What is… the measurement variety?

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Matroids, Rigidity, and Algebraic Statistics

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The plan

The **measurement variety** $M_{d,G}$ of a graph G is (informally) the variety of d-dimensional squared edge measurements of G.

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The **measurement variety** $M_{d,G}$ of a graph G is (informally) the variety of d-dimensional squared edge measurements of G.

I will talk about:

- How does $M_{d,G}$ encode whether G is (globally) d-rigid?
- The dual variety of $M_{d,G}$ (the **generic stress variety**) and its algebraic matroid (the **generic stress matroid**).

The measurement variety

Algebraic matroids recap

An irreducible affine variety $X \subseteq \mathbb{C}^E$ determines an **algebraic matroid** $\mathcal{M}(X)$ on the set E of coordinate axes in which a subset $E' \subseteq E$ is

- independent $\Leftrightarrow$ projection $X \to \mathbb{C}^{E'}$ is dominant
- spanning $\Leftrightarrow$ generic fiber of projection $X \to \mathbb{C}^{E'}$ is finite

The rank of $\mathcal{M}(X)$ is $\dim(X)$.

Important fact

For generic $x \in X$, $\mathcal{M}(X) = \mathcal{M}(T_x(X))$.

The measurement map

Fix a positive integer d and a graph $G = (V, E)$.

We have

- the (complex) **configuration space** $(\mathbb{C}^d)^V$,
- the **measurement space** $\mathbb{C}^E$.

Definition

The (squared) **measurement map** $m_{d,G} : (\mathbb{C}^d)^V \to \mathbb{C}^E$ is defined by

$$m_{d,G} : p \mapsto \left((p_u - p_v)^T (p_u - p_v) \right)_{uv \in E}$$

The measurement variety

Definition

The d-dimensional **measurement variety** of G is

$$M_{d,G} = \overline{m_{d,G}((\mathbb{C}^d)^V)}.$$

For the complete graph K_V, M_{d,K_V} is the d-dimensional **Cayley-Menger variety** on vertex set V.

Definition

The d-dimensional **generic rigidity matroid** of G is

$$\mathcal{R}_d(G) = \mathcal{M}(M_{d,G})$$

Rigid graphs

Definition

We say that $G = (V, E)$ is **d-rigid** if the projection

$$\pi : M_{d,K_V} \to M_{d,G}$$

has finite fibers generically.

Rigid graphs

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has finite fibers generically.

If $|V| \geq d + 1$, then

$$G \text{ is } d\text{-rigid} \iff \dim(M_{d,G}) = d|V| - \binom{d+1}{2}.$$

Thus rigidity is intrinsic to the measurement variety.
(At least if we know the number of vertices.)

Globally rigid graphs

Definition

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Globally rigid graphs

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We say that G is **globally d-rigid** if the projection

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has fibers of size one generically.

If $|V| \geq d+2$, then

$$G \text{ is globally } d\text{-rigid} \iff ???$$

Is there an intrinsic characterization?

(Spoiler: there is.)

The rigidity matrix and tangent spaces

Recall

$$\mathcal{R}_d(G) = \mathcal{M}(M_{d,G}) = \mathcal{M}(T_x(M_{d,G}))$$

for generic $x \in M_{d,G}$.

What are the tangent spaces of $M_{d,G}$?

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Fix $p \in (\mathbb{C}^d)^V$. The **rigidity matrix** of (G, p) is

$$R(G, p) = Jac(m_{d,G})_p.$$

Set $x = m_{d,G}(p)$.

Fact

$$\operatorname{col}(R(G, p)) \subseteq T_x(M_{d,G}),$$

with **equality** if p is generic.

Stresses and tangent hyperplanes

Definition

A vector $\omega \in \mathbb{C}^E$ is a **stress** of (G,p) if $R(G,p)^T\omega = 0$.

Using the previous slide and some linear algebra we get:

Important fact

For generic p, ω is a stress of $(G,p) \Leftrightarrow T_x(M_{d,G}) \subseteq \omega^\perp$.

Mantra:

stresses of generic realizations

$\updownarrow$

tangent hyperplanes of $M_{d,G}$ at generic points

A life lesson

stresses are **normal** (vectors)

The contact locus

One last definition

For a vector $\omega \in \mathbb{C}^E$, the **contact locus** of ω is

$$L(\omega) = \overline{\{x \in M_{d,G} : x \text{ is smooth and } T_x(M_{d,G}) \subseteq \omega^\perp\}}$$

One can show that for a generic stress ω of a generic (G, p),

$$L(\omega) = \overline{\{m_{d,G}(q) : q \in (\mathbb{C}^d)^V, \omega \text{ is a stress of } (G, q)\}}$$

Another important fact

For a "generic" tangent hyperplane $\omega \in \mathbb{C}^E$ (whatever that means), $L(\omega) \subseteq M_{d,G}$ is a linear subspace.

Geometric Gortler-Healy-Thurston

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Theorem (Gortler-Healy-Thurston 2010)

A graph G on at least $d+2$ vertices is globally d-rigid $\Leftrightarrow$ G is d-rigid and $\dim(L(\omega)) = \binom{d+1}{2}$ for a "generic" tangent hyperplane $\omega \in \mathbb{C}^E$.

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Maybe the first condition can be replaced by $M_{d,G} \neq \mathbb{C}^E$.

(True for $d = 1, 2$, open in general.)

The generic stress matroid

Duality for affine varieties

Let $X \subseteq \mathbb{C}^E$ be irreducible and homogeneous.

Definition

The **dual** of X, denoted by X^*, is the Zariski-closure of

$$\{\omega \in \mathbb{C}^E : T_x(X) \subseteq \omega^\perp \text{ for some } x \in X_{sm}\}$$

Intuitively, X^* is the variety of hyperplanes tangent to X.

Fact

This is indeed a duality: $(X^*)^* = X$.

The generic stress variety

Definition

The d-dimensional **generic stress variety** of G is

$$S_{d,G} = (M_{d,G})^*$$

Recall: tangent hyperplanes to $M_{d,G} \approx$ stresses of generic realizations.

Let us say that $\omega \in \mathbb{C}^E$ is an **almost generic d-stress** of G if ω is the stress of some generic realization (G, p) in $\mathbb{C}^d$. Then

$$S_{d,G} = \overline{\{\omega \in \mathbb{C}^E : \omega \text{ is an almost generic } d\text{-stress of } G\}}$$

The generic stress matroid

Definition

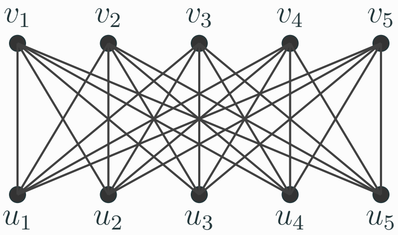
The d-dimensional **generic stress matroid** of G is

$$\mathcal{S}_d(G) = \mathcal{M}(S_{d,G})$$

A set $E' \subseteq E$ is

- *independent* in $\mathcal{S}_d(G)$ if for almost all $\omega' \in \mathbb{C}^{E'}$, there is an almost generic d-stress ω of G with $\omega|_{E'} = \omega'$;
- *spanning* in $\mathcal{S}_d(G)$ if for almost all $\omega' \in \mathbb{C}^{E'}$, there are finitely many almost generic d-stresses ω of G with $\omega|_{E'} = \omega'$.

Example: stresses of $K_{5,5}$ in $\mathbb{C}^3$



Take a generic realization $(K_{5,5}, p)$ in $\mathbb{C}^3$.

Choose affine dependencies $\sum_{i=1}^{5} a_i p(u_i) = \sum_{i=1}^{5} b_i p(v_i) = 0$.

Then $\omega = (a_i b_j)_{i,j=1}^{5}$ is a stress of $(K_{5,5}, p)$.

By a result of Bolker and Roth, every almost generic 3-stress of $K_{5,5}$ arises in this way.

Example: the stress matroid of $K_{5,5}$

Corollary

The stress variety $S_{3,K_{5,5}}$ is

$$\{ab^T : a, b \in \mathbb{C}^5, \sum_{i=1}^{5} a_i = \sum_{j=1}^{5} b_j = 0\} \subseteq \mathbb{C}^{5\times 5} \cong \mathbb{C}^E$$

Equivalently, $S_{3,K_{5,5}}$ is the variety of 5×5 matrices of rank 1 with zero row and column sums.

- $\mathcal{S}_3(K_{5,5}) \leq \mathcal{R}_1(K_{5,5})$ (= graphic matroid of $K_{5,5}$),
- $\mathrm{rank}(\mathcal{S}_3(K_{5,5})) = \dim(S_{3,K_{5,5}}) = 7$,
- vertex stars are dependent in $\mathcal{S}_3(K_{5,5})$,
- complements of vertex stars are spanning in $\mathcal{S}_3(K_{5,5})$.

Why study the stress matroid?

The stress matroid $\mathcal{S}_d(G)$ describes the global geometry of stresses of generic realizations.

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Question (Connelly 2011)

Suppose G is obtained by first gluing two redundantly d-rigid graphs (each on at least $d+2$ vertices) along $d+1$ vertices, and then deleting a common edge. Is G redundantly d-rigid?

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Theorem (G 2023+)

Yes!

Independent sets in the stress matroid

Note that if $e \in E$ is a bridge in $\mathcal{R}_d(G)$ (i.e., it is generically unstressable), then $S_{d,G} \subseteq \{\omega_e = 0\}$, and e is a loop in $\mathcal{S}_d(G)$.

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Theorem (G 2023+)

Fix $G = (V, E)$ and let $E' \subseteq E$ be a set of edges. If

- $G[E']$ is a forest on at most $d + 2$ vertices, and
- none of the edges in E' are bridges in $\mathcal{R}_d(G)$,

then E' is independent in $\mathcal{S}_d(G)$.

This is tight: vertex stars are dependent in $\mathcal{S}_3(K_{5,5})$.

Proof idea: we can use projective transformations to prescribe stresses on a small number of edges.

Spanning sets in the stress matroid

Theorem (G 2023+)

Fix $G = (V, E)$ and $E' \subseteq E$. If $E - E'$ spans at most $d+1$ vertices in G, then E' is spanning in $\mathcal{S}_d(G)$.

It follows that $\mathcal{S}_d(G)$ has no bridges.

Spanning sets in the stress matroid

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In fact, we have a **global rigidity** result.

Theorem (G 2023+)

If $\omega \in S_{d,G}$ is generic, $\omega' \in \mathbb{C}^E$ is a stress of some realization in general position, and $\{e \in E : \omega(e) \neq \omega'(e)\}$ spans at most $d + 1$ vertices, then $\omega = \omega'$.

Is this surprising?

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Example:

- Take a generic realization of K_4 in $\mathbb{R}^2$. It has a nonzero stress ω.
- Change the value of ω on a single edge uv arbitrarily to obtain another vector ω'.
- Then in any realization (K_4, q) that has ω' as a stress, we must have $q(u) = q(v)$.

A mysterious duality

Recall

$$\mathcal{S}_d(G) = \mathcal{M}(S_{d,G}) = \mathcal{M}(T_\omega(S_{d,G}))$$

for generic $\omega \in S_{d,G}$.

What are the tangent spaces of $S_{d,G}$?

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What are the tangent spaces of $S_{d,G}$?

I do not know, but:

Mysterious fact

For generic $\omega \in S_{d,G}$, we have

$$T_\omega(S_{d,G})^\perp = L(\omega).$$

By taking algebraic matroids, this implies

$$\mathcal{S}_d(G)^* = \mathcal{M}(L(\omega)).$$

The stress matroid of globally rigid graphs

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A graph G on at least $d+2$ vertices is globally d-rigid $\Leftrightarrow$ $\dim(M_{d,G}) = d|V| - \binom{d+1}{2}$ and $\dim(L(\omega)) = \binom{d+1}{2}$ for generic $\omega \in S_{d,G}$.

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When $d = 1$ or $d = 2$, $\mathcal{S}_d(G)$ is a uniform matroid for every globally d-rigid graph.

This fails for $d \geq 3$!

Characterizing these matroids is closely related to deciding whether the edge directions of a generic realization lie on a conic at infinity.

Some open questions

Measurement variety:

- Intrinsic characterization of global rigidity?
- What is the geometric relationship between $M_{d,G}$ and M_{d+1,G^*}, where G^* is the cone graph of G?

The stress matroid:

- Characterize $\mathcal{S}_2(G)$ for all G.
- Characterize $\mathcal{S}_d(G)$ for globally d-rigid G when $d \geq 3$.

References

Thank you!

Some references:

- Gortler, Theran, Thurston, **Generic unlabeled global rigidity**, *Forum of Mathematics, Sigma*, 2019.
- Garamvölgyi, **Stress-linked pairs of vertices and the generic stress matroid**, 2023. *arXiv:2308.16851*
- Rosen, Sidman, Theran, **Algebraic matroids in action**, *The American Mathematical Monthly*, 2020.