The tropical variety of symmetric rank 2 matrices

(Joint work with May Cai and Josephine Yu)

Kisun Lee - Clemson University
Matroids, Rigidity, and Algebraic Statistics

Intro 1 - Matrix completion problems

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gcr depends on the positions of known entries

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Assuming the diagonal entries are always known, it has an application to the maximum likelihood threshold:

- **Gross-Sullivant 2018**
- **Blekherman-Sinn 2018**
- **Bernstein-Dewar-Gortler-Nixon-Sitharam-Theran 2024**

In 2018...

Nonlinear Algebra

Sep 5 - Dec 7, 2018
Semester Program

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We define the collection of independent sets:
$\mathcal{I} = \{S \subset E \mid P \cap k[S] = \langle 0 \rangle\}$
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🌴 An independent set of algebraic matroid of $\mathcal{S}_n^{r \leq m}(\mathbb{C})$ corresponds to a partial matrix that can be completed to rank $\leq m$.

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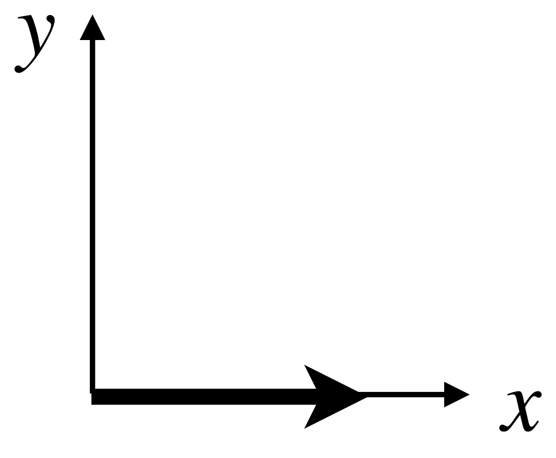

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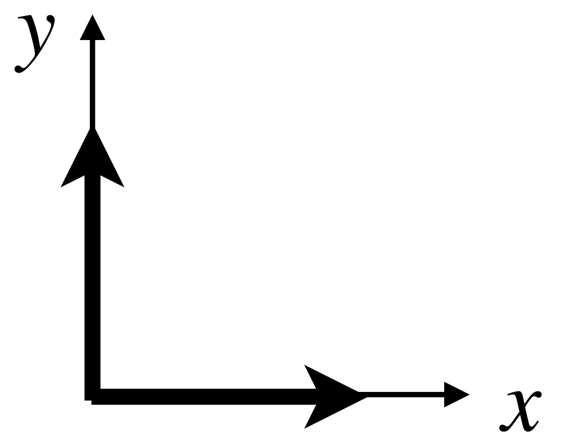

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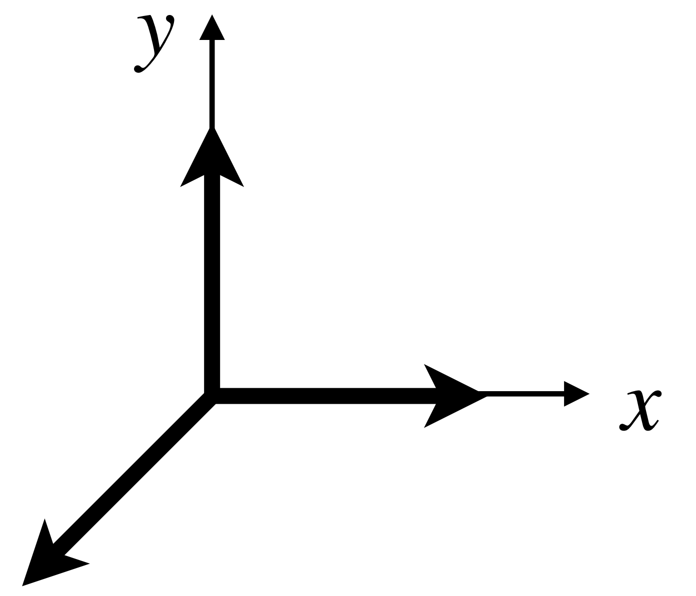

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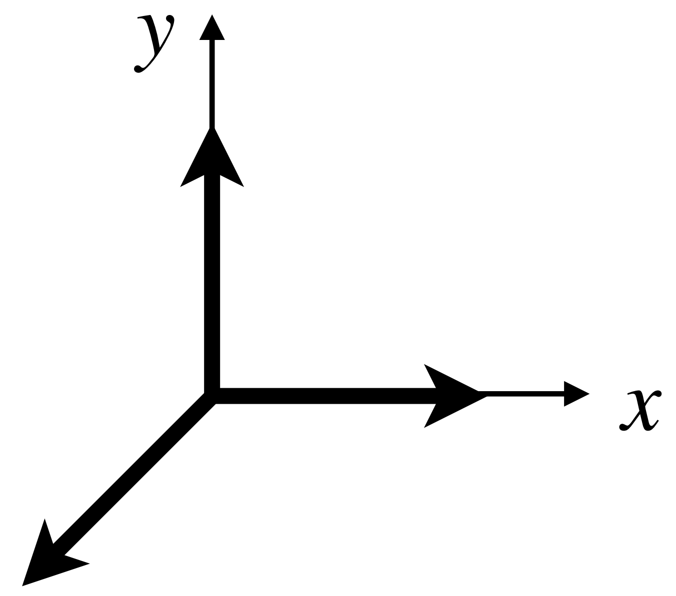

I: an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ with $V = V(I)$

$\mathrm{trop}(V) = \bigcap_{f \in I} \mathrm{trop}(V(f))$ (a tropical variety)

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If V is an irreducible variety of d-dimensional, then trop(V) is the support of a balanced fan of pure dimension d connected through codimension 1.

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Q. What can we say about the tropicalization of rank 2 symmetric matrices?


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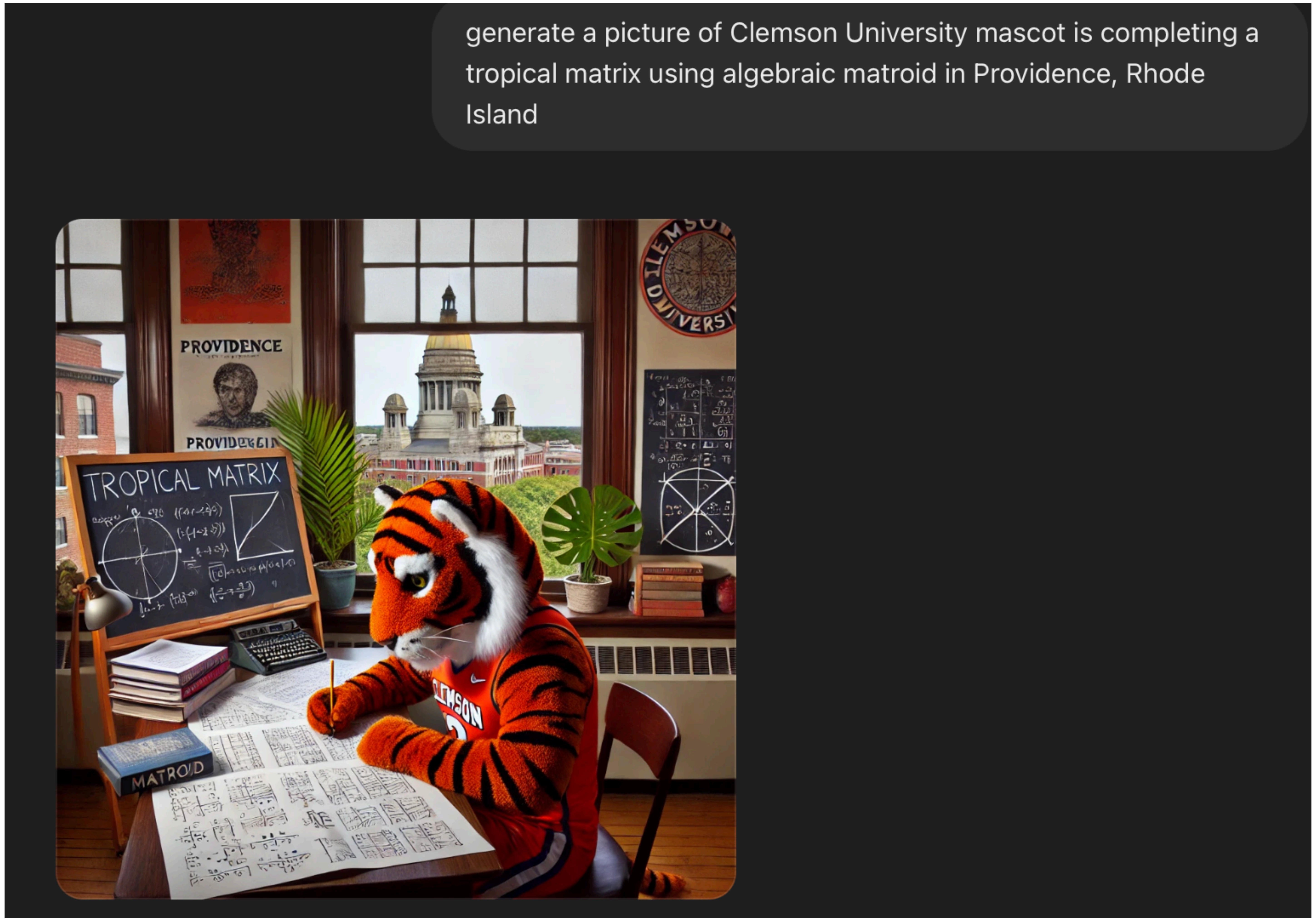
TROPICAL MATRIX
MATROID

Study tropical matrix completion using algebraic matroids



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Ranks in tropical matrices

An $m \times n$ matrix M has rank r if

- Smallest r such that $A = BC$ for an $m \times r$ matrix B and $r \times n$ matrix C

- Smallest dimensional linear space containing the columns of M

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(Barvinok rank) $\geq$ (Kapranov rank) $\geq$ (tropical rank)

Tropical rank

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The 3×3 minor

$$-x_{13}x_{22}x_{31} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{11}x_{22}x_{33}$$

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The tropical 3×3 minor vanishes if

$x_{13} \odot x_{22} \odot x_{31} \oplus x_{12} \odot x_{23} \odot x_{31} \oplus x_{13} \odot x_{21} \odot x_{32} \oplus x_{11} \odot x_{23} \odot x_{32} \oplus x_{12} \odot x_{21} \odot x_{33} \oplus x_{11} \odot x_{22} \odot x_{33}$

attains its minimum at least twice

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So, symmetric tropical rank 3

(even though it has tropical rank 2)

Ranks for symmetric matrices

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studied by **Cartwright-Chan 2012**

- Symmetric Barvinok rank
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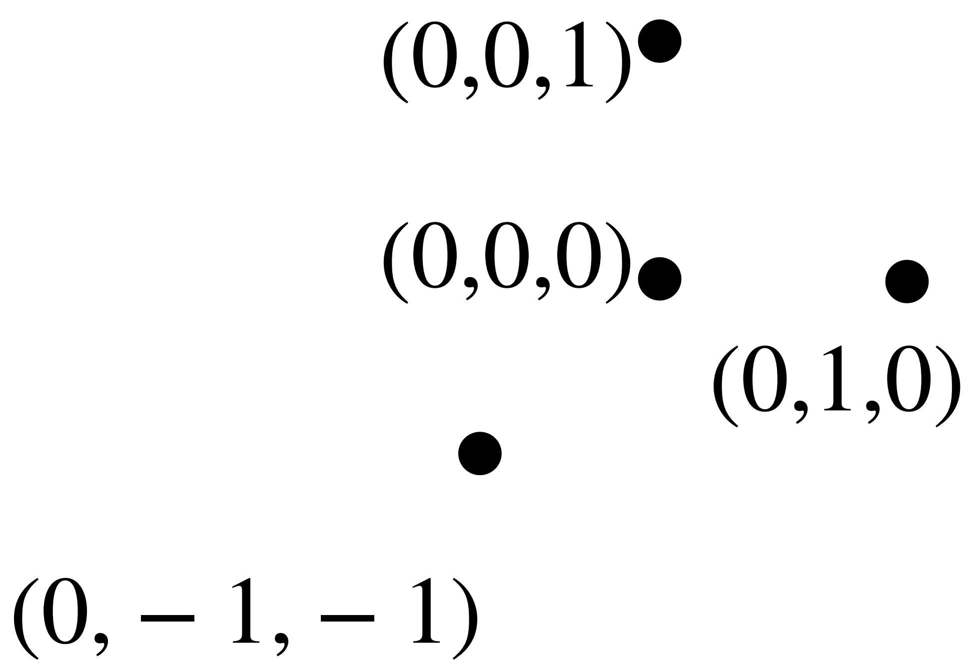

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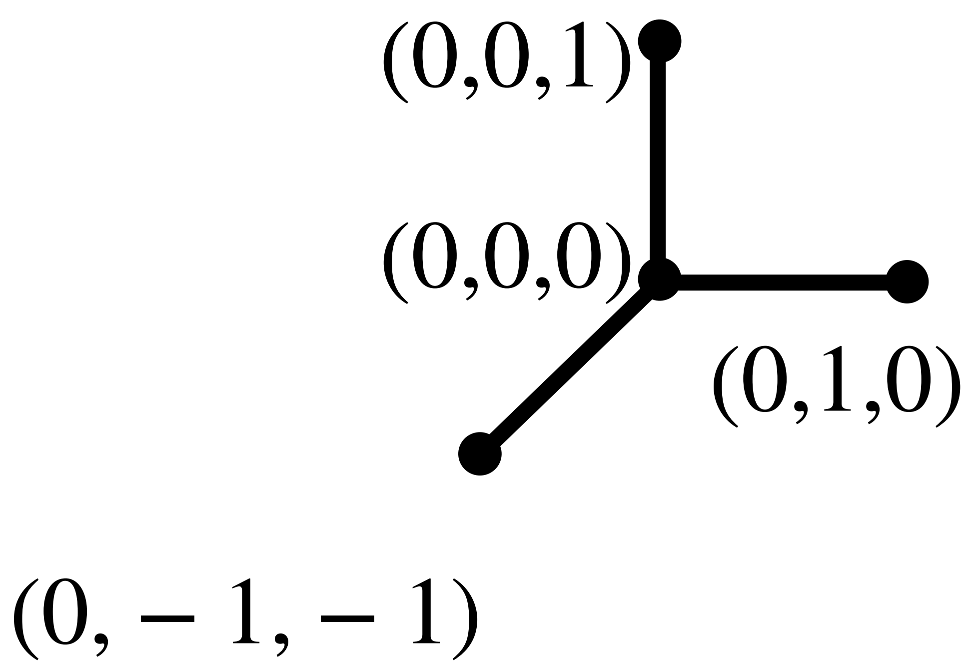

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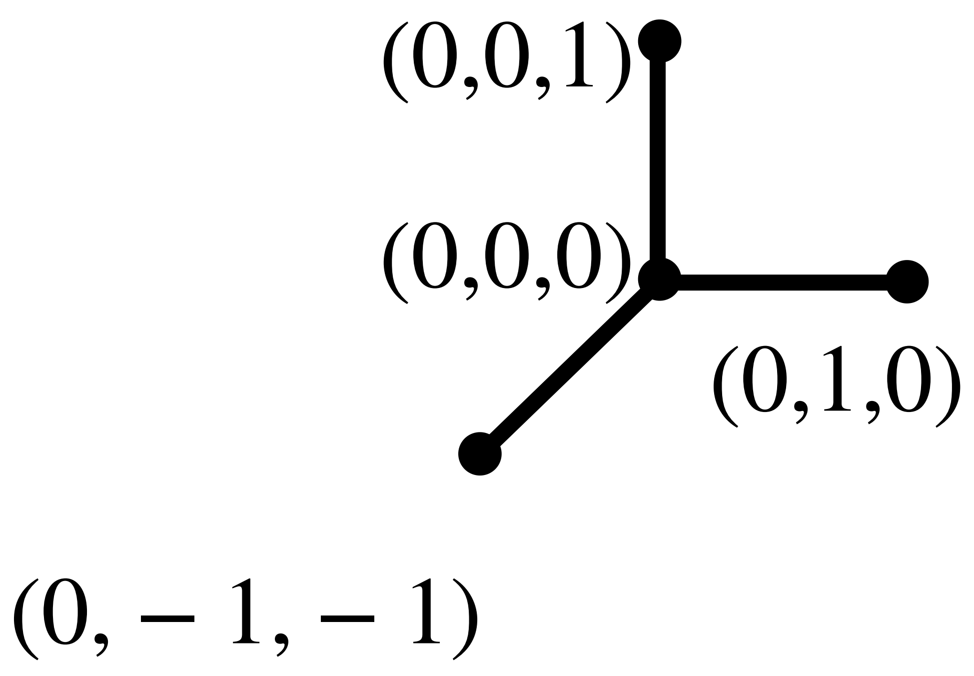

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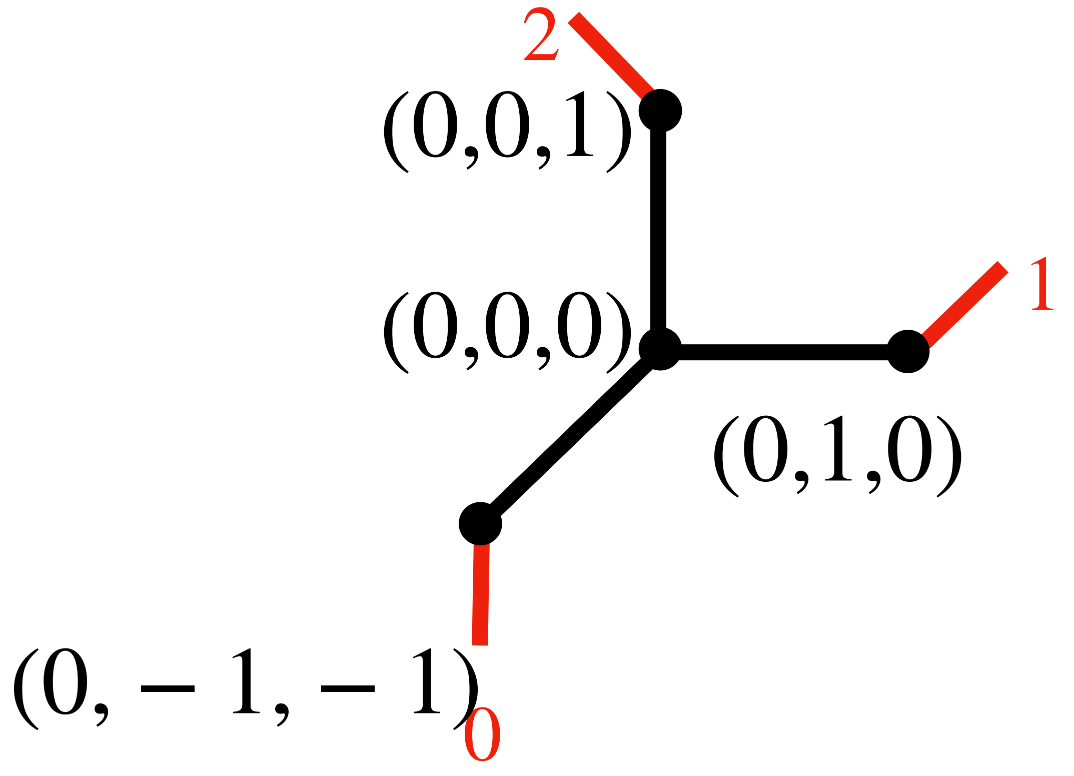

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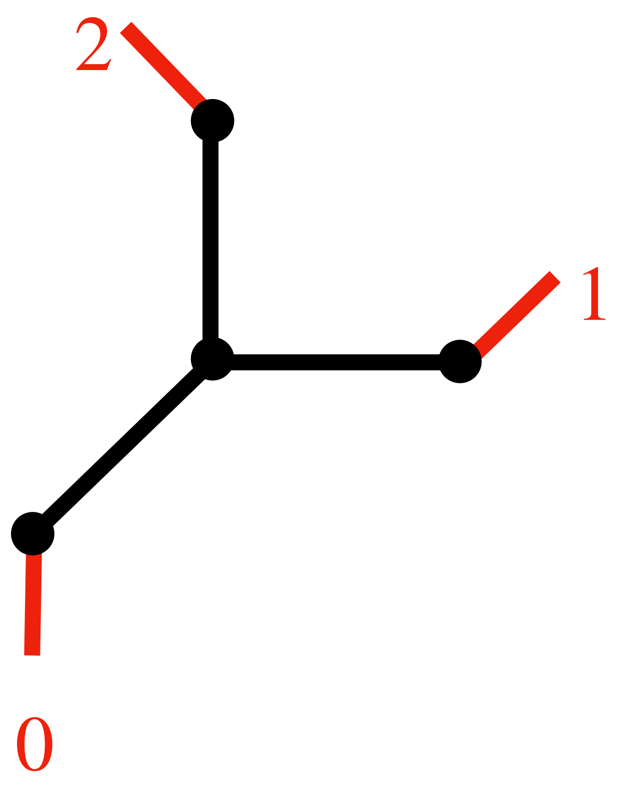

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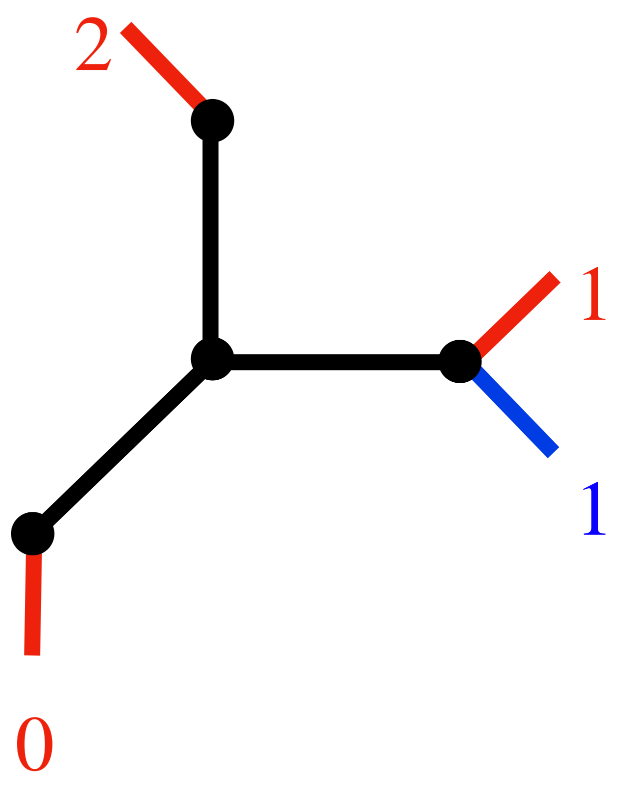

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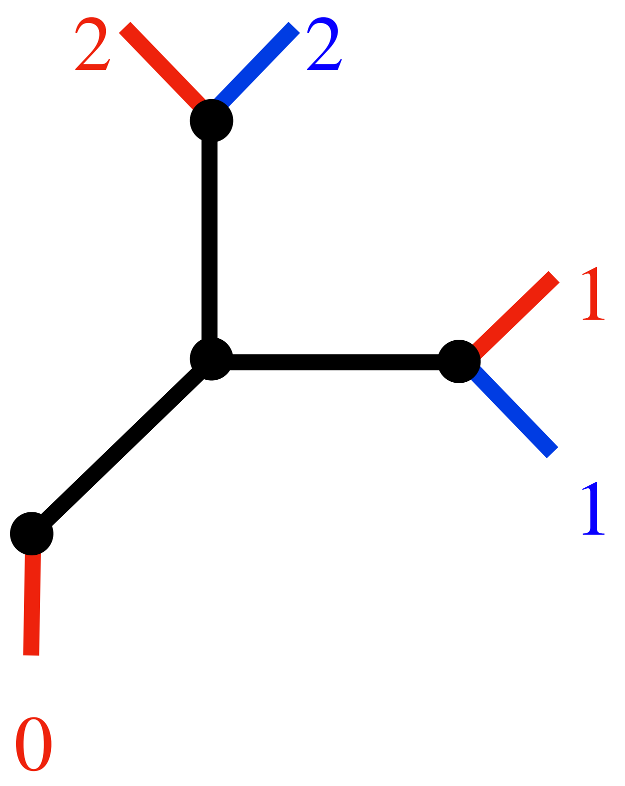

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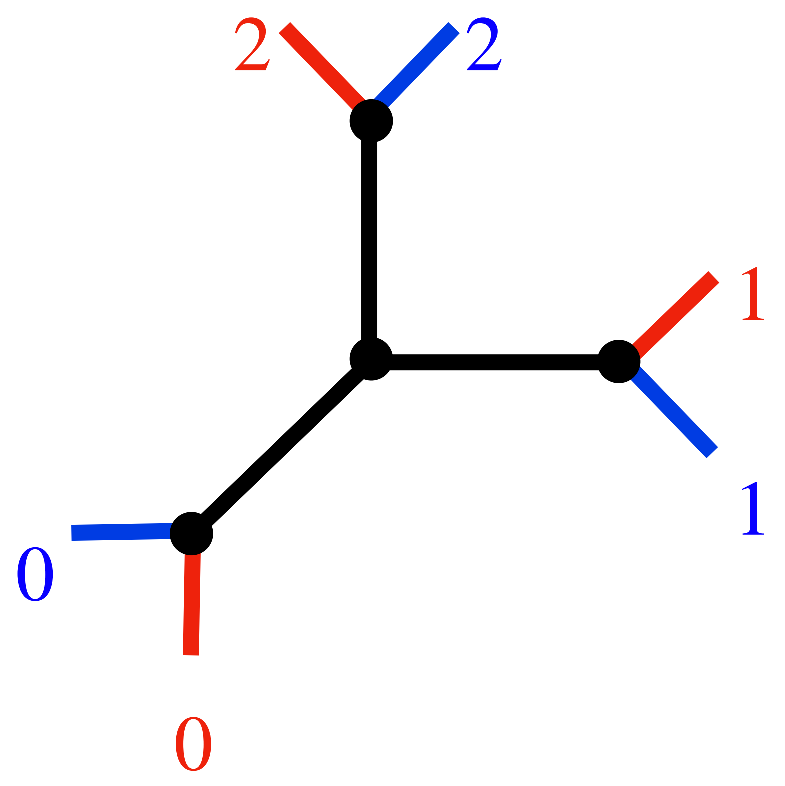

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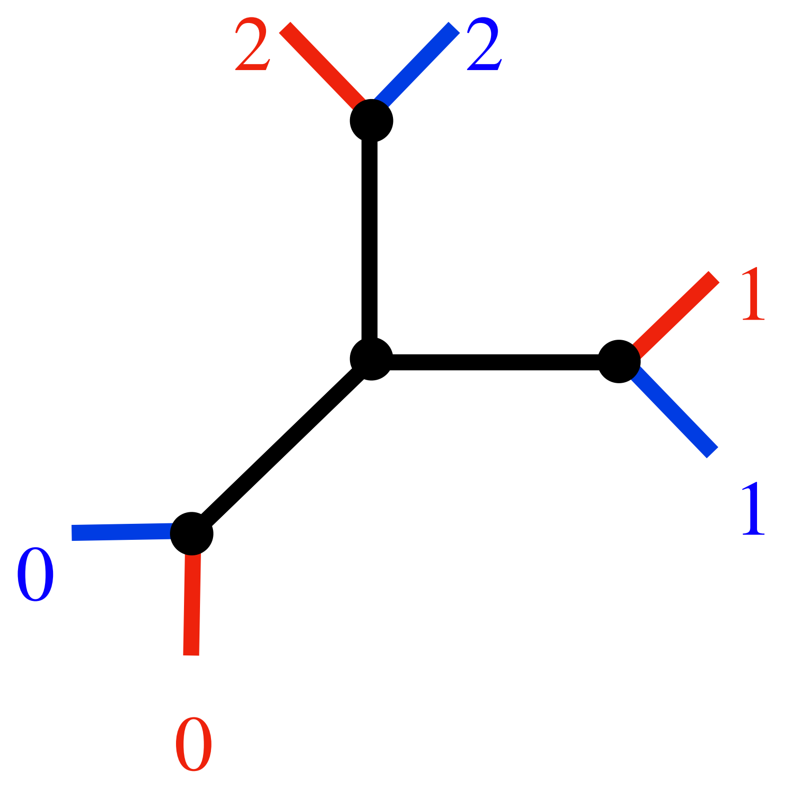

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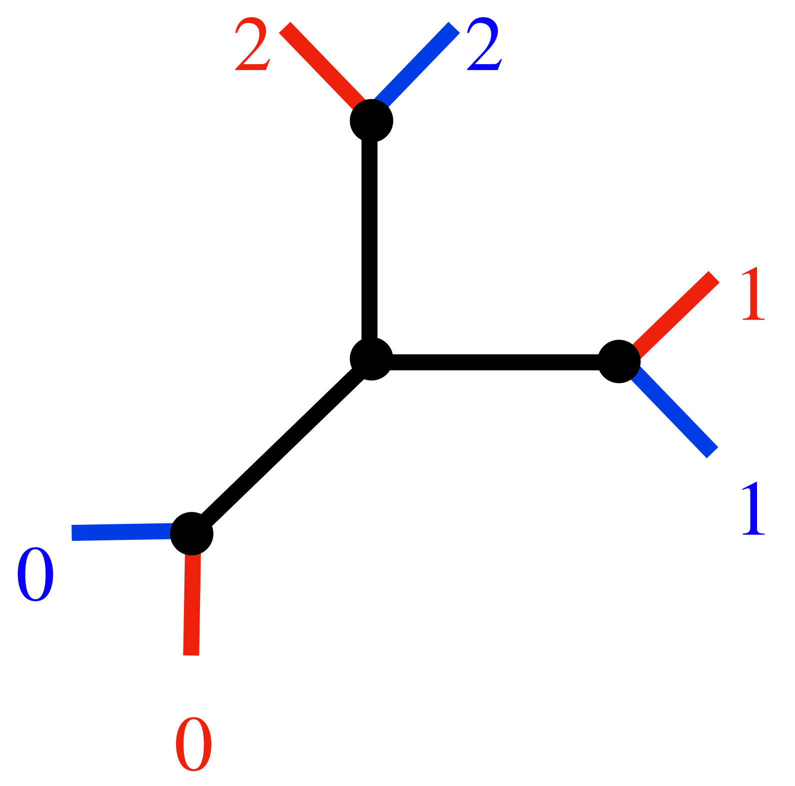

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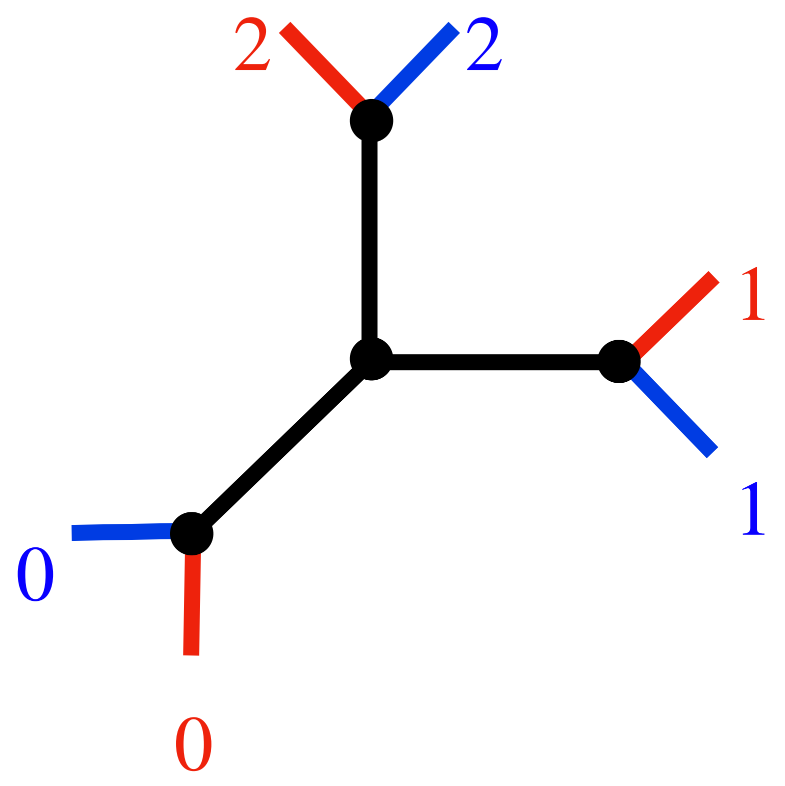

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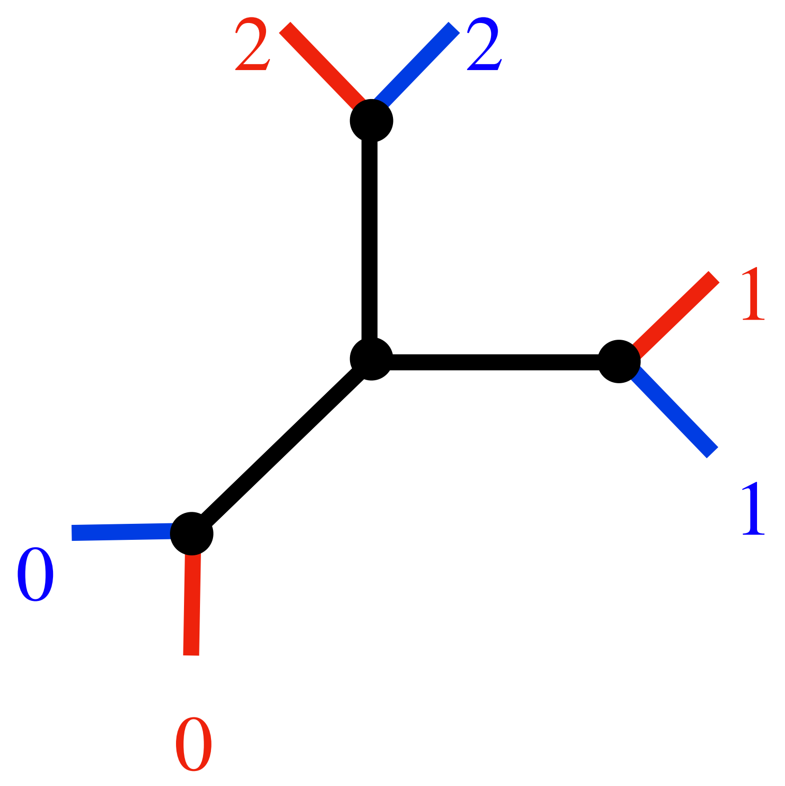

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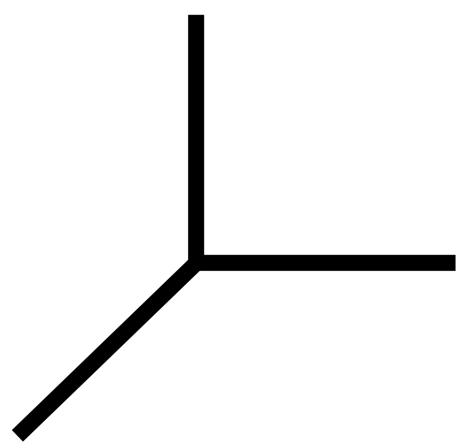

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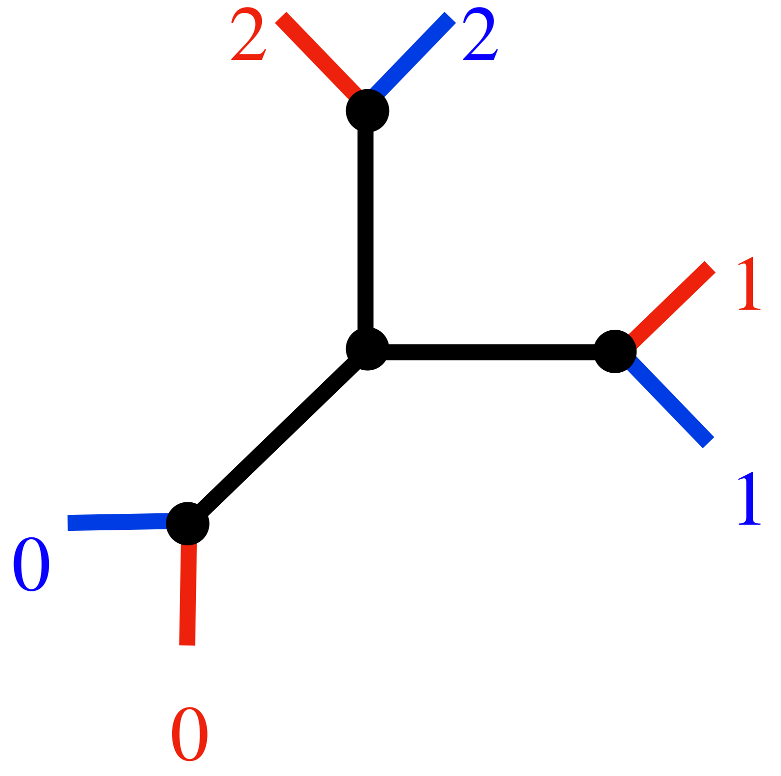

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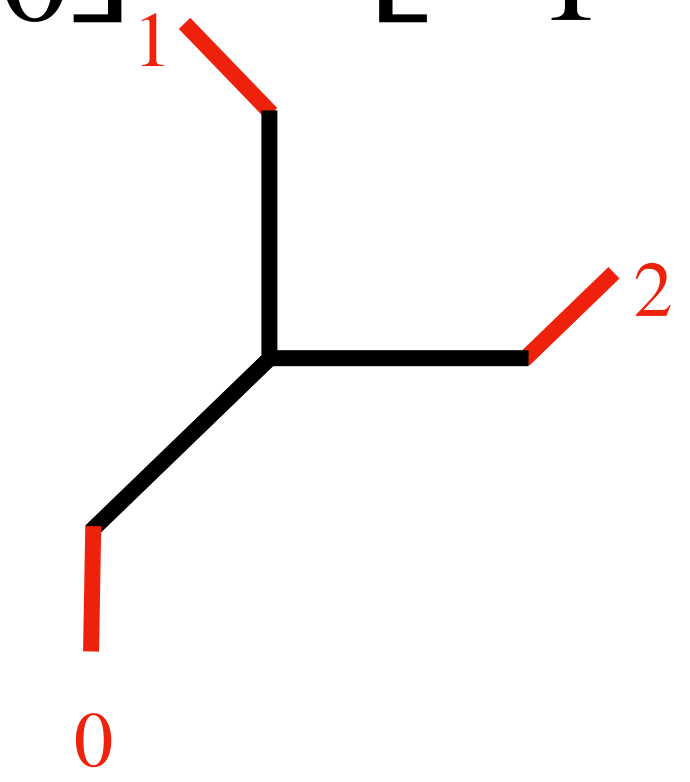

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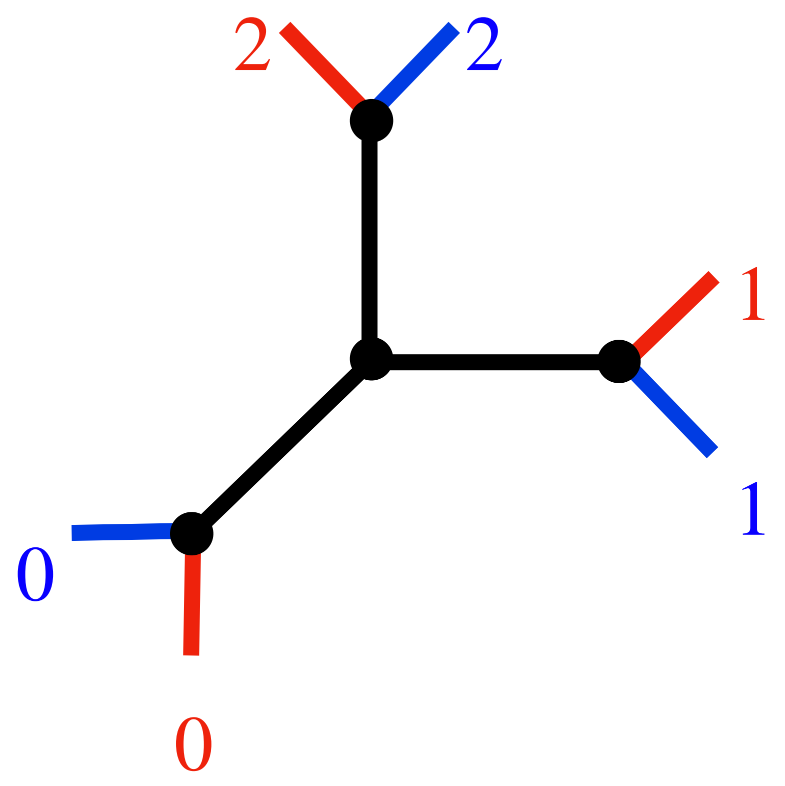

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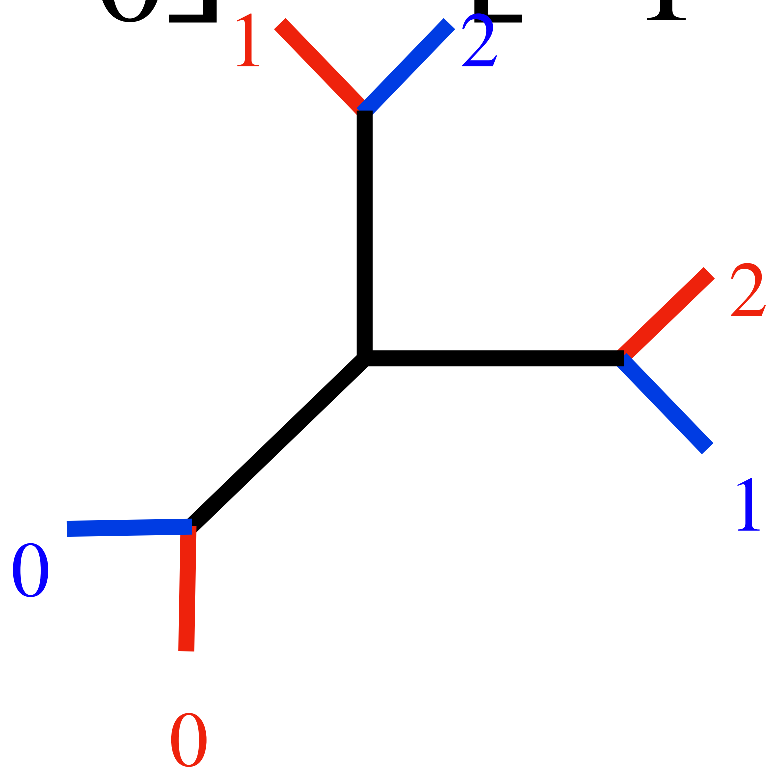

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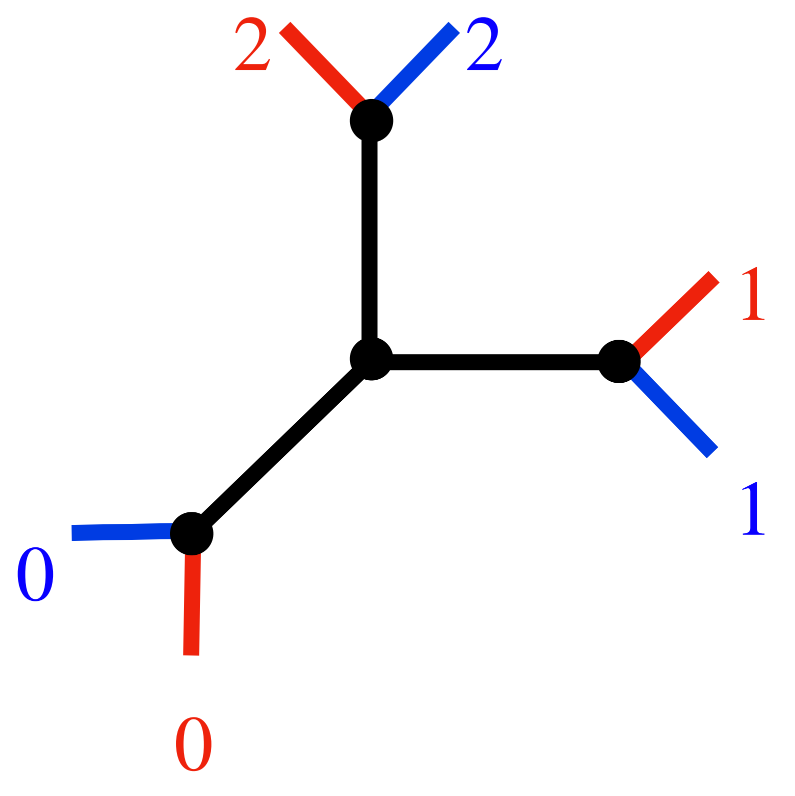
$$\text{ex) } M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} \;v_0 & \;\;v_1 & \;v_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}$$



Theorem (Cai-L.-Yu 2025) The space of symmetric tropical rank 2 matrices forms a simplicial fan structure of symmetric bicolored trees (symbic trees).

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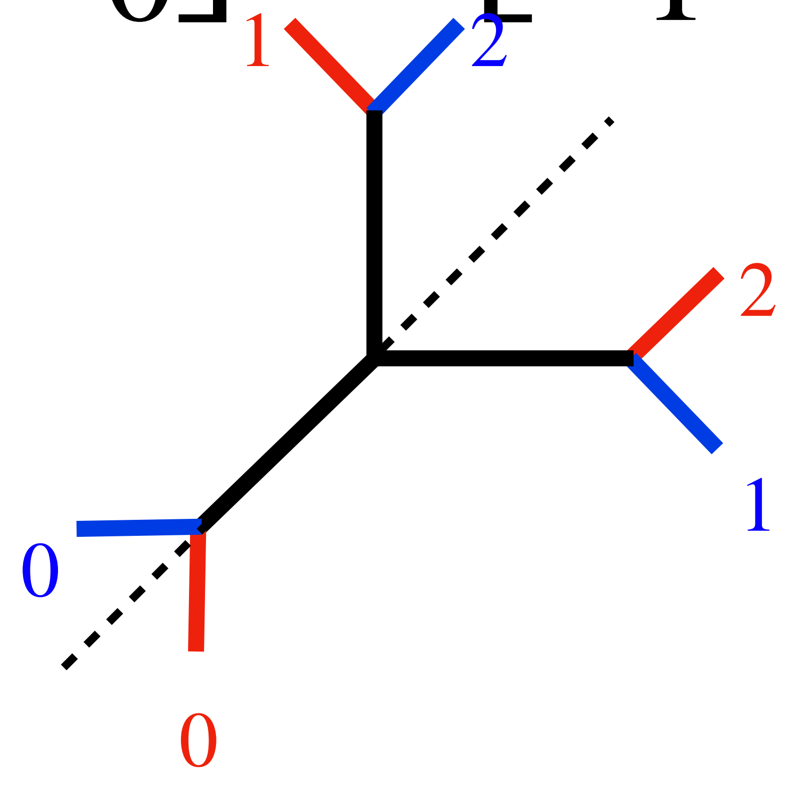

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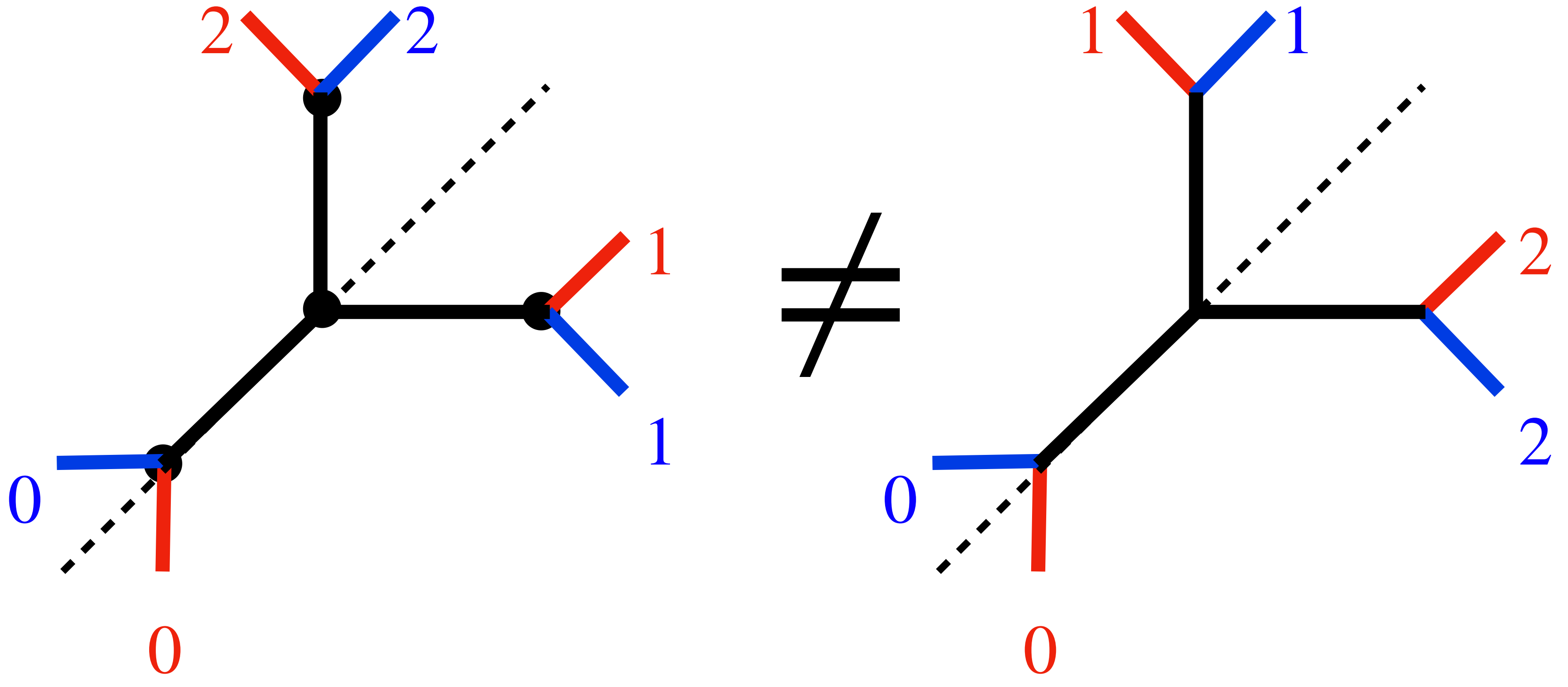

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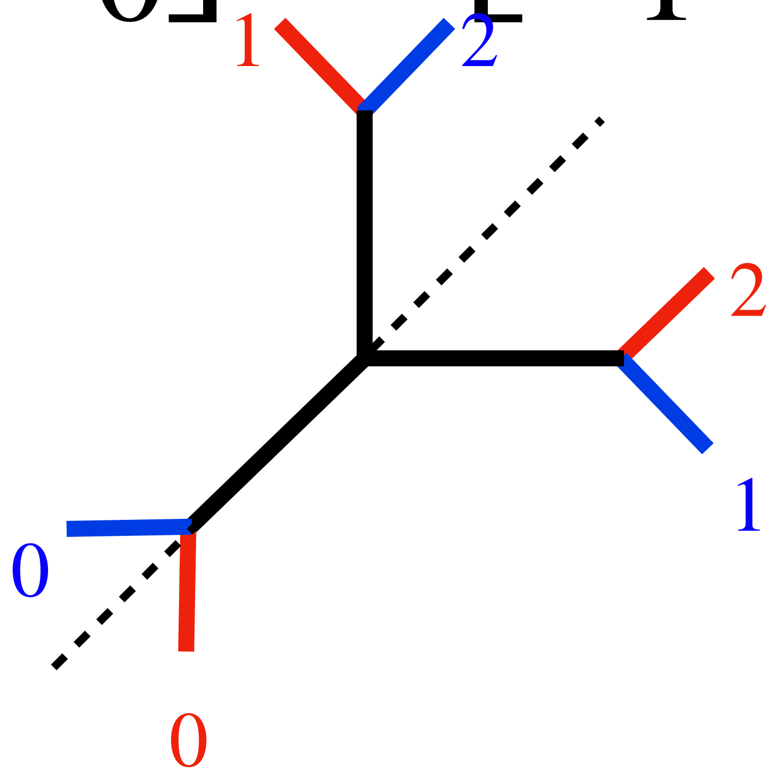

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ex)

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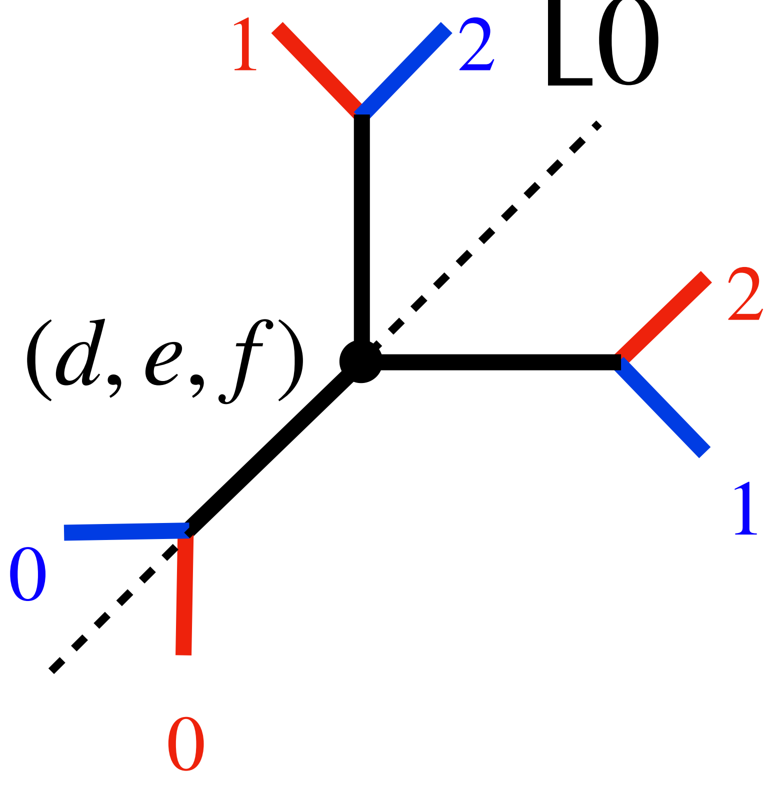
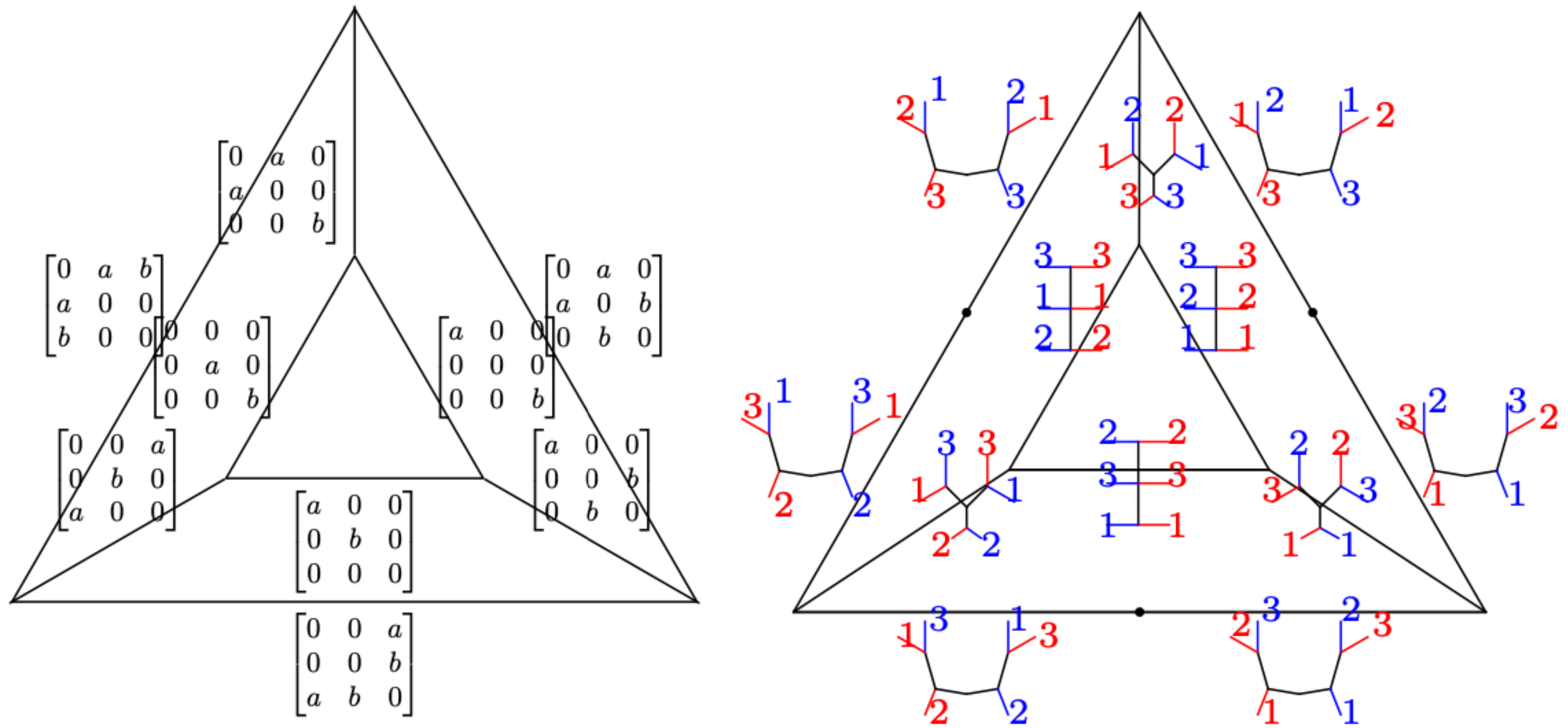
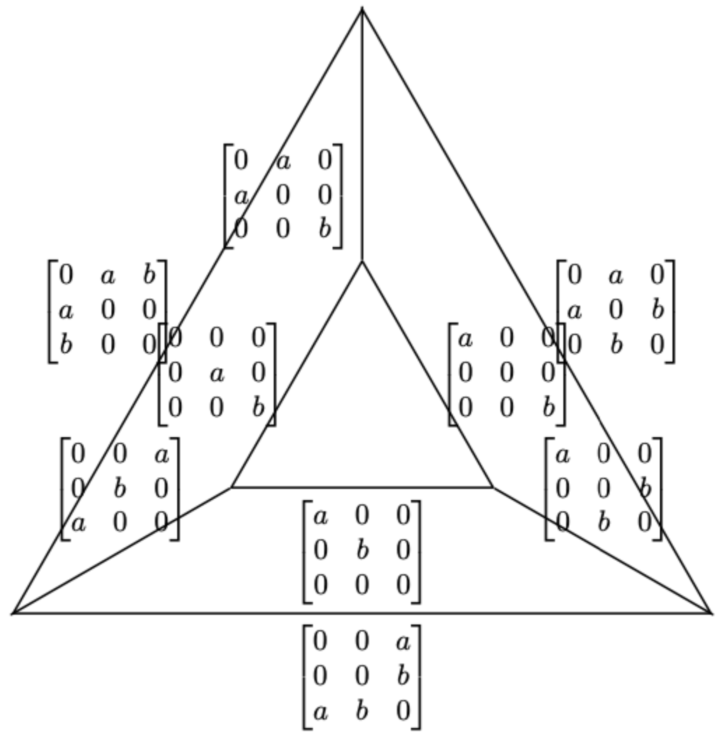


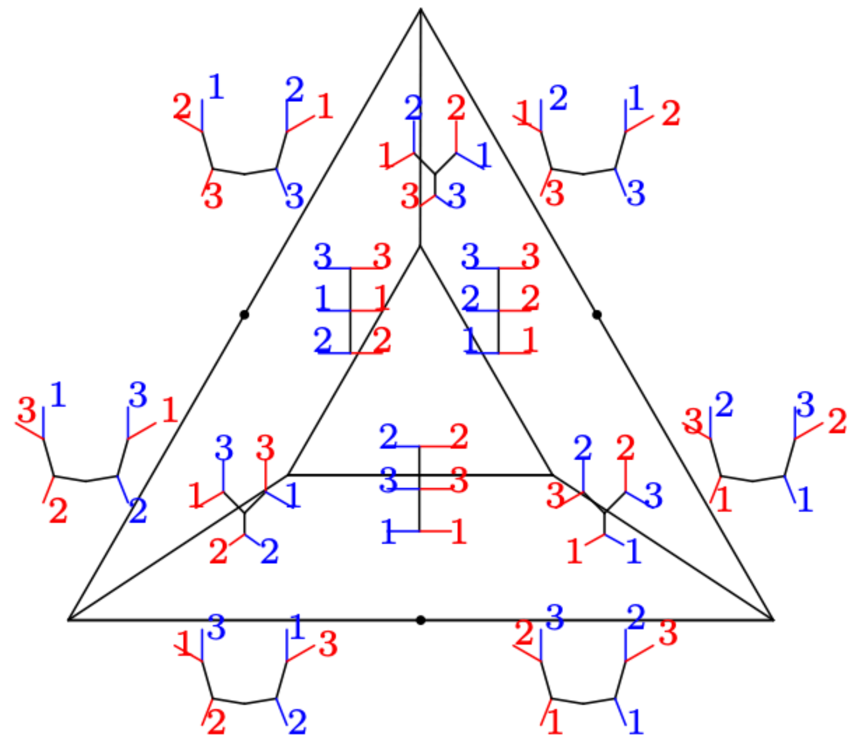
The tree can be translated by adding a proper matrix.



The space of 3×3 symmetric tropical rank 2 matrices

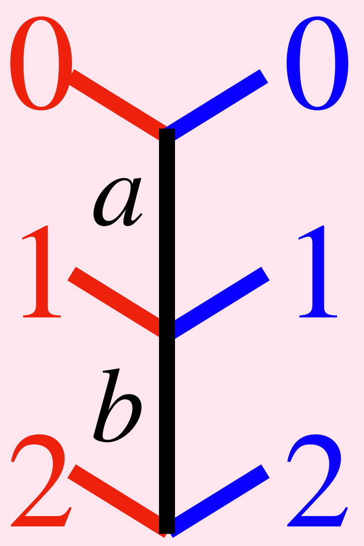
Tropical rank 2 matrices and bicolored trees





Regular symbic trees (all inner edges have positive length) correspond to top-dimensional cells

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & a \\ 0 & a & a+b \end{bmatrix}$$



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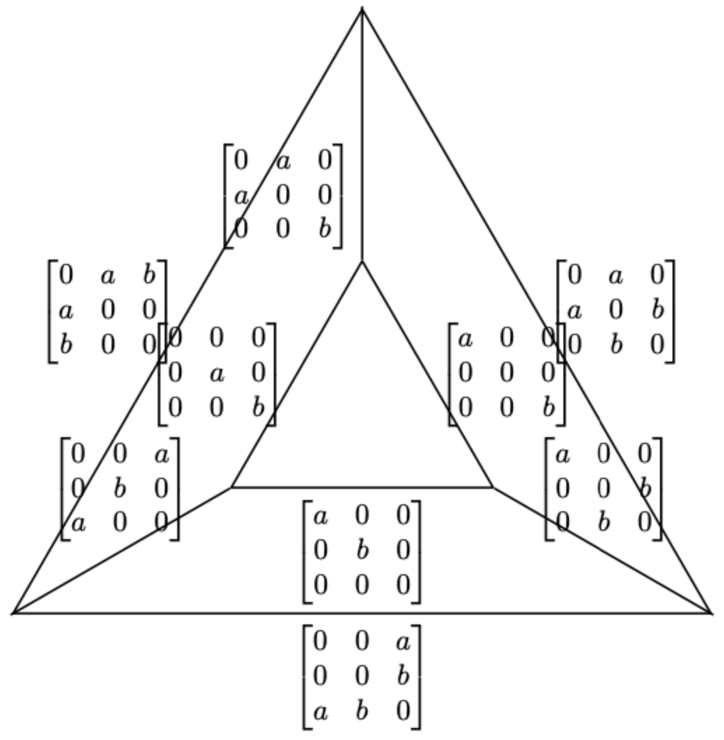

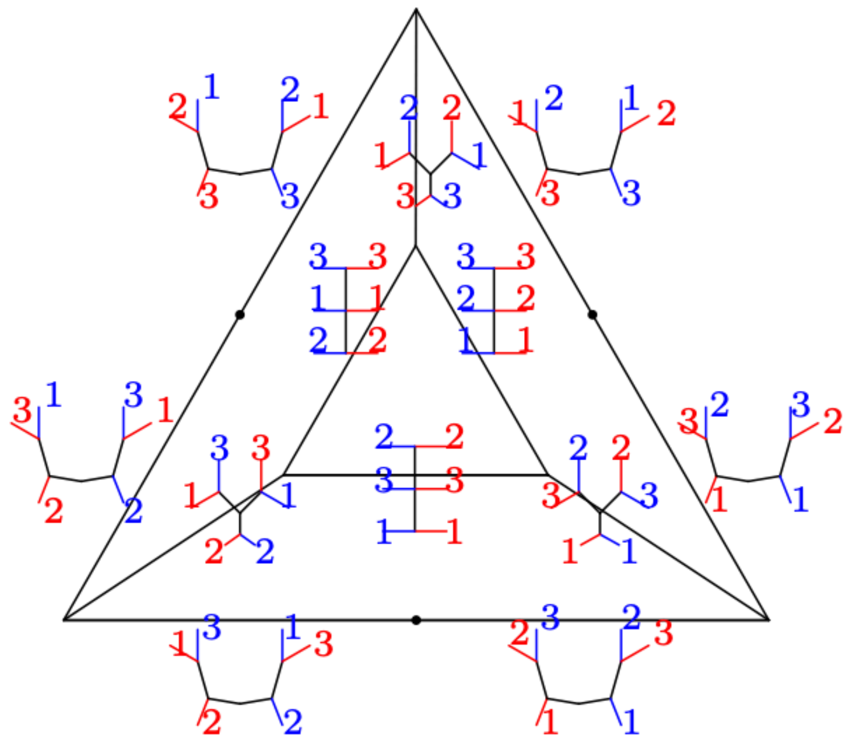

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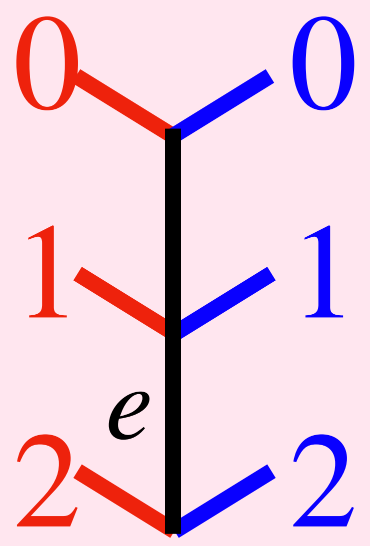
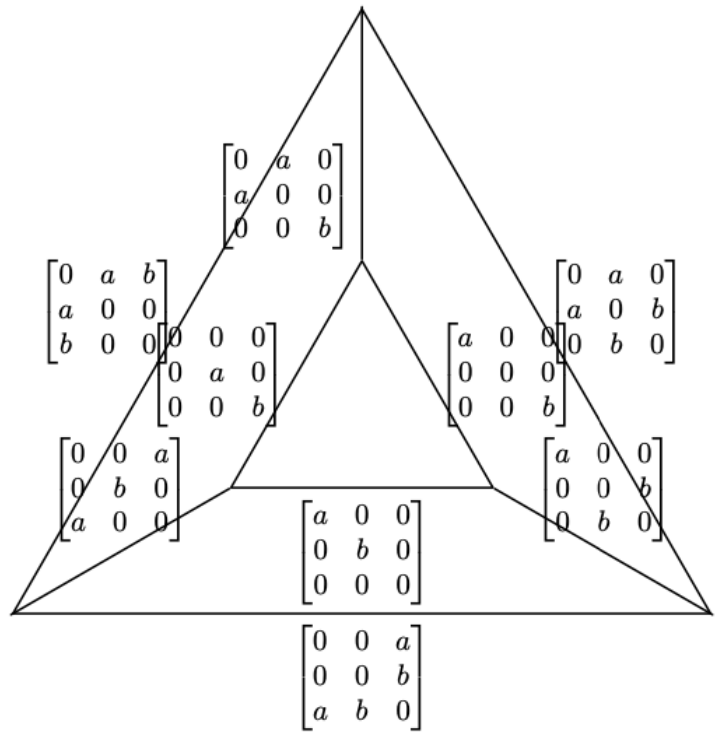
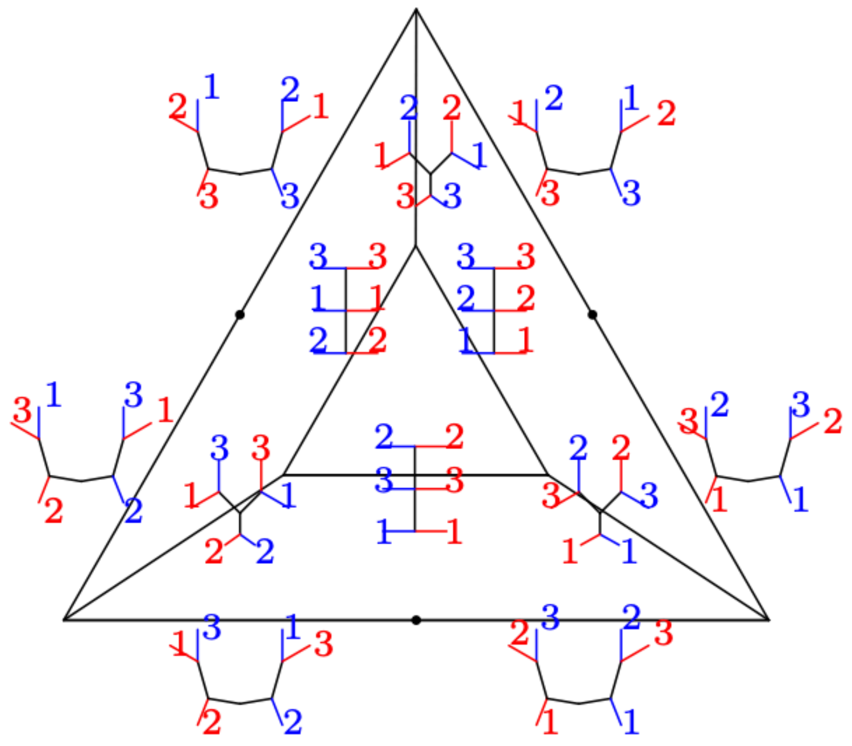


Shrink the edge e

$b \to 0$

Tropical rank 2 matrices and bicolored trees





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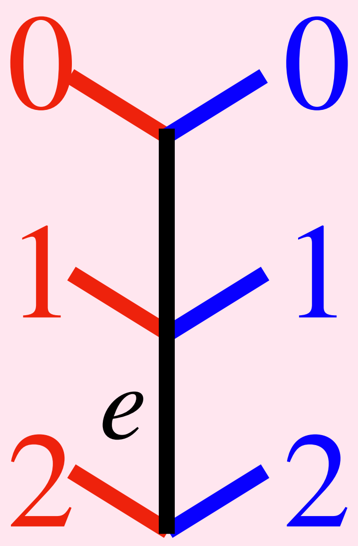
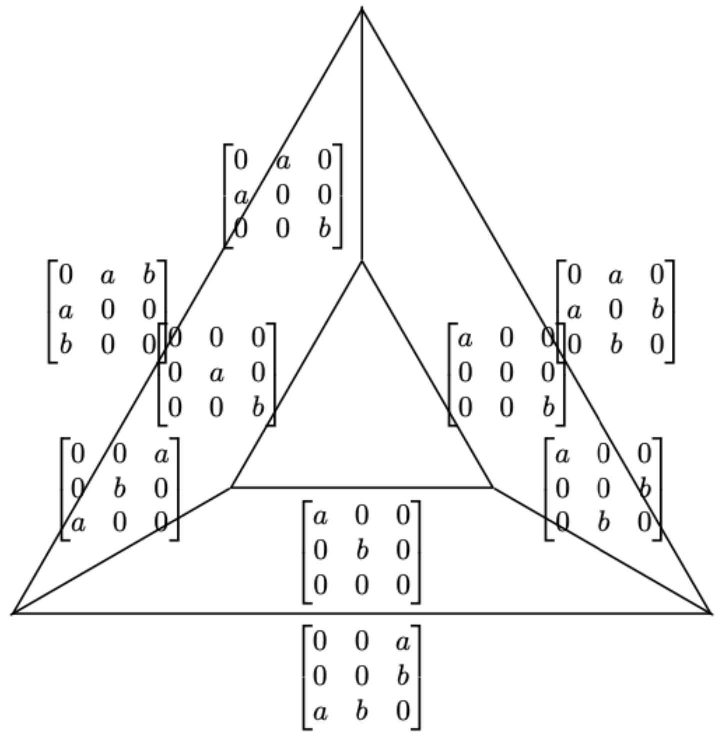


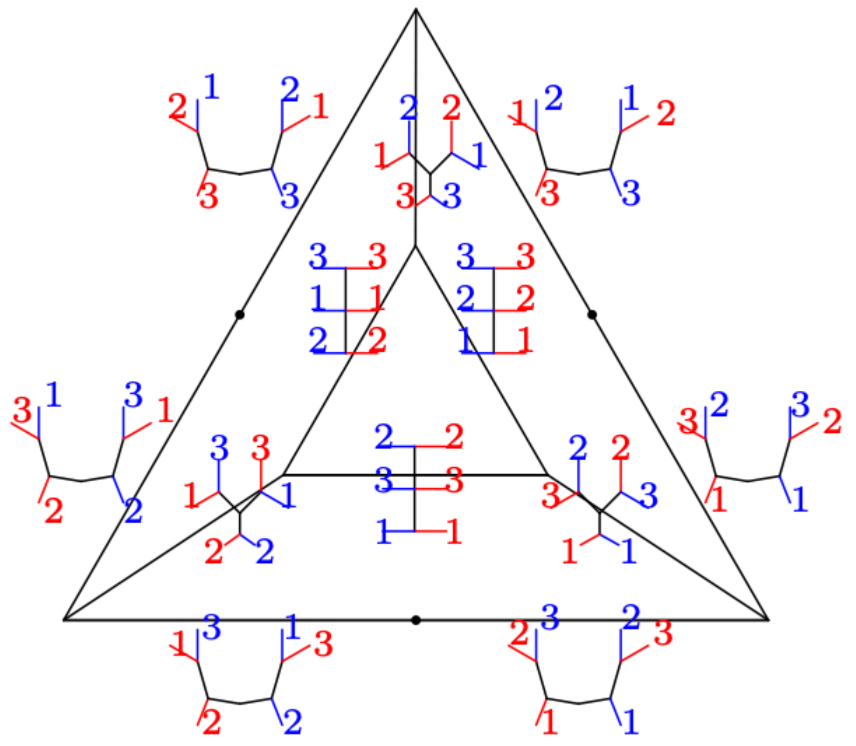
Shrink the edge e

$b \to 0$

codimension-1 cell

Tropical rank 2 matrices and bicolored trees





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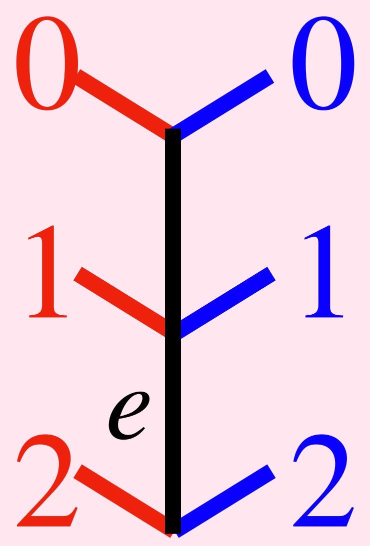
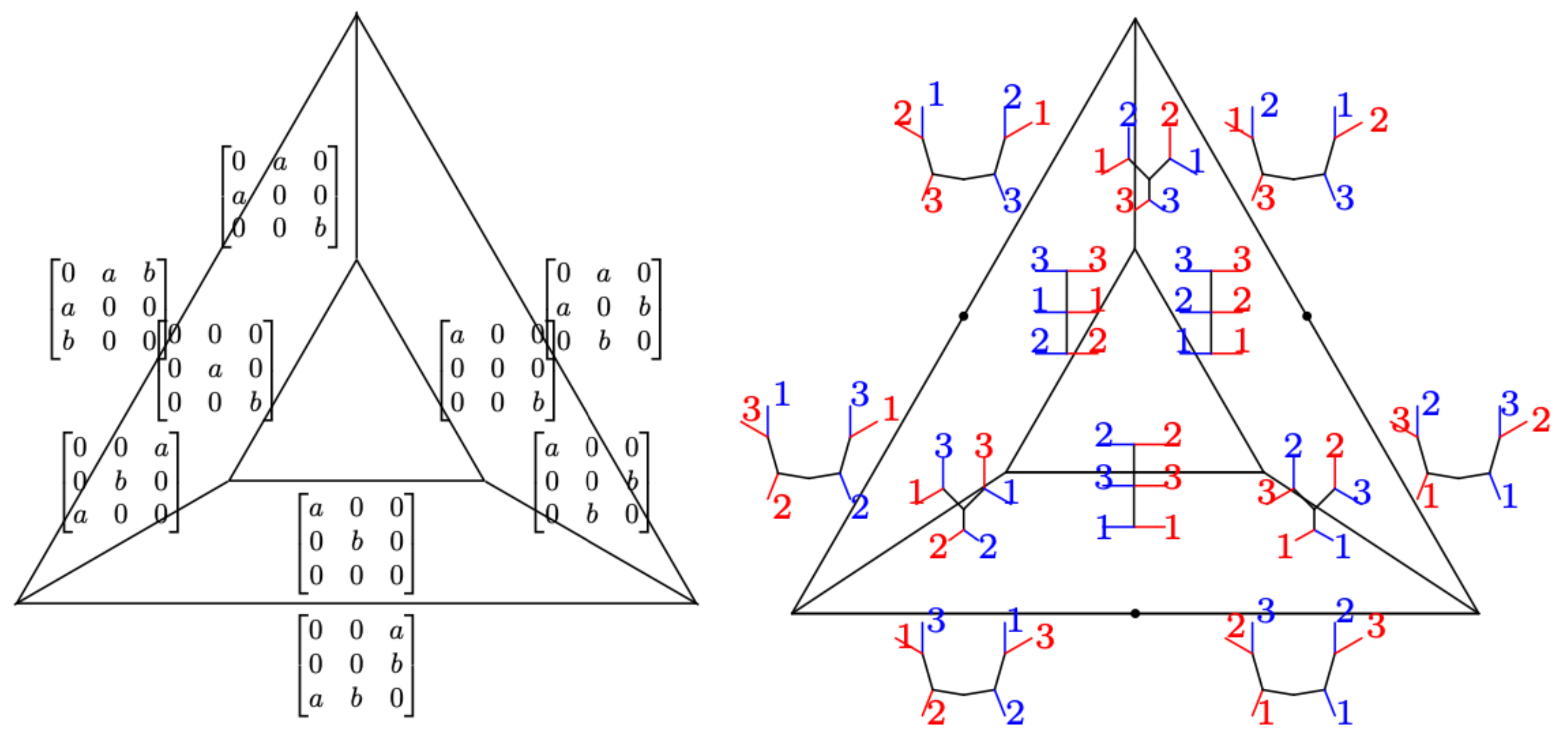


codimension-1 cell

Tropical rank 2 matrices and bicolored trees



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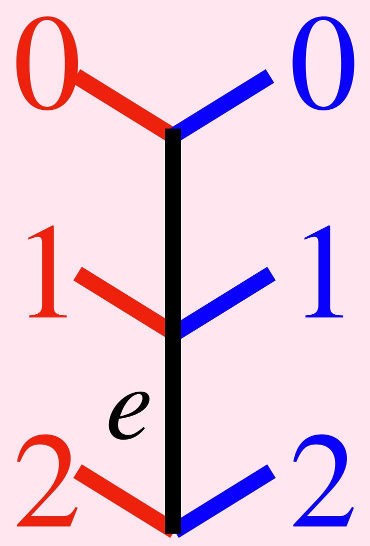

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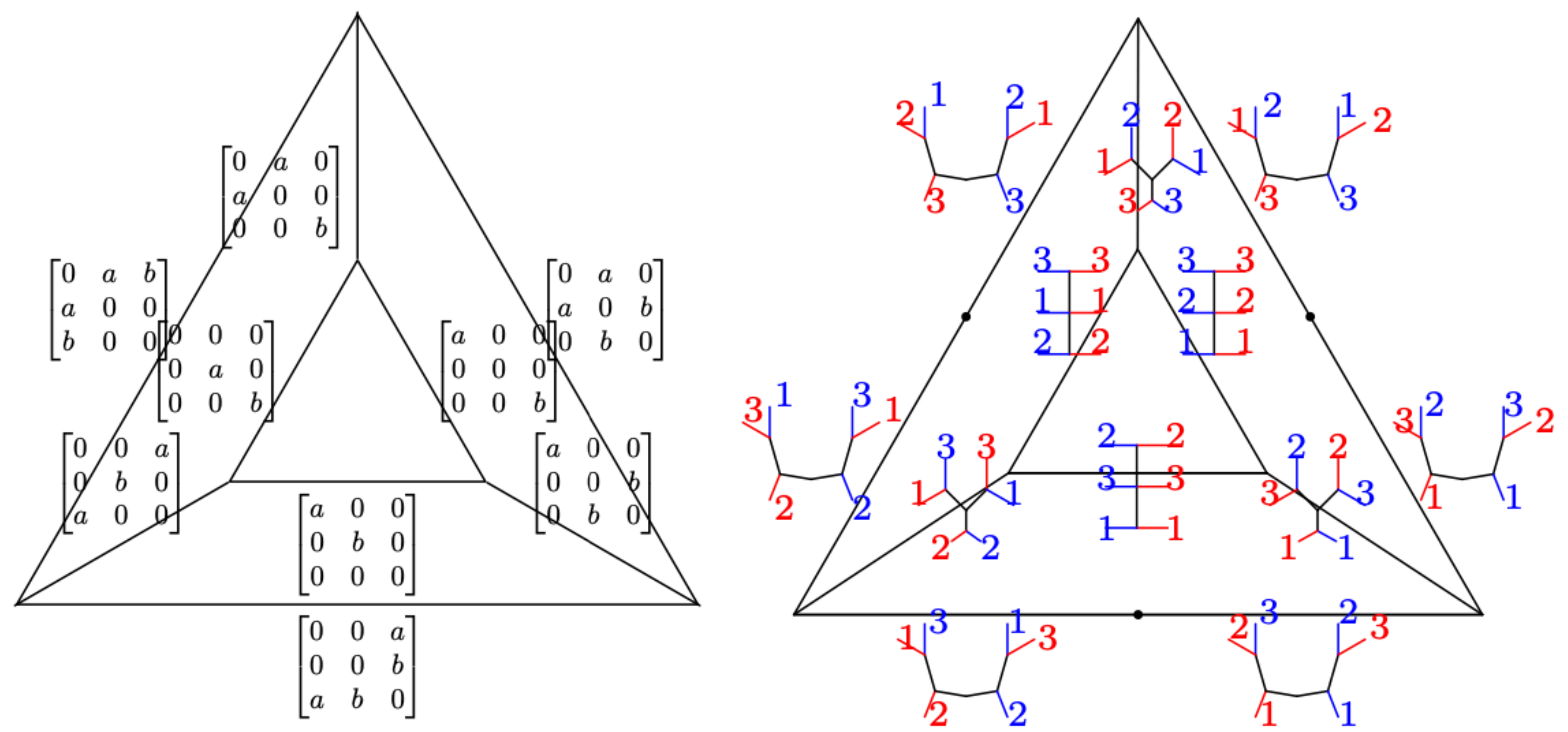
codimension-1 cell

Tropical rank 2 matrices and bicolored trees



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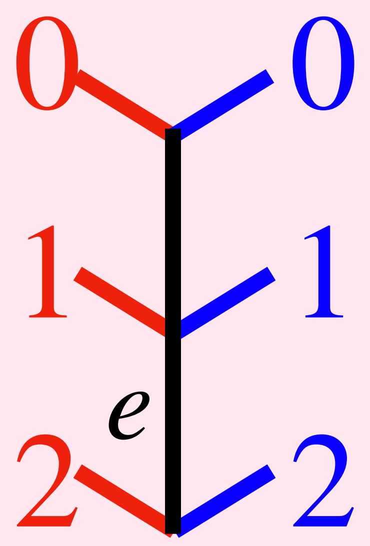
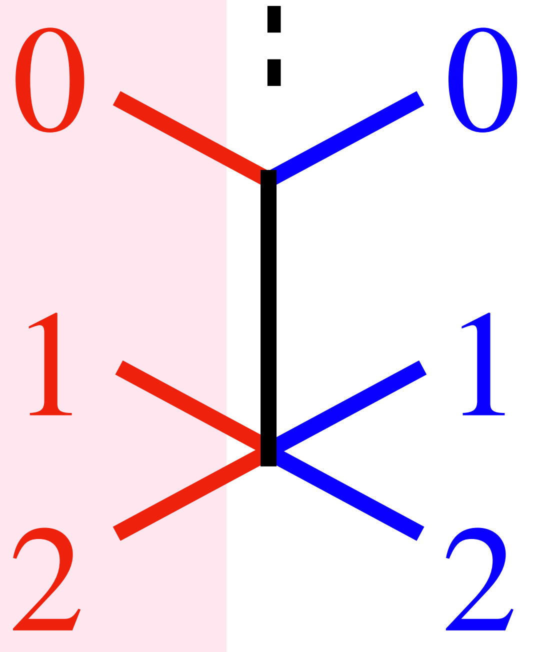


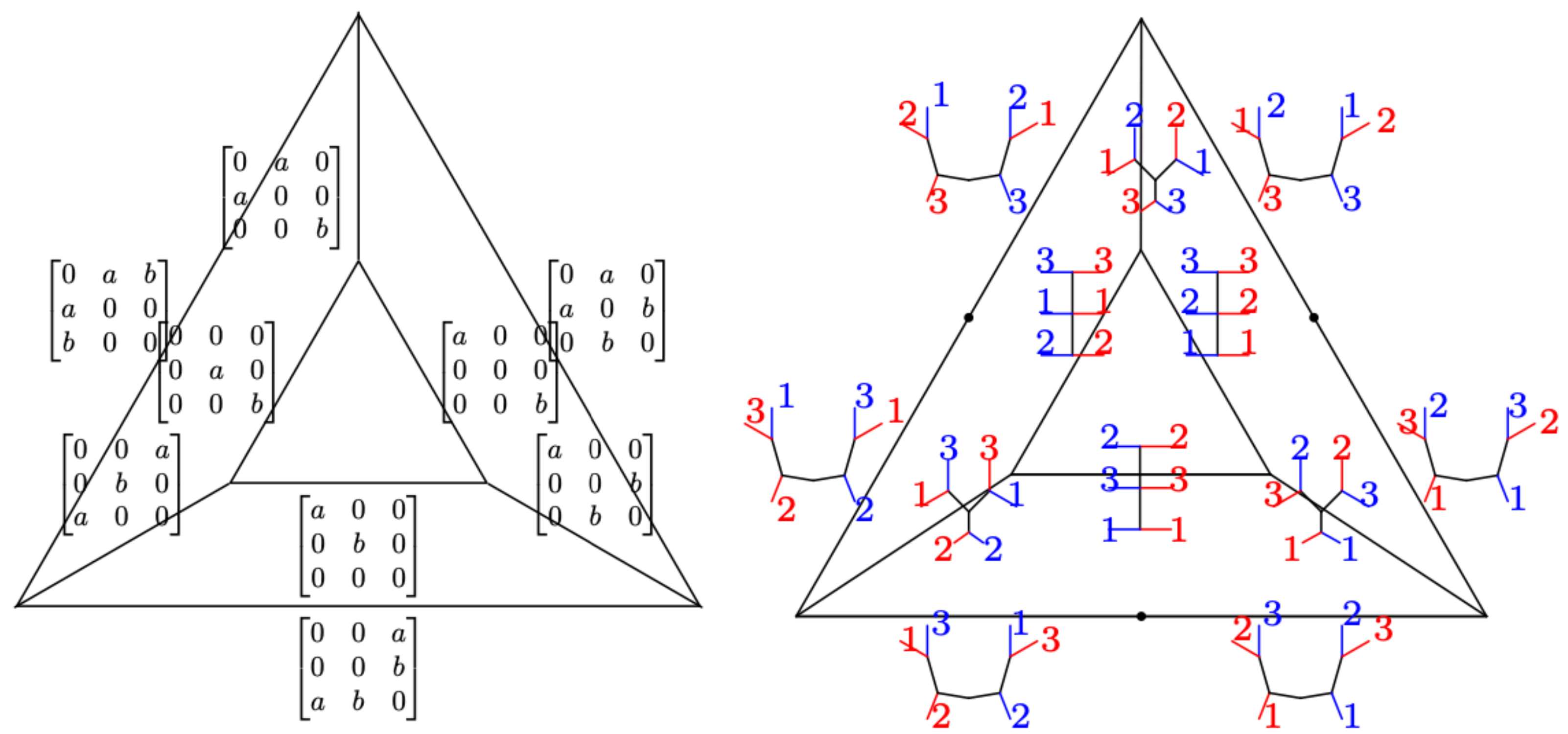
codimension-1 cell



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & a \\ 0 & a & a \end{bmatrix}$$

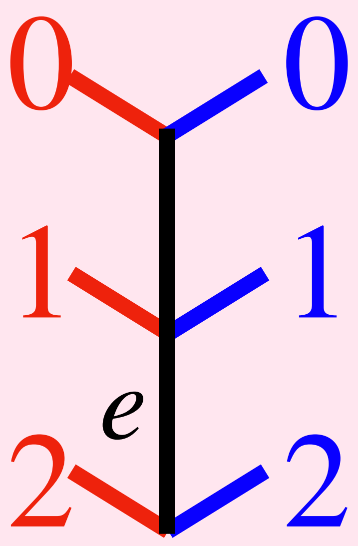
Singular symbic trees correspond to lower-dimensional cells

Tropical rank 2 matrices and bicolored trees

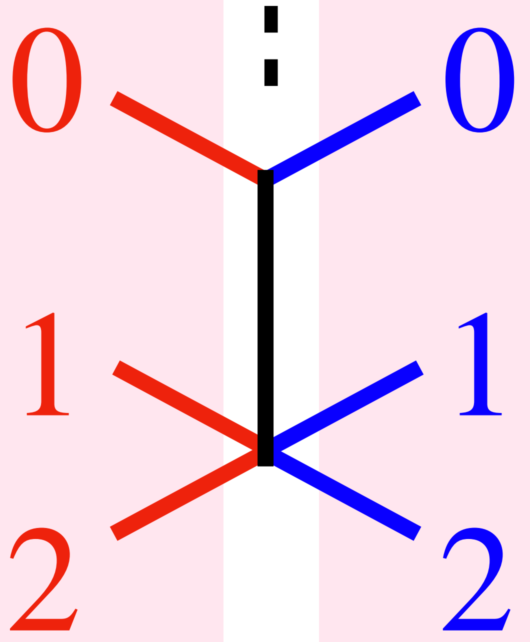


Regular symbic trees (all inner edges have positive length) correspond to top-dimensional cells

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & a \\ 0 & a & a+b \end{bmatrix}$$

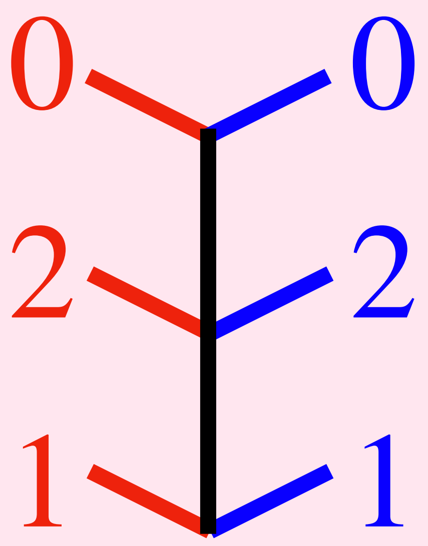


codimension-1 cell

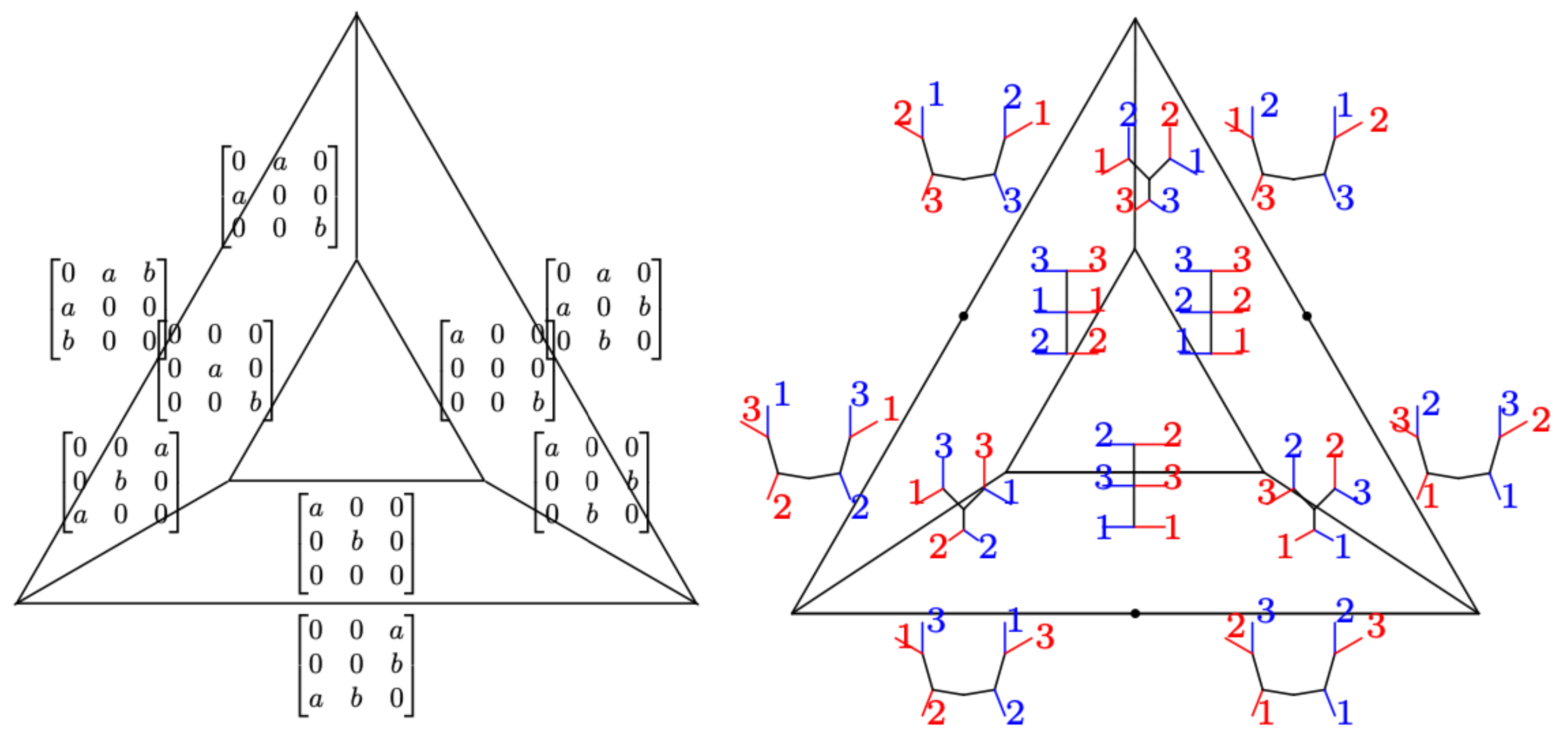


$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & a \\ 0 & a & a \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a+b & a \\ 0 & a & a \end{bmatrix}$$



Tropical rank 2 matrices and bicolored trees



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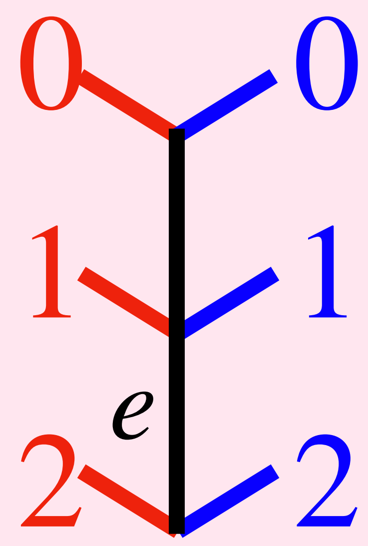
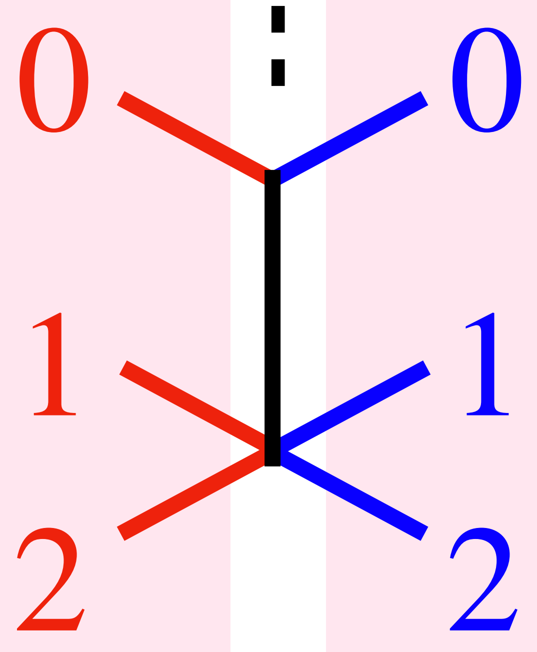
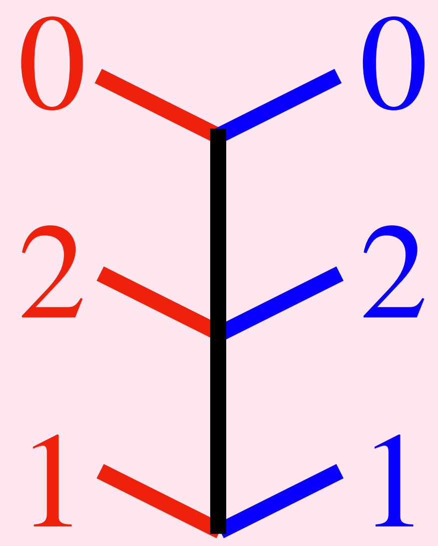


$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & a \\ 0 & a & a \end{bmatrix}$$



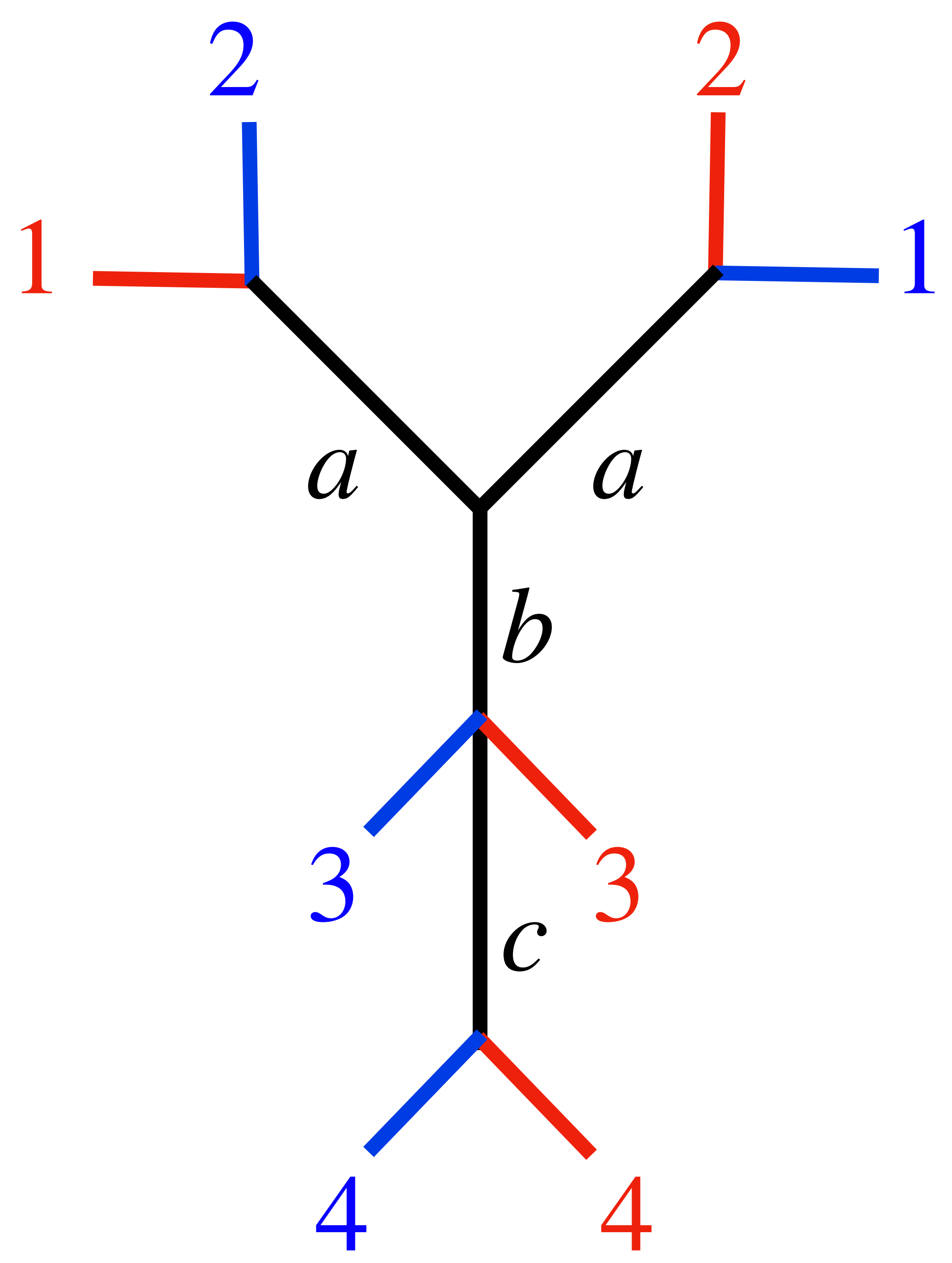
codimension-1 cell

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a+b & a \\ 0 & a & a \end{bmatrix}$$



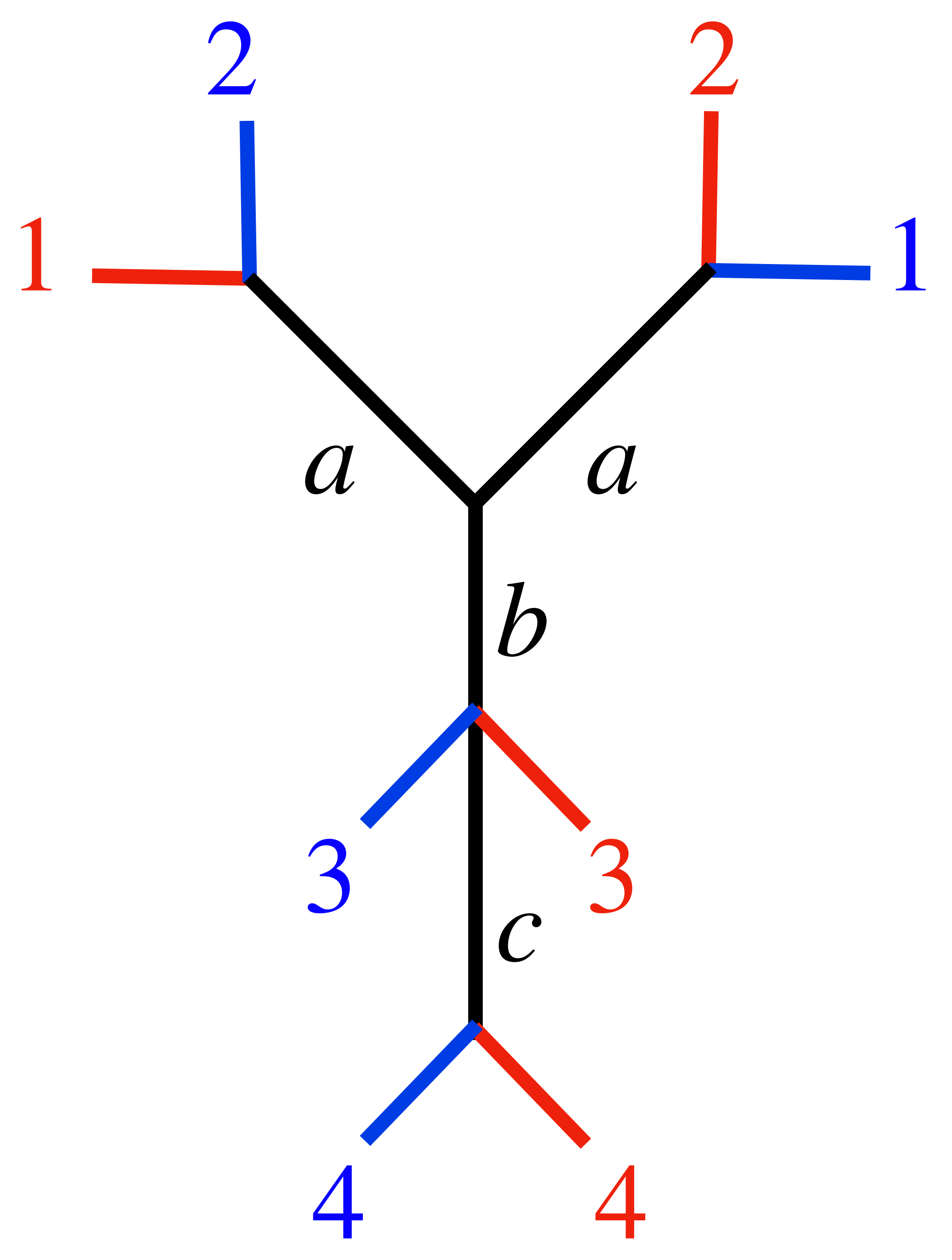
Transition of edges

The matroids of symbic trees



$$M = \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & b & b+c \end{bmatrix} + \begin{bmatrix} 2d & d+e & d+f & d+g \\ d+e & 2e & e+f & e+g \\ d+f & e+f & 2f & f+g \\ d+g & e+g & f+g & 2g \end{bmatrix}$$

The matroids of symbic trees



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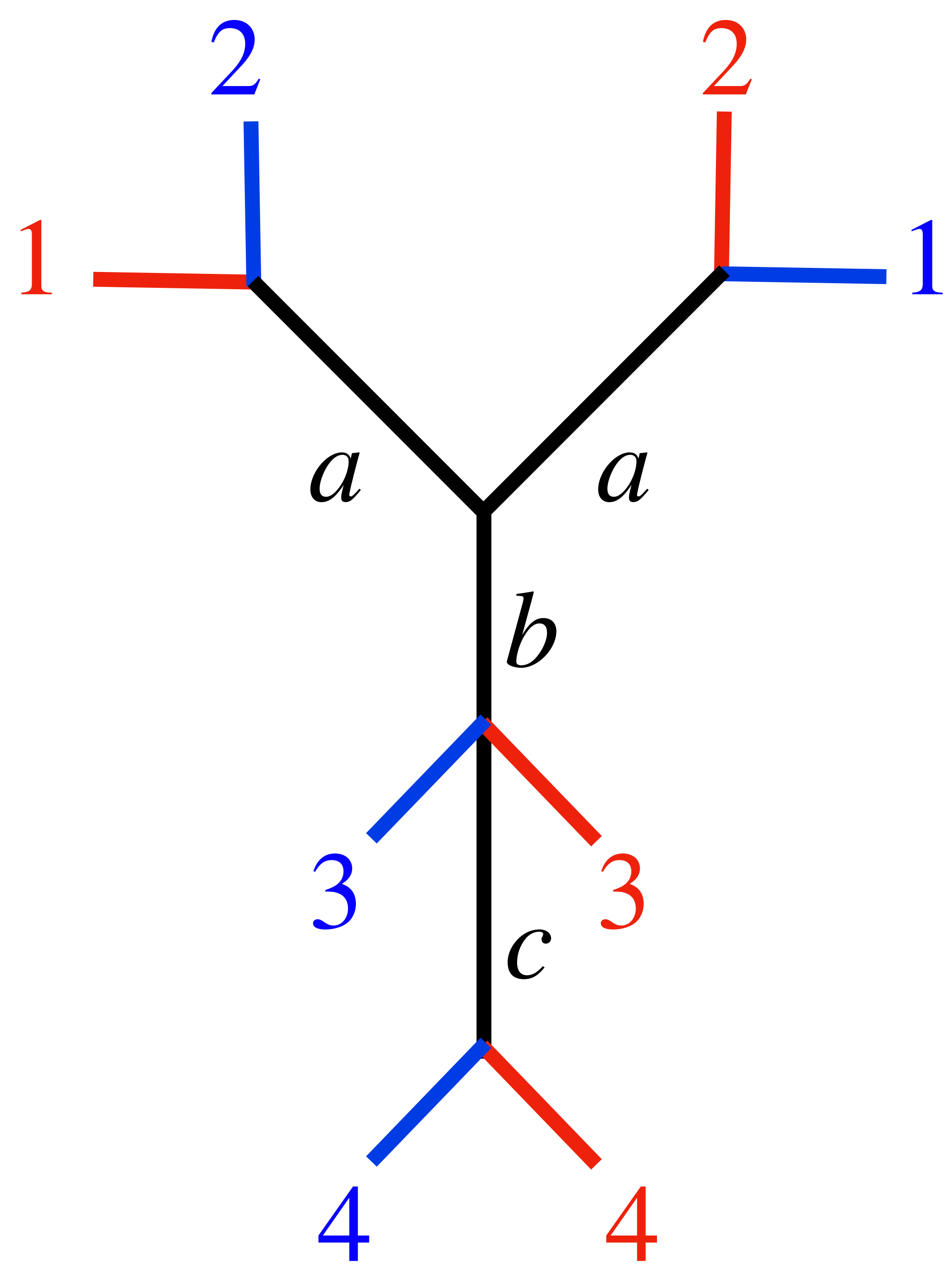
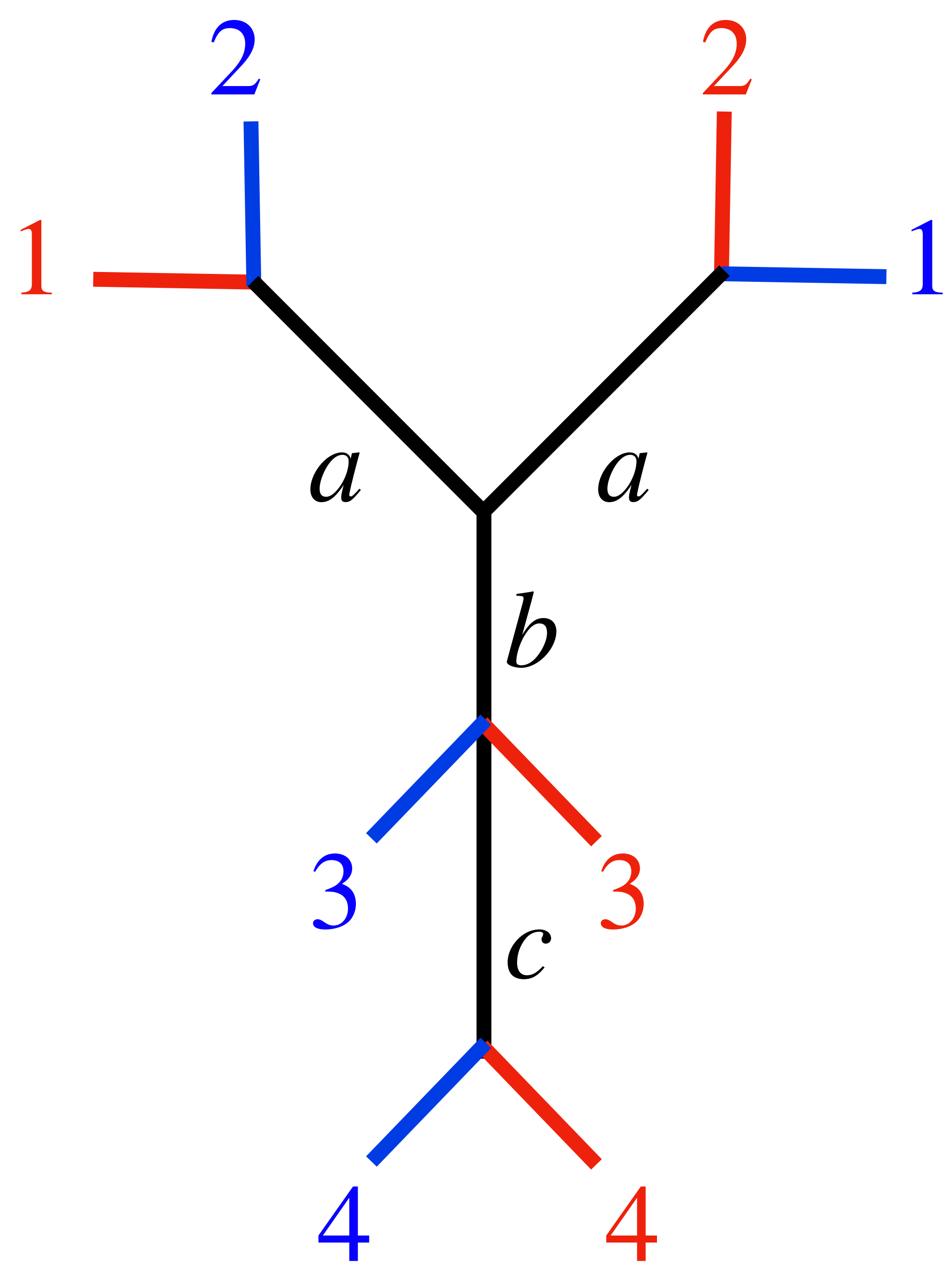
$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

The matroids of symbic trees



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

The matroids of symbic trees



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

We define a matroid of a symbic tree by a linear matroid of this parameter matrix.

The matroids of symbic trees

🌴 Can we use symbic trees to study algebraic matroids of rank 2 symmetric matrices?

The matroids of symbic trees

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Technical lemmas (cf. Bernstein 2017)

- I is independent in the algebraic matroid of $\mathcal{S}_n^{r\leq 2}(\mathbb{C})$ if and only if I is independent in a matroid of a regular symbic tree with n vertices.

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Can we find a family of symbic trees that determine the algebraic matroid of rank 2 symmetric matrices?

The matroids of symbic trees

Theorem (Cai-L.-Yu 2025) The collection of bases in the algebraic matroid of rank 2 symmetric matrices is the union of bases of matroids of trees with caterpillar branches.

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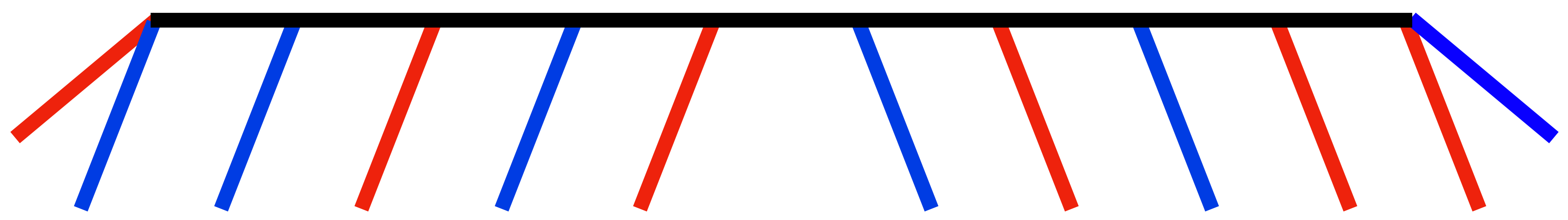

caterpillar tree

The matroids of symbic trees

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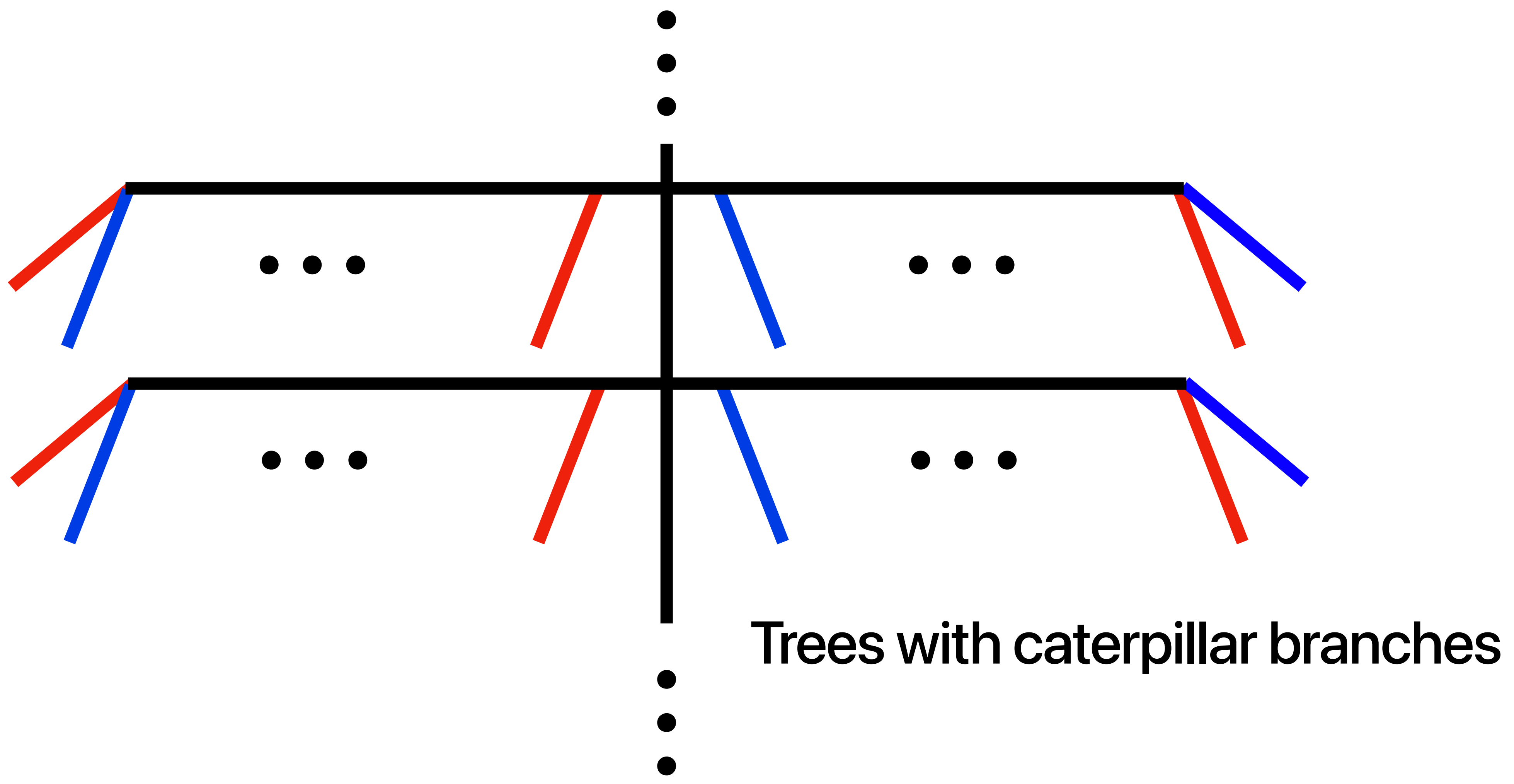

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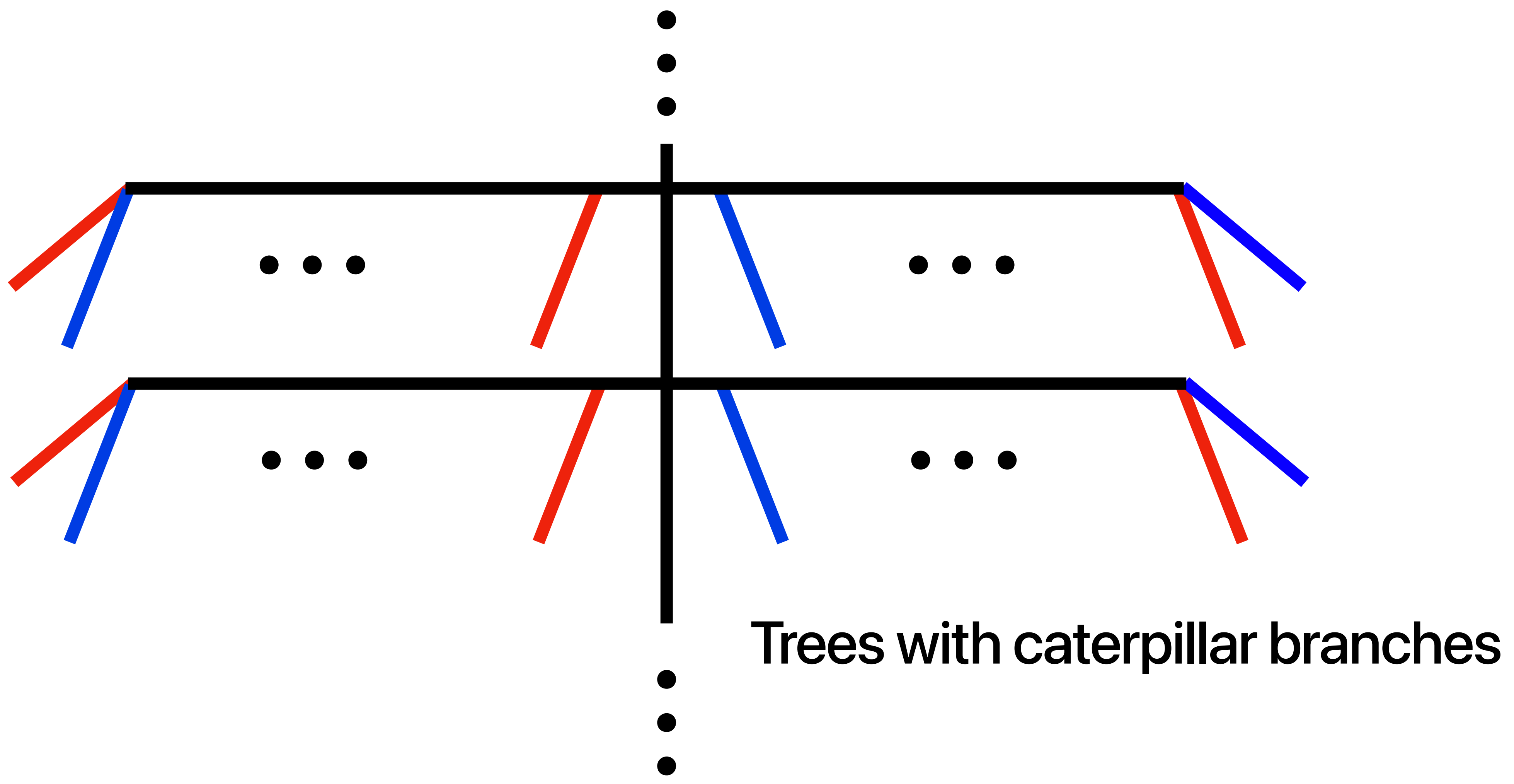

Conjecture Caterpillar symbic trees are sufficient to determine the algebraic matroid of rank 2 symmetric matrices.

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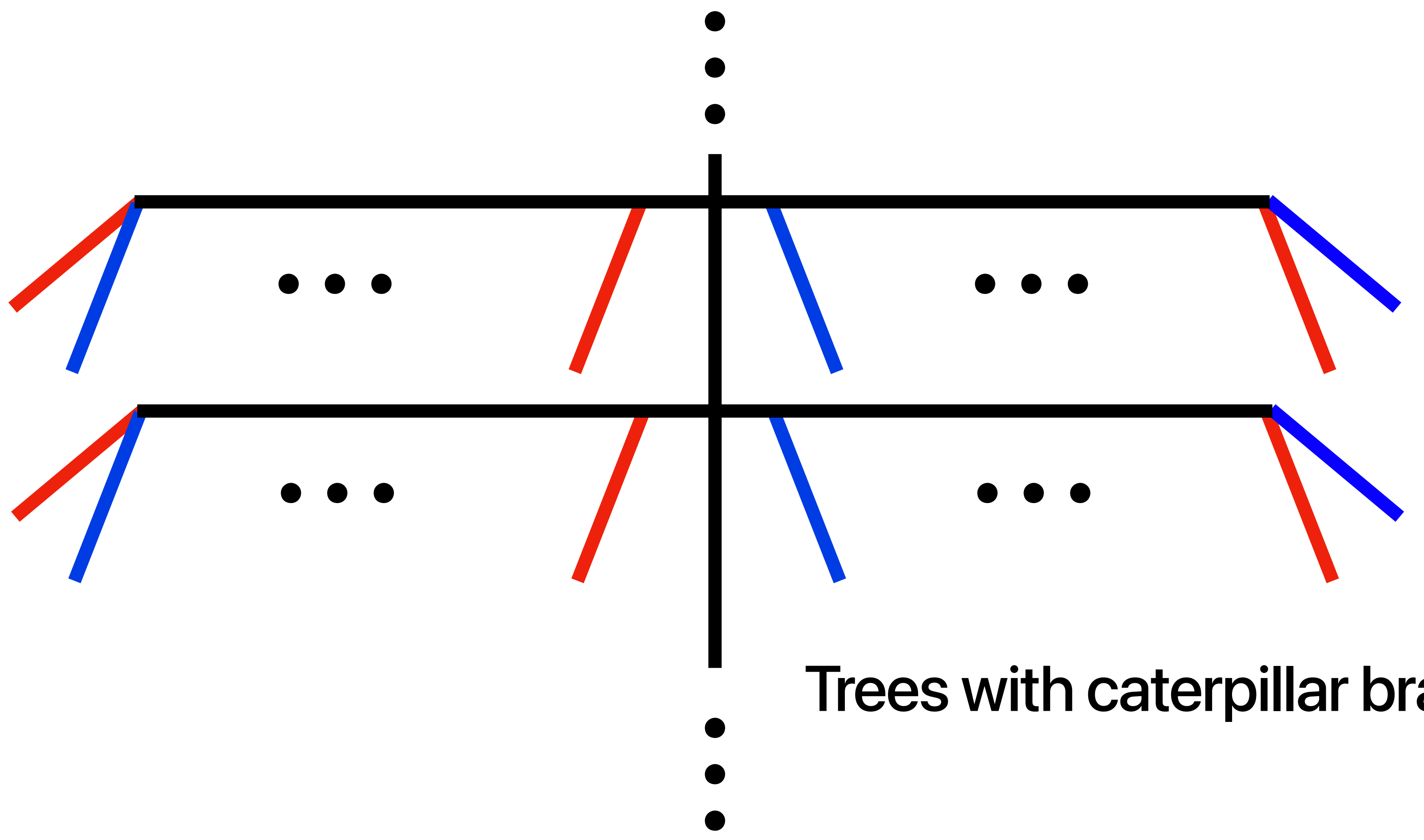


Trees with caterpillar branches

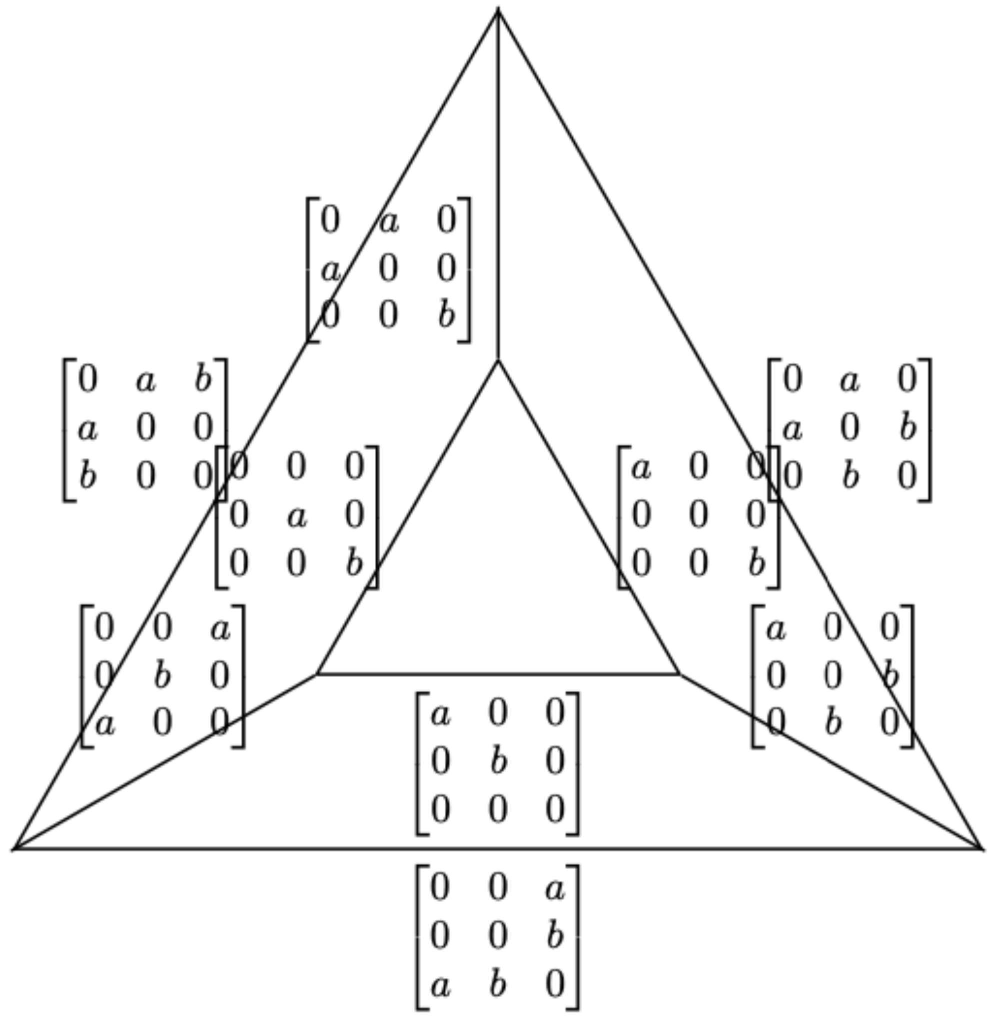
Conjecture Caterpillar symbic trees are sufficient to determine the algebraic matroid of rank 2 symmetric matrices.

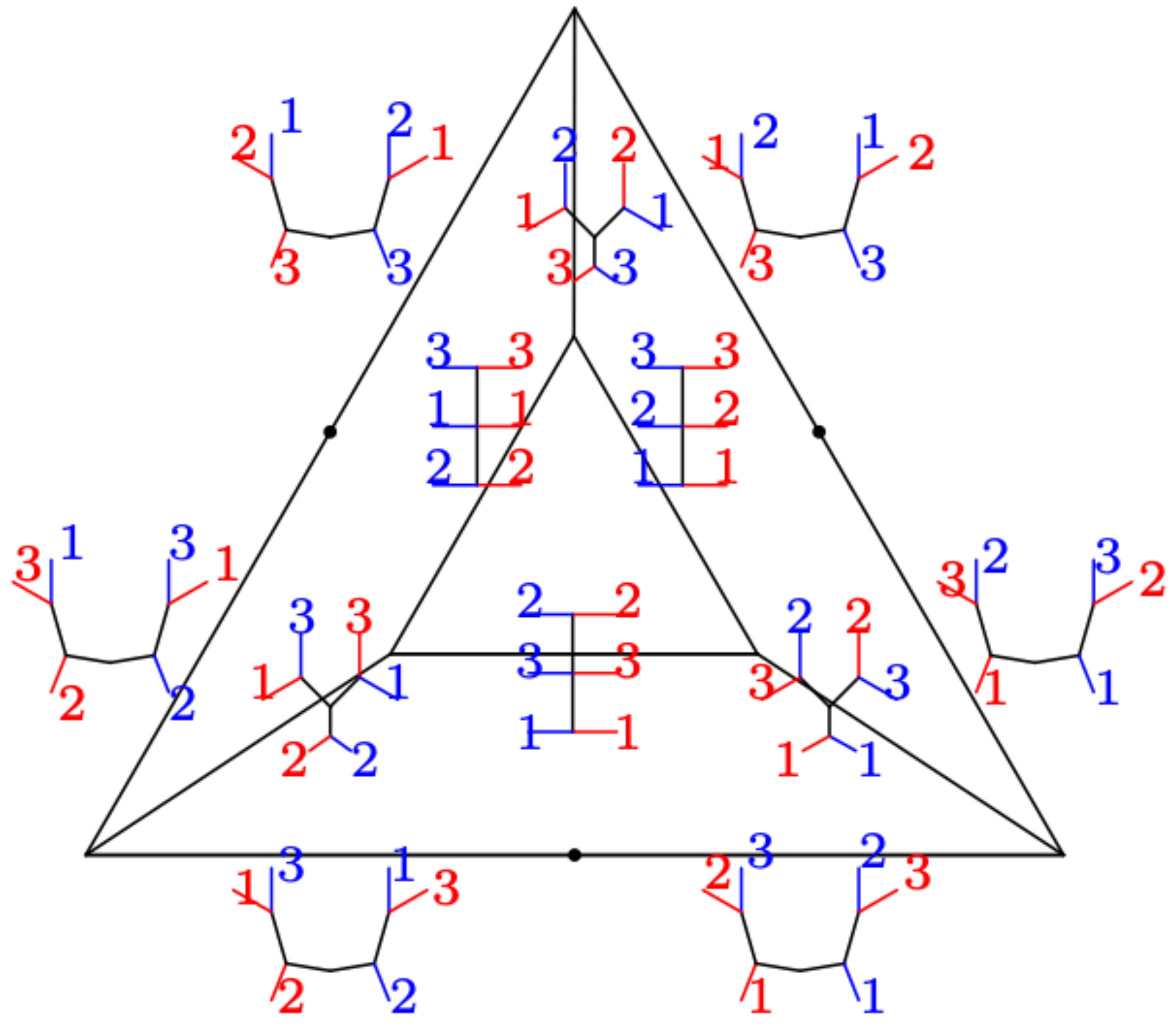
(One evidence…?)

Theorem (Al Amadieh-Cai-Yu)
A symbic tree is caterpillar if and only if its corresponding matrix has symmetric Barvinok rank 2.

For a more generalized conjecture, see **Conjecture 5.4 of Brakensiek-Dhar-Gao-Gopi-Larson**

References





Thank you for your attention!

(https://arxiv.org/abs/2404.08121)

References

Tropical convexity

A set $S \subset \mathbb{R}^n$ is called tropically convex if for any $x, y \in S$ and $a, b \in \mathbb{R}$, $a \odot x \oplus b \odot y \in S$.

Remark If S is tropically convex, then $S + \mathbb{R}\mathbf{1} \subset S$. Hence, it is natural to work on $\mathbb{R}^n / \mathbb{R}\mathbf{1}$.

Theorem (Develin-Santos-Sturmfels 2005) Let M be a $d \times n$ matrix. Then the troprank$(M) = \dim$ tconv(columns of M) $+ 1$