$x_1, \ldots, x_n \in \mathbb{R}^d$ w/ Mohammadi, Theran

$m : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}^{\binom{n}{2}}$

$m(x_1, \ldots, x_n) = (\|x_i - x_j\|^2)$

Obs: $(y_{ij}) \in \mathrm{Im}(m)$ satisfies poly. rel'ns.

$$m^* : \underbrace{\mathbb{C}[y_{ij} \mid 1 \le i < j \le n]}_{\binom{n}{2}} \longrightarrow \underbrace{\mathbb{C}[x_i]}_{dn}$$

$$m^*(y_{ij}) = \underbrace{(x_{i1} - x_{j1})^2}_{\in CM_{1,n}} + \cdots + \underbrace{(x_{id} - x_{jd})^2}_{\in CM_{1,n}}$$

$I_{d,n} :=$ Ker m^* ideal of polys in $\mathbb{C}[y_{ij}]$

$= (d+1)$-minors of an $(n-1)\times(n-1)$ Gram matrix

Ex $n=4$

$$\begin{pmatrix} 2y_{14} & y_{14}+y_{24}-y_{12} & y_{14}+y_{34}-y_{13} \\ y_{14}+y_{24}-y_{12} & 2y_{24} & y_{24}+y_{34}-y_{23} \\ y_{14}+y_{34}-y_{13} & y_{24}+y_{34}-y_{23} & 2y_{34} \end{pmatrix}$$

$I_{1,4} = \langle 2\times 2 \text{ minors} \rangle$

$I_{2,4} = \langle \det \rangle$

Def: (Borcea) The Cayley-Menger variety $CM_{d,n}$ is $\{ p \in \mathbb{C}^{\binom{n}{2}} \mid f(p) = 0 \ \forall f \in I_{d,n} \}$.
$(\subseteq \mathbb{P}^{\binom{n}{2}-1})$

Q: What does $CM_{d,n}$ tell us about rigidity?

$G = ([n], E)$ graph

$\mathbb{C}[E] = \mathbb{C}[y_{ij} \mid ij \in E]$

G is independent in $R_{d,n}$ $\iff$

$I_{d,n} \cap \mathbb{C}[E] = \langle 0 \rangle$

Ex: $d=2$, $|E| = 2n-3$

$I_{2,n} \cap \mathbb{C}[E] = \langle 0 \rangle \Rightarrow G$ is min. rigid.

In general,

$X \subseteq \mathbb{C}^N$

$\mathcal{I}(X) = \{ f \in \mathbb{C}[z_1, \dots, z_N] \mid f(p) = 0 \ \forall p \in X \}$

If X is an irr. alg. variety of dim r

$\mathcal{M}(X)$ = algebraic matroid of X

ground set $= [N]$

$I \subseteq [N]$ is indep if $\mathcal{I}(X) \cap \mathbb{C}[I] = \langle 0 \rangle$

Ex: X = coord. plane $\subseteq \mathbb{C}^3$

$\mathcal{I}(X) = \langle z_1 \rangle \subseteq \mathbb{C}[z_1, z_2, z_3]$

bases $= \{ \{2,3\} \}$

rank $\mathcal{M}(X) = \dim X$

X = generic plane $\subseteq \mathbb{C}^3$

$\mathcal{I}(X) = \langle z_1 + 3z_2 + 35 z_3 \rangle$

bases $= \{ \{1,2\}, \{1,3\}, \{2,3\} \}$

Obs: A gen'l pt of $CM_{d,n}$ is a sum of d pts on $CM_{1,n}$.

Def: $X \subseteq \mathbb{C}^N$ irred. variety of dim r

$(X \subseteq \mathbb{P}^{N-1})$

$X^{\{d\}}$ = dth <u>secant variety</u> of X

= a gen'l pt is a sum of d pts of X

If $\dim X^{\{d\}} < \min\{dr, N\}$, then X is <u>d-defective</u>.

<u>Q</u>: How is $\mathcal{M}(X^{\{d\}})$ related to $\mathcal{M}(X)$?

<u>Def</u>: $\mathcal{M}, \mathcal{N}$ matroids on E
- $\mathcal{M} \le \mathcal{N}$ if every ind. subset of $\mathcal{M}$ is ind in $\mathcal{N}$
- $\mathcal{M} \cup \mathcal{N}$ = matroid w/ ind sets

  $I \cup J$ I is ind in $\mathcal{M}$
  J " " $\mathcal{N}$

- $\underbrace{\mathcal{M} \cup \cdots \cup \mathcal{M}}_{d} = d\mathcal{M}$

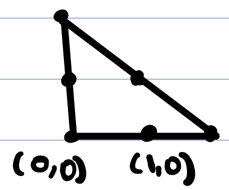
<u>Obs</u>: $\mathcal{M}(X) \le \mathcal{M}(X^{\{2\}}) \le \cdots \le \mathcal{M}(X^{\{N+1\}}) = \mathcal{M}(\mathbb{C}^N)$
if X spans $\mathbb{C}^N$.

inspired by Cruickshank, Mohammadi, Nixon, Tanigawa
g-rigidity

<u>Thm1</u>: $\mathcal{M}(X^{\{d\}}) \le d\mathcal{M}(X)$

<u>Thm2</u>: $X \subseteq \mathbb{C}^N$, irred (not a cone $\subseteq \mathbb{P}^{N-1}$),
X spans $\mathbb{C}^N$
$\mathcal{M}(X^{\{d\}}) = d\mathcal{M}(X) \iff \exists B$ of $d\mathcal{M}(X)$ s.t.
projection of X to $\mathbb{C}^B$ is
d-defective.

Ex: · projective curves never have defective projections
· 2-defective toric surfaces are rare <u>but</u> 2-defective projections are plentiful

$2\Delta =$ 

$\longleftrightarrow X_{2\Delta}$ parametrized by $\varphi : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^5$

$\varphi(s,t) = [1 : s : t : st : s^2 : t^2]$

If P is a lattice polygon & $P \supseteq 2\Delta$, then X_P has a 2-defective projection.