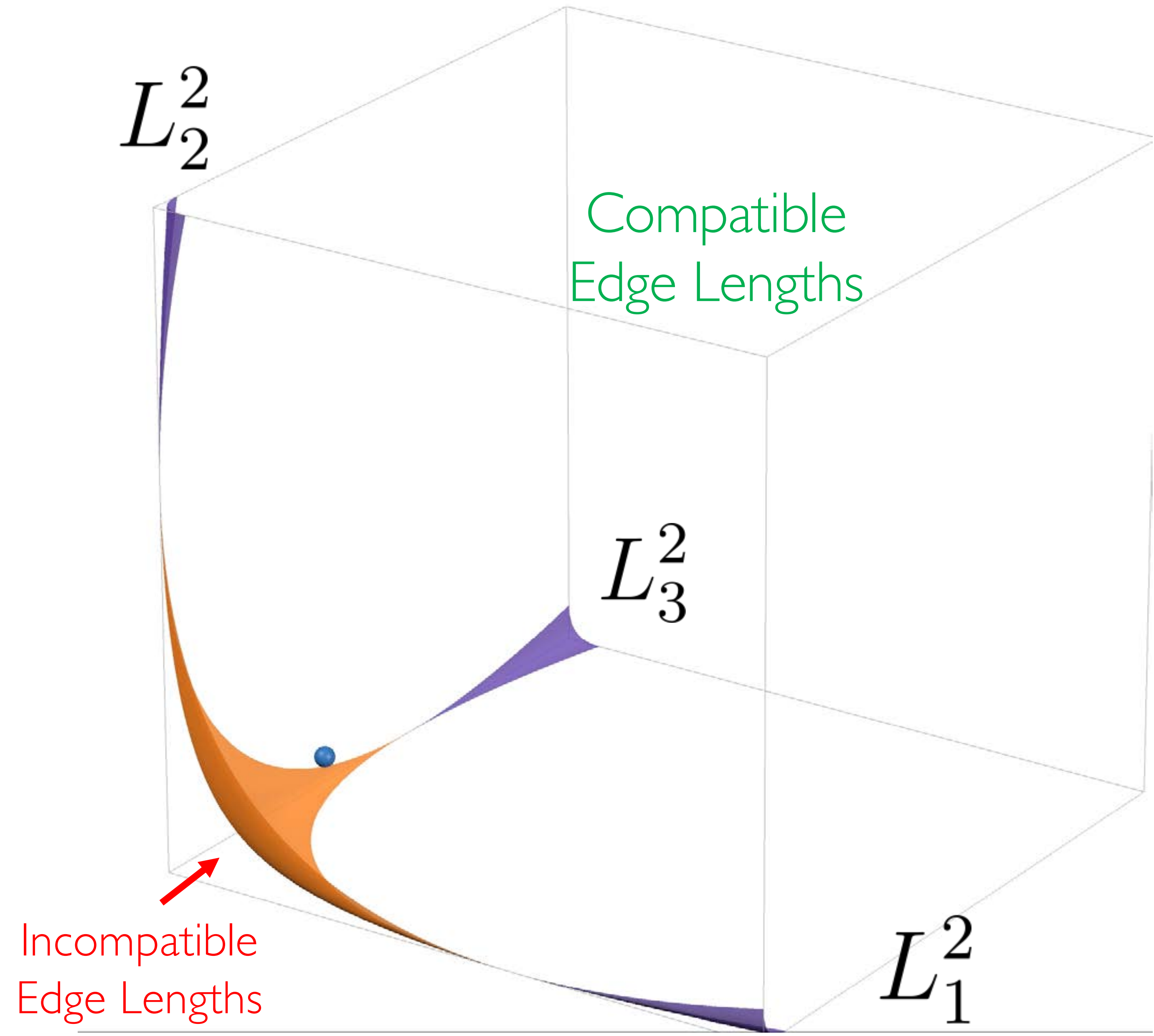
Lubensky et al., *Rep. Prog. Phys.* (2015).

Critical Manifold of the 3-Bar Linkage



The critical manifold forms the boundary between regions with compatible/incompatible geometry





Dilatational strain: e.g. changing distance between the pivot points

We'd like to show that something similar happens for complex, large, disordered networks.

Key idea: the critical manifold contains all configurations that satisfy force balance while having internal stresses (in the limit that the overall magnitude of the internal stresses approach zero):

Each state on the critical manifold corresponds to a generalized state of self stress $\boldsymbol{\sigma}$

$$F(\sigma, x) = 0$$

Net forces on vertices → F

Generalized self stresses → σ

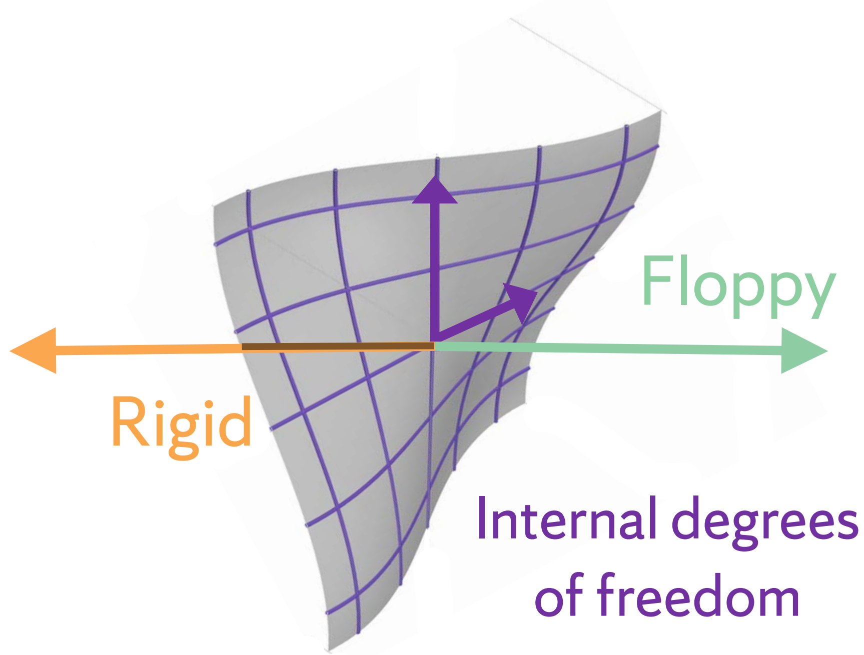
Coordinates of vertices → x

Can we make a new set of degrees of freedom that parameterizes the critical manifold?

Net forces on vertices → $$F(\sigma, x) = 0$$

Generalized self stresses → σ

Coordinates of vertices → x



For a given $\boldsymbol{\sigma}$ can we find a corresponding set of vertex configurations?

$$x(\sigma)$$

Describe internal stresses by coarse-graining the lowest level degrees of freedom (node coordinates) into higher geometric quantities (lengths, areas, etc)

Geometric relationship

$$F(\sigma, x) = \frac{\partial E}{\partial x_i} = \sum_\alpha \frac{\partial E}{\partial h_\alpha} \frac{\partial h_\alpha}{\partial x_i} = \sum_\alpha \sigma_\alpha \frac{\partial h_\alpha}{\partial x_i}$$

Generalized stress associated with h_α

If we choose $h_\alpha = L_\alpha^2/2$
the force balance equation is linear and therefore uniquely solvable!

$$\sigma_\alpha = \frac{T_\alpha}{L_\alpha}$$

Tension on edge α

Length of edge α

Generalized stress is a force density

$$F(\sigma, x) = P\vec{x} + \vec{b} = 0$$

Prestress Matrix

Quantifies boundary conditions

$$P = \sigma_\alpha \frac{\partial^2 h_\alpha}{\partial x_{i\mu} \partial x_{j\nu}} = [\sigma_\alpha g_{\alpha i} g_{\alpha j}]\delta_{\mu\nu}$$

Schek, Comput. Methods Appl. Mech. Eng. (1974)

Yes, we can parameterize the critical manifold!

If you give me your favorite

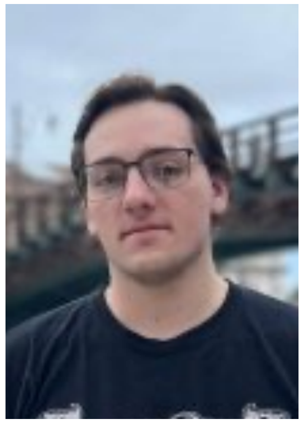
- Network structure: g
- Boundary conditions: b
- Geometric stress: σ

we can give you the coordinates of the corresponding critical configuration:

$$x_{i\mu}(\sigma) = -\sum_{\alpha j} P^{-1}_{ij} g_{\alpha j} \sigma_\alpha b_{\alpha\mu}$$

An h set that gives us a complete and linear mapping from vertex model coordinates to generalized self stress:

Kelly Aspinwall



Tyler Hain

Vertex model Energy Functional:

$$E = \sum_f \left[\frac{K_P}{2}(P_f - P_{0f})^2 + \frac{K_A}{2}(A_f - A_{0f})^2 \right]$$

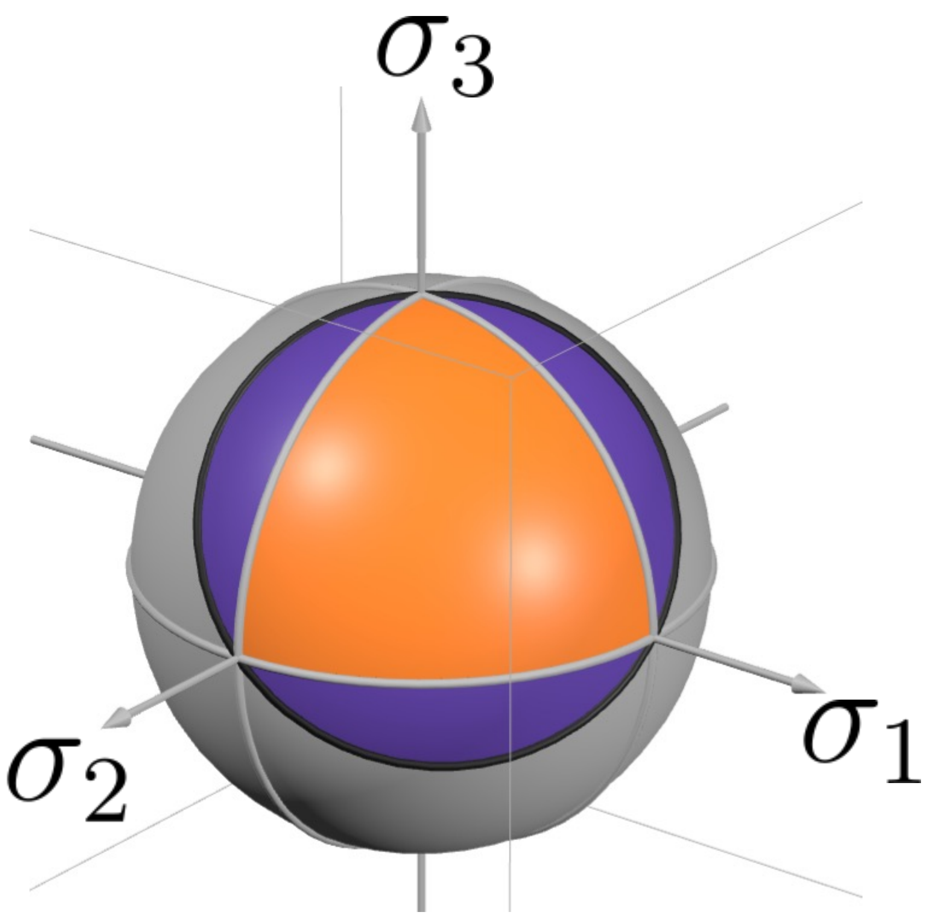
Perimeter Term — Area Term

$$h = \{\frac{1}{2}L_\alpha^2, A_f\}$$

$$\sigma_\alpha^P = \frac{\tau_\alpha}{L_\alpha} \qquad \sigma_f^A = K_A(A_f - A_{0f})$$

$$R_{\alpha i \mu}^P = L_{\alpha\mu} g_{\alpha i} \qquad R_{f i \mu}^A = \frac{g_{\alpha i}}{2} \begin{pmatrix} G_{\alpha\beta}^f L_{\beta y} \\ -G_{\alpha\beta}^f L_{\beta x} \end{pmatrix}$$

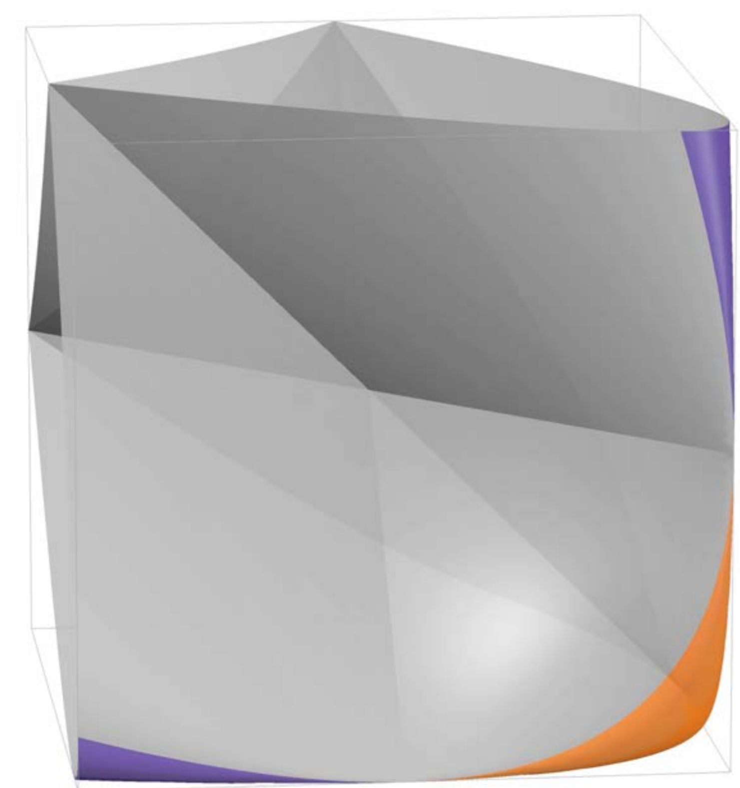
Critical Manifold of the 3-Bar Linkage

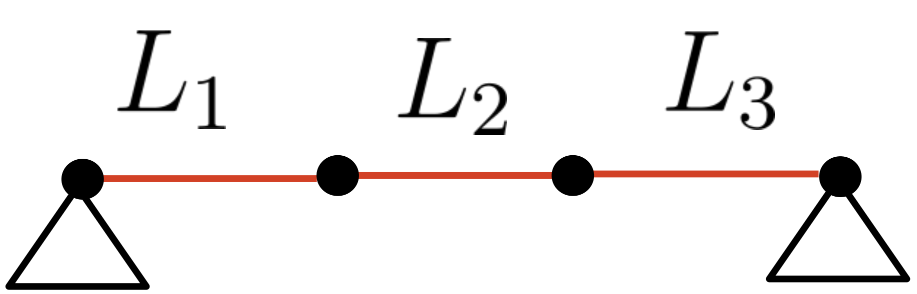


Space of Self-Stresses

$$x_{i\mu}(\sigma)$$

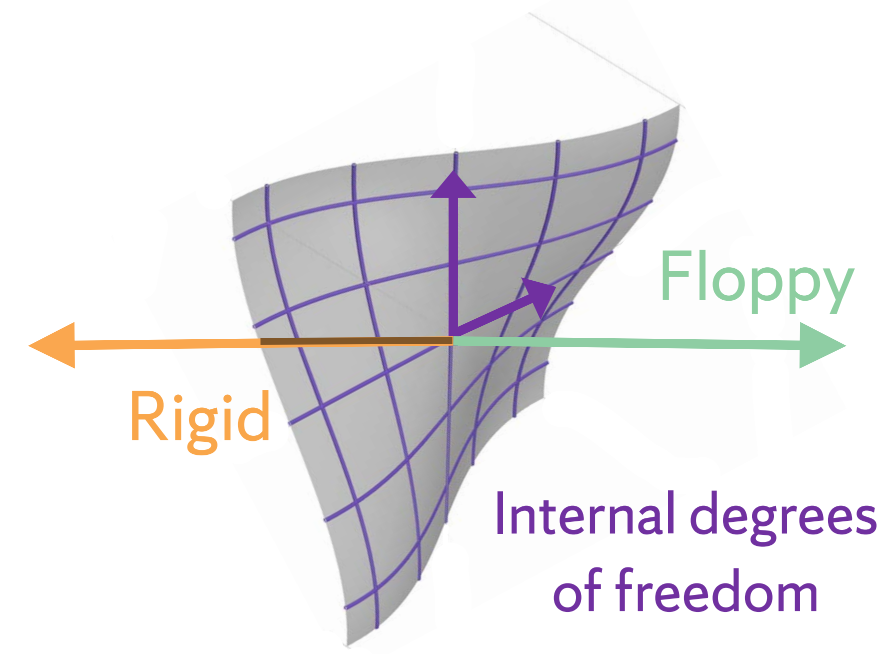
$$P\vec{x} + \vec{b} = 0$$





For disordered central force networks, we can use this parameterization

$$x_{i\mu}(\sigma) = -\sum_{\alpha j} P^{-1}_{ij} g_{\alpha j} \sigma_\alpha b_{\alpha\mu}$$



to demonstrate:

- the critical manifold is codimension one in the space of vertices
- the generalized self-stress vector is normal to the critical manifold
- therefore it is easy to move around on the critical manifold by looking at how geometric stresses change

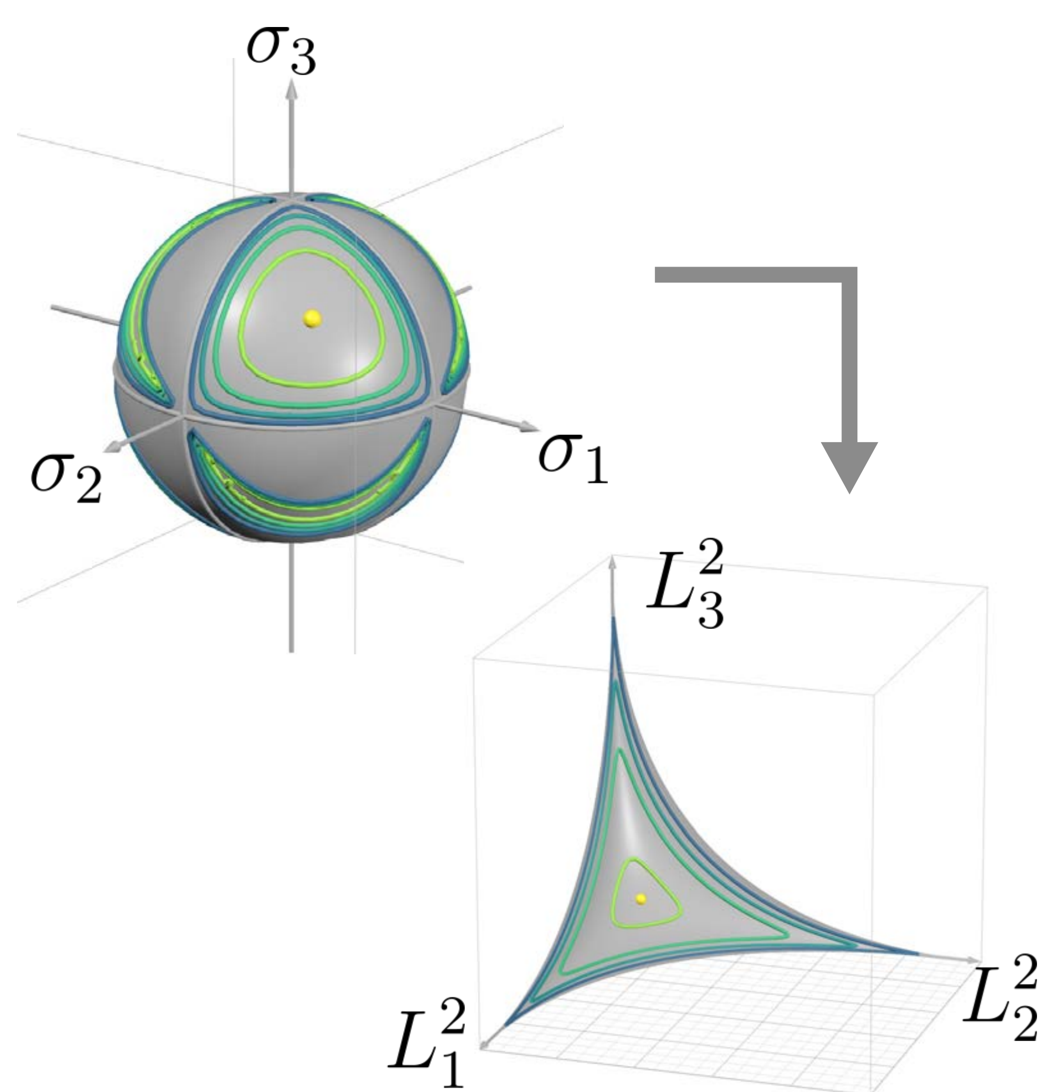
Hain, Santangelo, Manning, PRE 2025

With this self-stress parameterization, we can rationally search the critical manifold for special configurations

Use gradient descent to traverse the space of self-stresses to optimize any objective function

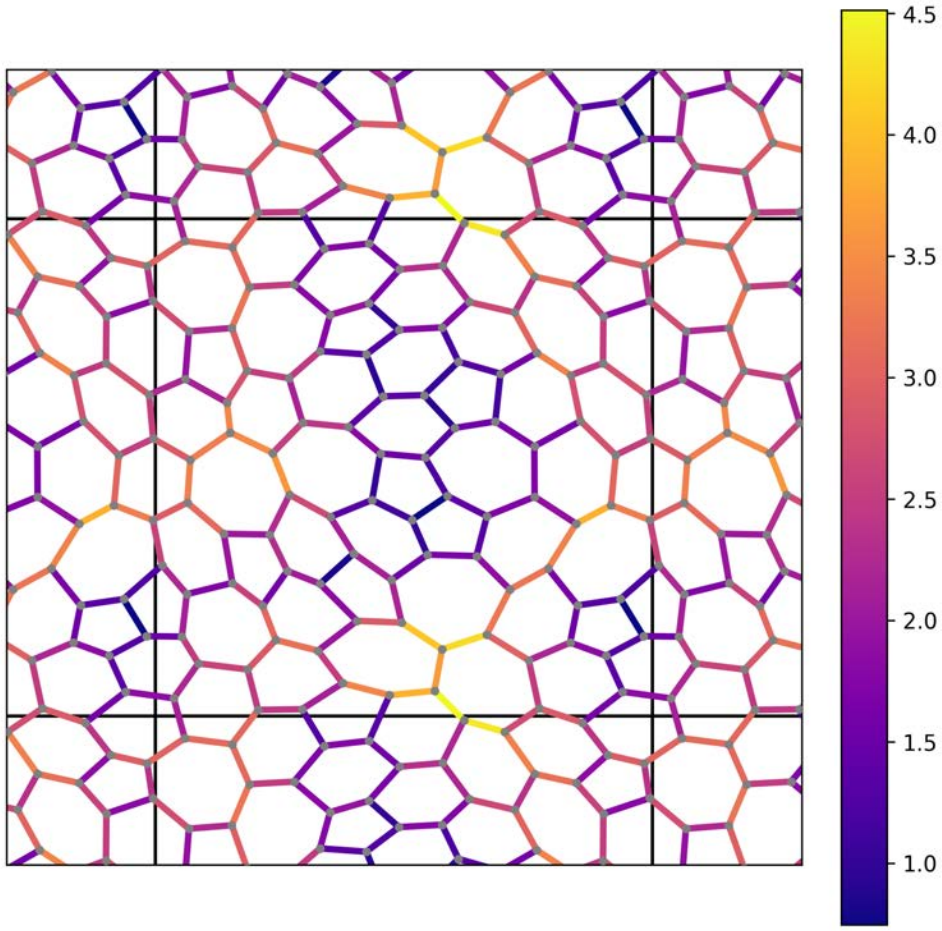
$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}\sigma_\alpha} = \frac{\partial \mathcal{O}}{\partial \sigma_\alpha} + \sum_{\beta\mu} \frac{\partial \mathcal{O}}{\partial L_{\beta\mu}} \frac{\partial L_{\beta\mu}}{\partial \sigma_\alpha}$$

Self-stress parameterization lets us take a total derivative!

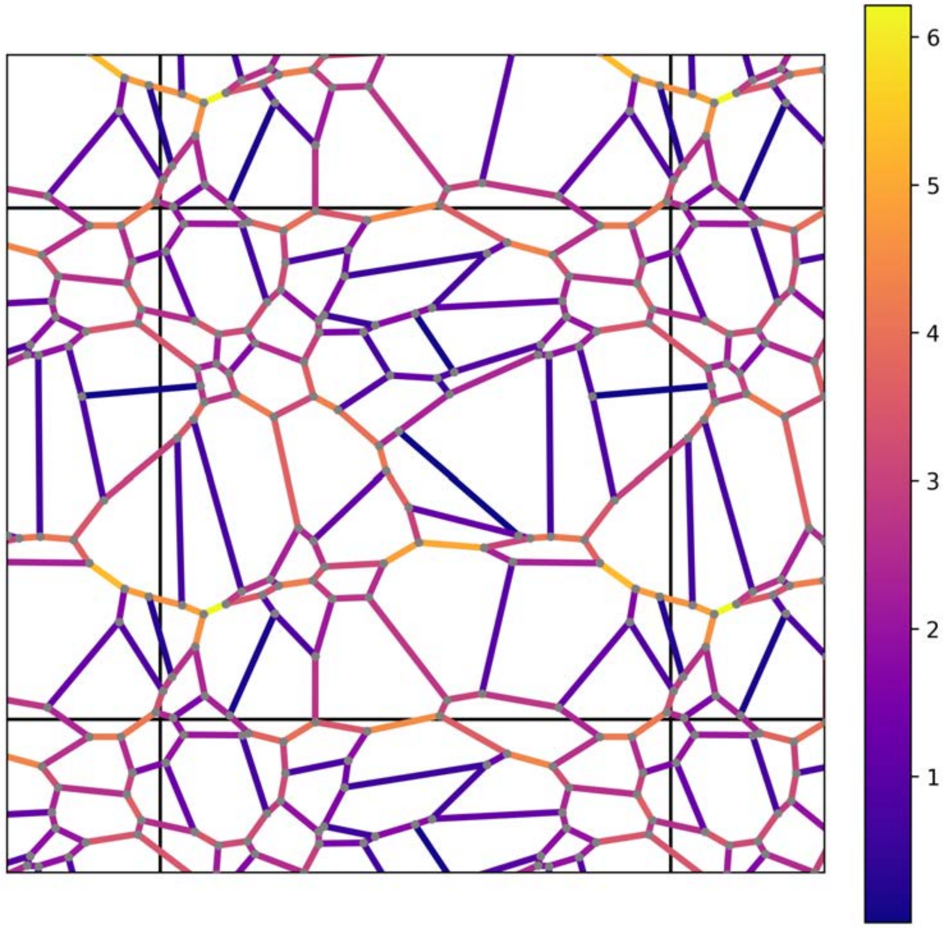


Randomly generated configurations

Randomly assign rest lengths, strain to the critical manifold



Randomly assign self-stresses, use parameterization to get configuration



With this self-stress parameterization, we can rationally search the critical manifold for special configurations

Structure-Based Objective Functions:

Say we want to find rigid networks with regular structure: e.g. all edges have equal lengths or equal tensions

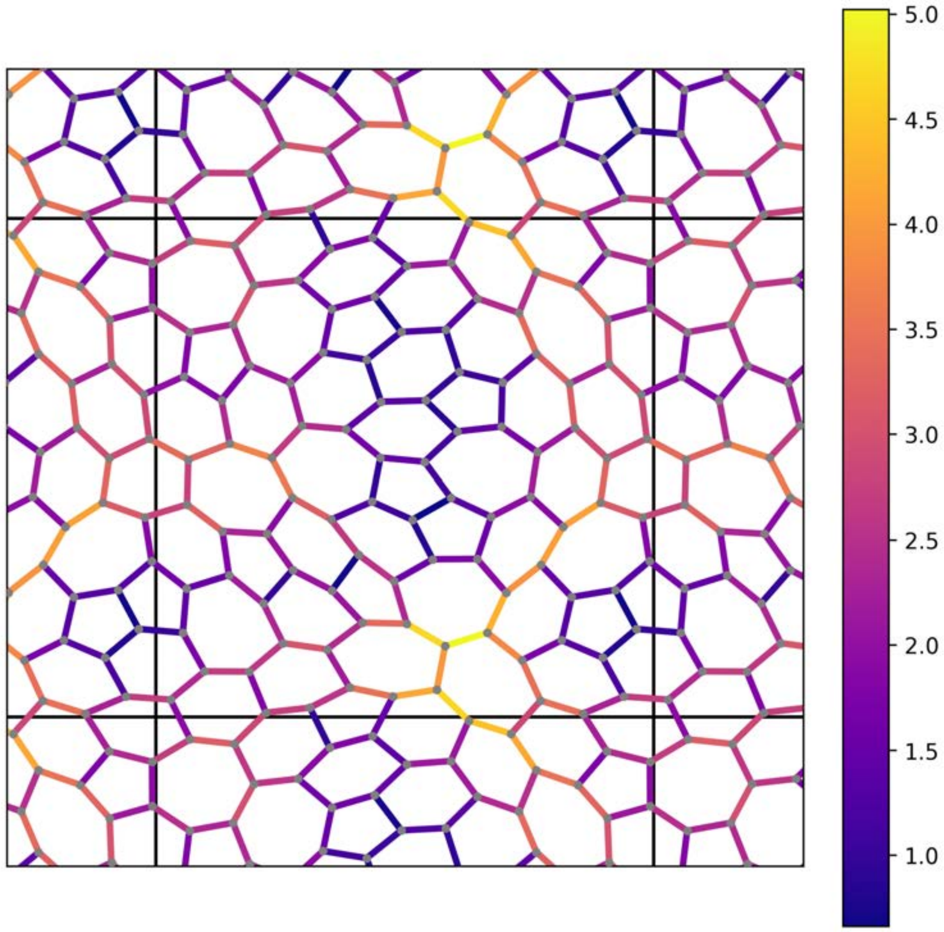
We can minimize the fluctuations of these quantities

$$V_L = \frac{\langle (L - \langle L \rangle)^2 \rangle}{\langle L \rangle^2}$$

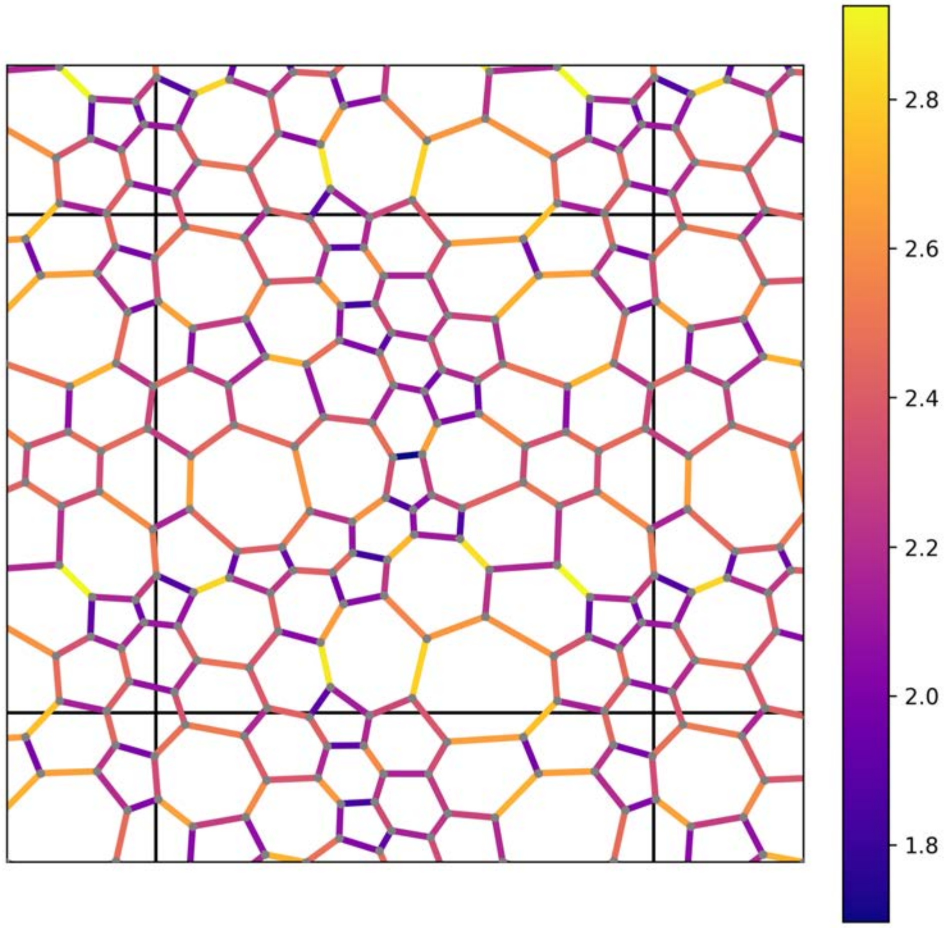
$$V_\tau = \frac{\langle (\tau - \langle \tau \rangle)^2 \rangle}{\langle \tau \rangle^2}$$

Structural Objective Functions

Minimized length fluctuations



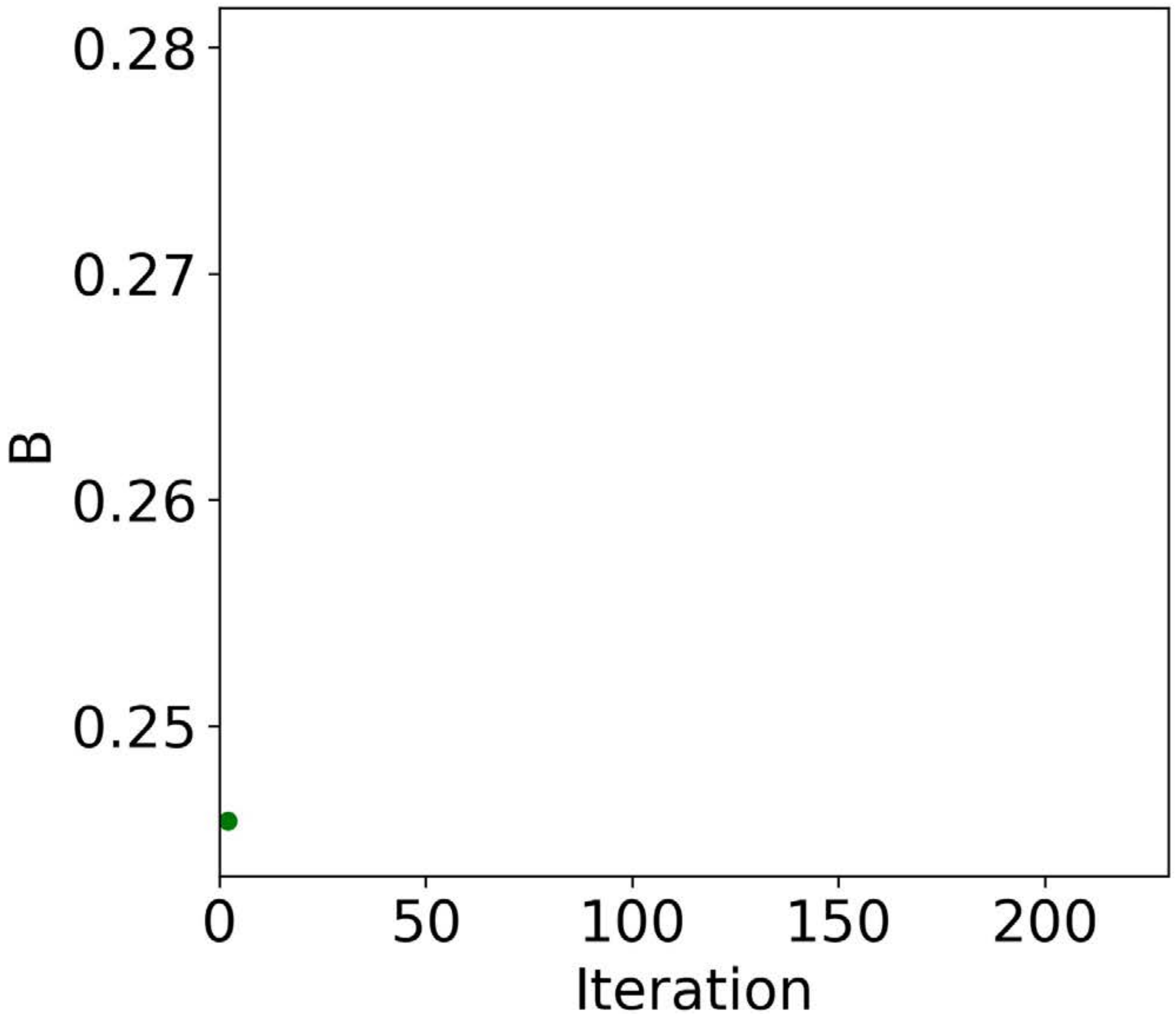
Minimized tension fluctuations

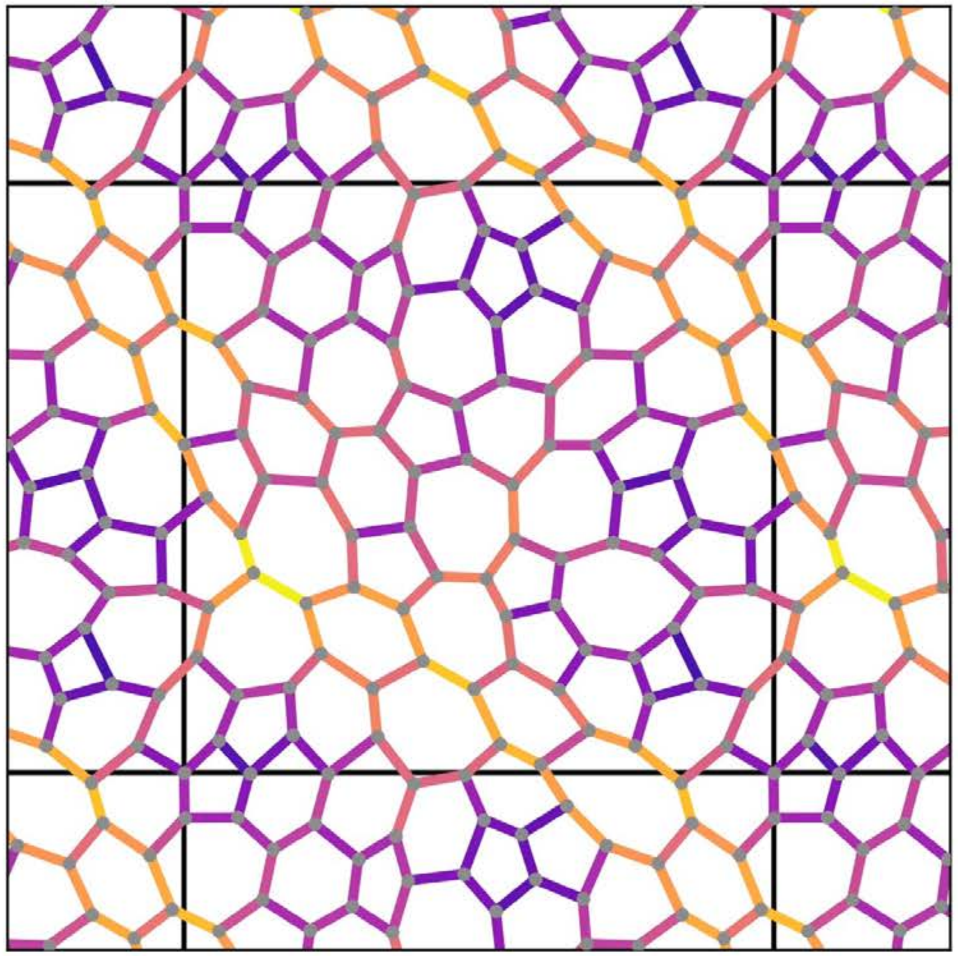


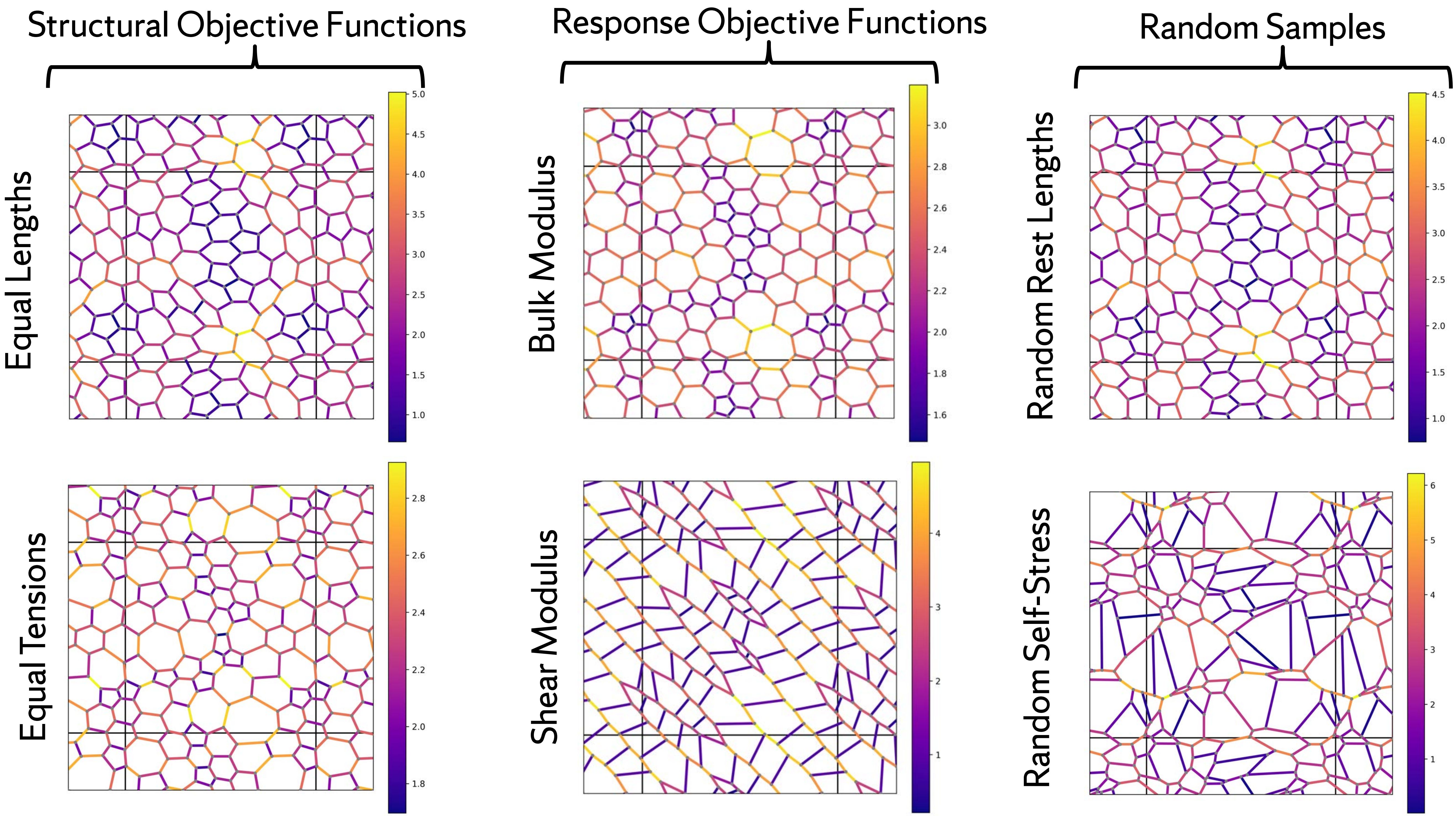
For the graphs we studied, we could ALWAYS find a network where all the edges were of equal length!
Such networks can be self assembled!
We are currently trying to build them with DNA origami in collaboration with Ben Rodgers (Brandeis)

Maximize bulk modulus at the transition

$$\dot{h}_\alpha = L_\alpha^2$$

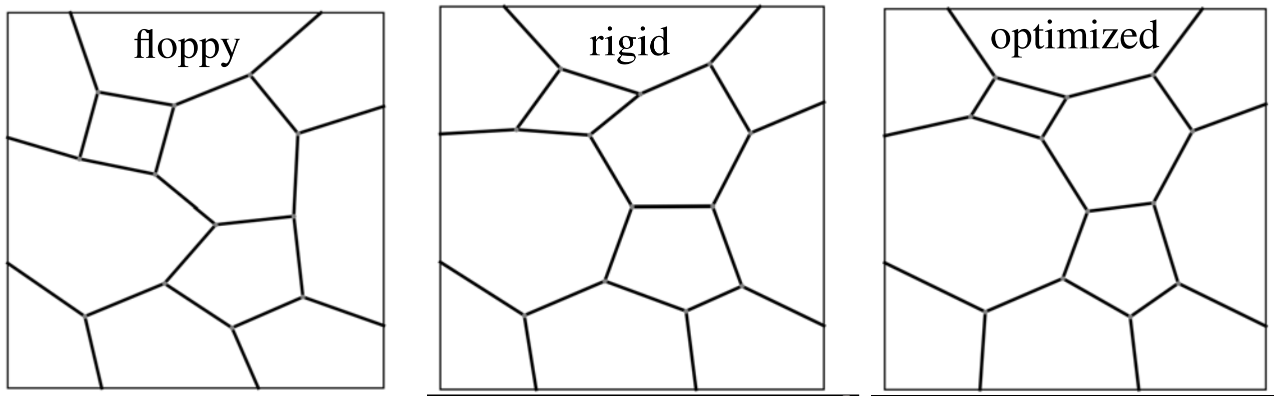


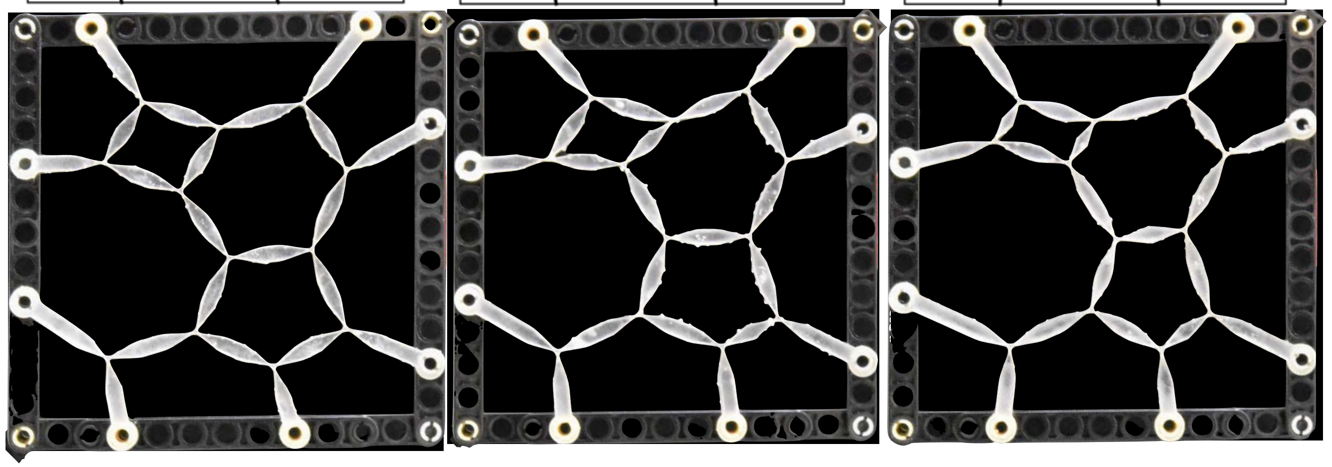




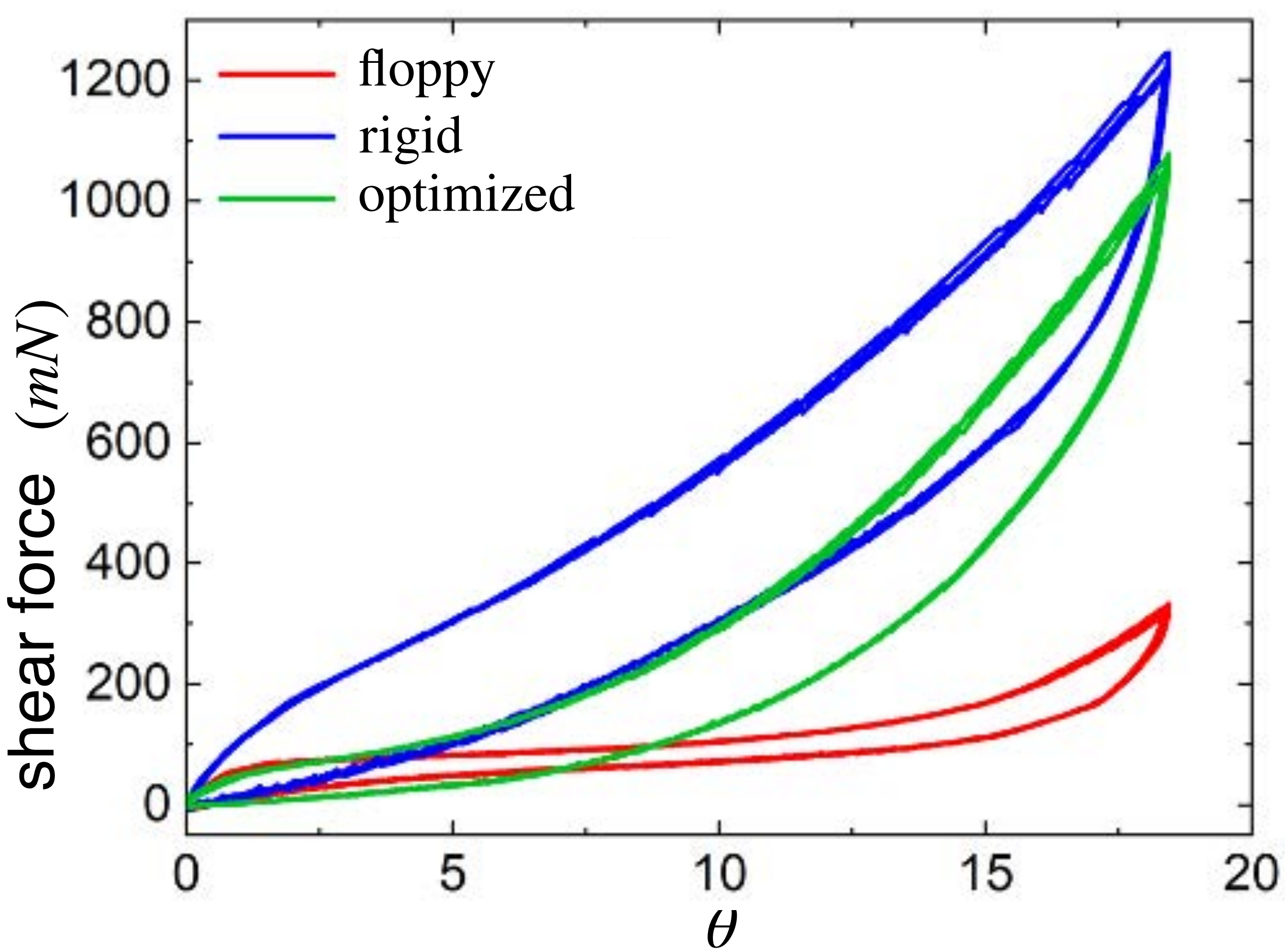
Structural Objective Functions
Response Objective Functions
Random Samples
Equal Lengths
Bulk Modulus
Random Rest Lengths
Equal Tensions
Shear Modulus
Random Self-Stress

optimized: maximum bulk stiffness / minimum shear stiffness





floppy rigid optimized



Joe Roback Ryan Hayward

Preliminary: Examples of small designed networks

3D printed hydrogel

Summary:

- Disordered mechanical networks (jammed packings, fiber networks, cells) can use multiple pathways to regulate rigidity/fluidity:
  - first order rigidity (changing packing/density)
  - second order rigidity (confluent tissues and ECM networks + mechanical metamaterials)
  - also fluctuations/temperature (didn't emphasize today)
- We have developed a method to design disordered mechanical metamaterials poised at a second-order rigidity transition
  - by demonstrating that there is a manifold of states at the critical point, which we can parameterize
  - we can search this manifold to find configurations that optimize an additional objective function.

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