Matrix completion and tensor codes

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Matrix completion

Definition (Matrix completion matroid)

Fix $m \leq n$ and $0 \leq d \leq m$. The matrix completion matroid $\mathcal{B}_{m,n}(d,d)$ is the matroid on $[m] \times [n]$ whose bases are the subsets S of size $dm + dn - d^2$ such that, if you fill in the entries in an $m \times n$ matrix labeled by S with generic complex numbers, you can fill in the remaining entries so the matrix has rank at most d.

Theorem (Bernstein)

S is independent in $\mathcal{B}_{m,n}(2,2)$ if and only if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles.

Bipartite rigidity

- The bipartite rigidity matroid $\mathcal{B}_{m,n}(a,b)$ is a matroid on $[m] \times [n]$ of rank $na + mb - ab$, introduced by Kalai–Nevo–Novik.
- Contraction of a matrix completion matroid.
- Hyperconnectivity matroid (Kalai): matroid on $\binom{[n]}{2}$, generalizing skew symmetric matrix completion.
- Symmetric matrix completion matroid: matroid on $\binom{[n]}{2} \sqcup [n]$.

Tensor codes

- Suppose we have an $m \times n$ array of servers. The data on each server is an element of a field k.
- For redundancy, each column is required to lie in a fixed subspace of k^m, and each row is required to lie in fixed subspace of k^n.
- Suppose the servers labeled by S fail. Can we recover all of the data?
- For simplicity, we will assume that the subspaces are *generic*.

Tensor codes

- Let k a field of characteristic $p \geq 0$. Let $v_1, \ldots, v_m$ be m *generic* vectors in k^s and $w_1, \ldots, w_n$ be n generic vectors in k^r.
- We have mn vectors $v_i \otimes w_j$ in $k^s \otimes k^r$.

Definition (Tensor matroid)

Let $\mathrm{T}_{m,n}(s,r,p)$ be the matroid on $[m] \times [n]$ whose bases are the sets S of size rs for which $\{v_i \otimes w_j : (i,j) \in S\}$ is a basis for $k^s \otimes k^r$.

- S is spanning in $\mathrm{T}_{m,n}(s,r,p)$ if and only if no data is lost when the servers labeled by S^c fail.

Equivalence

- $\mathrm{T}_{m,n}(s,r,p)$: vectors in $k^s \otimes k^r$, where k has characteristic p.

Theorem (Brakensiek–Dhar–Gao–Gopi–L.)

$\mathcal{B}_{m,n}(a,b)$ is the matroid dual of $\mathrm{T}_{m,n}(m-a, n-b, 0)$.

- No data is lost when the servers labeled by S fail if and only if S is independent in $\mathcal{B}_{m,n}(a,b)$.
- The hyperconnectivity matroid is dual to a $\wedge^2$ matroid, and the symmetric matrix completion matroid is dual to a Sym^2 matroid.

Applications to $m - a$ small

- If $m - a$ is small, then we can analyze $\mathcal{B}_{m,n}(a, b)$ using $\mathsf{T}_{m,n}(m - a, n - b, 0)$.
- We describe cocircuits in $\mathcal{B}_{m,n}(a, b)$ when $m - a \leq 3$ and give a polynomial time algorithm to check independence.

Theorem (Brakensiek–Dhar–Gao–Gopi–L.)

Write $S = \bigcup_{i=1}^{m} \{i\} \times A_i \subseteq [m] \otimes [n]$. Let S_k be the set of $j \in [n]$ which appear in exactly k of the A_i. S is independent in $\mathrm{T}_{m,n}(3, r, 0)$ if and only if

$$|A_i \cap A_j \cap A_k \cap A_\ell| = 0$$

$$|A_i \setminus S_3| + |S_3| \leq r$$

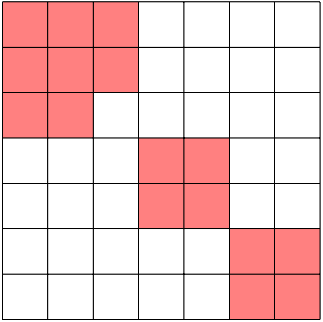
$$|(A_i \cap A_j) \setminus S_3| + |(A_k \cap A_\ell) \setminus S_3| + |S_3| \leq r$$

$$|A_i \setminus S_3| + |A_j \setminus S_3| + |S_2 \setminus (S_3 \cup A_i \cup A_j)| + 2|S_3| \leq 2r$$

$$|S_1| + 2|S_2| + 3|S_3| \leq 3r$$

Applications to $m - a$ small

- We do not know a nice description of the independent sets or circuits of $\mathcal{B}_{m,n}(a, b)$ when $m - a = 3$.
- $\mathcal{B}_{m,n}(a, b)$ has a Laman-like description when $m - a \leq 2$.



- A circuit of $\mathrm{T}_{m,n}(4, 4, p)$ for any p.

Positive characteristic

- Applications use $\mathrm{T}_{m,n}(s, r, p)$ when $p > 0$, especially $p = 2$.
- We have

$$\mathrm{T}_{m,n}(m-d, n-d, p)^{\perp} \subseteq \text{matrix completion in char } p \subseteq \mathcal{B}_{m,n}(d, d).$$

Theorem (Brakensiek–Dhar–Gao–Gopi–L.)

If $s \leq 3, m - s \leq 1$, or $m - s = n - r = 2$, then $\mathrm{T}_{m,n}(s, r, p)$ is independent of p.

Rank 2 matrix completion in characteristic p

Theorem (Bernstein)

S is independent in $\mathcal{B}_{m,n}(2,2)$ if and only if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles.

- We show that if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles, then S is independent in the dual of $\mathrm{T}_{m,n}(m-2, n-2, p)$ for any p.
- We prove a determinant is nonzero by constructing an explicit monomial with coefficient ± 1. Bicoloring the edges of a bipartite graph is equivalent to orienting the edges.

Linear algebraic matroids

- In all examples, $\mathrm{T}_{m,n}(s, r, p)$ is independent of $p \geq 0$.
- Also true for the $\wedge^2$ matroid, and for the Sym^2 matroid except when $p = 2$.

Theorem (Bernstein)

S is independent in the rank 2 skew-symmetric matrix completion matroid if and only if the corresponding graph has an edge orientation with no directed cycles or alternating closed trails.

- Our argument produces bases (in any characteristic) which satisfy a different-looking condition.