A tropical approach to counting realisations of frameworks

Oliver Clarke[*] Sean Dewar[†] Daniel Green-Tripp[†] James Maxwell[†]
Anthony Nixon[‡] Yue Ren[*] Ben Smith[‡]

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[*]University of Durham
[†]University of Bristol
[‡]Lancaster University

Outline of talk

1. Realisation numbers as root counts of polynomials
2. The Bergman fan of a graph
3. Proof by tropical geometry
4. Combinatorial upper bounds

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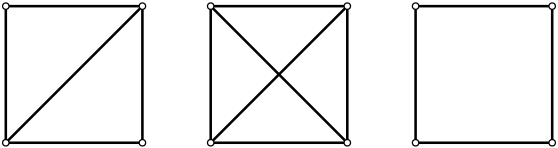
Generic rigidity

(G, p) *framework* where $G = (V, E)$ graph, $p : V \to \mathbb{F}^d$ realisation ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$).

- (G, p) is *rigid* if all length preserving continuous motions are trivial.
- G is *d-rigid* if there exists generic (G, p) in $\mathbb{F}^d$ that is rigid.
- G is *minimally d-rigid* if d-rigid and $G - e$ is not d-rigid $\forall e \in E$.



Question

How many frameworks with the same edge lengths are there?

Counting realisations of d-rigid graphs

Definition

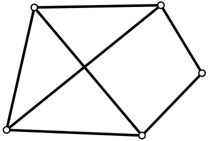
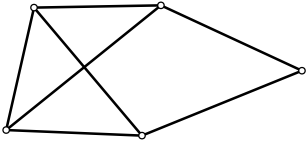
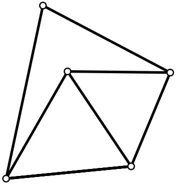
Given $G = ([n], E)$ with $n \geq d + 1$, its *rigidity map* is

$$f_{G,d} \colon \mathbb{F}^{n \cdot d} \to \mathbb{F}^{E}, \quad p \mapsto \left(\sum_{k=1}^{d} (p_{i,k} - p_{j,k})^2 \right)_{ij \in E} .$$

The *realisation space* of (G, p)

$$C_d(G,p) := f_{G,d}^{-1}(f_{G,d}(p))/\sim \qquad p \sim q \Leftrightarrow p_i = Aq_i + r \,,\, A \in O_d(\mathbb{F}) \,,\, r \in \mathbb{F}^d$$

- $\mathbb{F} = \mathbb{R}$: $|C_d(G,p)|$ is not constant over generic $p \in \mathbb{R}^{n \cdot d}$
- $\mathbb{F} = \mathbb{C}$: $|C_d(G,p)|$ is constant over generic $p \in \mathbb{C}^{n \cdot d}$.

Counting realisations of minimally d-rigid graphs

$$f_{G,d}\colon \mathbb{C}^{n\cdot d} \to \mathbb{C}^E, \quad \left(p_{i,k}\right)_{i\in[n],k\in[d]} \mapsto \left(\sum_{k=1}^{d}(p_{i,k}-p_{j,k})^2\right)_{ij\in E}.$$

Theorem (Folklore?)

For a graph G with $n \geq d+1$, the following are equivalent:

1. *G is d-rigid;*
2. *$C_d(G,p) = f_{G,d}^{-1}(f_{G,d}(p))/\sim$ is finite for generic $p \in \mathbb{C}^{n\cdot d}$;*
3. *$f_{G,d}^{-1}(\lambda)/\sim$ is finite for generic $\lambda \in \overline{f_{G,d}(\mathbb{C}^{n\cdot d})}$.*

Moreover, G is minimally d-rigid iff it is d-rigid and $\overline{f_{G,d}(\mathbb{C}^{n\cdot d})} = \mathbb{C}^E$.

Definition

The *d-realisation number* $c_d(G)$ of a minimally d-rigid graph G is

$$\begin{aligned} c_d(G) &:= |C_d(G,p)| \quad \text{for generic } p \in \mathbb{C}^{d\cdot n} \\ &= |f_{G,d}^{-1}(\lambda)/\sim| \quad \text{for generic } \lambda \in \mathbb{C}^E \end{aligned}$$

d-Realisation numbers as a root count

Proposition (Capco et al. '18, Clarke et al. '25+)

Let G a minimally d-rigid graph, and $I \subseteq \mathbb{C}[y_{ij,k}^{\pm} \mid ij \in E(G), k \in [d]]$ ideal generated by

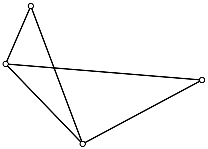
$$\text{(edge lengths)} \qquad f_{ij} := \sum_{k=1}^{d} y_{ij,k}^2 - a_{ij} \quad \text{for } ij \in E(G)$$

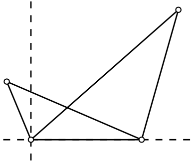
$$\text{(vertex pinning)} \qquad g_{i,\ell} := \sum_{k=1}^{d} c_{\ell,k} y_{1i,k} \quad \text{for } i \in [d] \setminus \{1\} \text{ and } \ell \in [d+1-i]$$

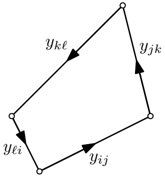
$$\text{(cycle sum)} \qquad h_{C,k} := \sum_{(s,t)\in C} y_{st,k} \quad \text{for each directed cycle } C \text{ of } G \text{ and } k \in [d],$$

where $a_{ij}, c_{\ell,k} \in \mathbb{C}$ generic. Then $2^d \cdot c_d(G) = |V(I)|$, where $V(I)$ solution set of I.

Idea: Write $y_{ij,k} := p_{i,k} - p_{j,k}$, gives 'isomorphism' between $V(I)$ and $C(G,p)$.







2-Realisation numbers as a root count

When $d = 2$, do change of variables:

$$\begin{aligned} y_{ij,1} &\mapsto y_{ij,1} + \mathrm{i} \cdot y_{ij,2} \\ y_{ij,2} &\mapsto y_{ij,1} - \mathrm{i} \cdot y_{ij,2} \end{aligned}, \qquad f_{ij} := y_{ij,1}^2 + y_{ij,2}^2 - a_{ij} \rightsquigarrow f'_{ij} := y_{ij,1} \cdot y_{ij,2} - a_{ij} .$$

We can kill these equations and half the variables by setting $y_{ij,2} = a_{ij} y_{ij,1}^{-1}$.

Corollary

Let G a minimally 2-rigid graph, and $I' \subseteq \mathbb{C}[y_{ij}^{\pm} \mid ij \in E(G)]$ be the ideal generated by

$$h'_{C,1} := \sum_{(i,j)\in C} y_{ij} \qquad \textit{for each directed cycle } C \textit{ of } G,$$

$$h'_{C,2} := \sum_{(i,j)\in C} a_{ij} y_{ij}^{-1} \qquad \textit{for each directed cycle } C \textit{ of } G,$$

$$g'_{12} := c_{1,1} y_{12}^2 + c_{1,2}.$$

where $a_{ij}, c_{1,k} \in \mathbb{C}$ generic. Then $4 \cdot c_2(G) = |V(I')|$.

Key Point

Realisation numbers are generic root counts of a polynomial system with special shape.

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1. Realisation numbers as root counts of polynomials
2. The Bergman fan of a graph
3. Proof by tropical geometry
4. Combinatorial upper bounds

Flats of a graph

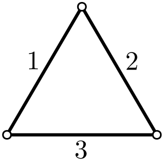
If you like matroids...

We specialise constructions to graphic matroids, all work for arbitrary matroids!

Given $G = (V, E)$ graph, define

$$\mathrm{rk}\colon 2^E \to \mathbb{Z}_{\geq 0} \quad , \quad \mathrm{rk}(A) = \max\{|T| \ : \ T \subseteq A \text{ forest}\}$$

- $F \subseteq E$ a *flat* if $\mathrm{rk}(F + e) = \mathrm{rk}(F) + 1$ for all $e \in E \setminus F$.
- The flats are G are disjoint unions of induced subgraphs.
- They form a graded lattice $\mathcal{L}_G$ with minimum $\emptyset$ and maximum E.



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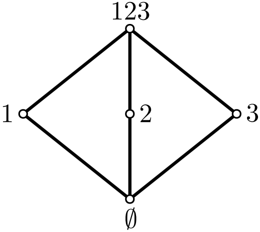

$\mathcal{L}_G$

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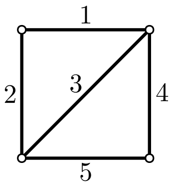

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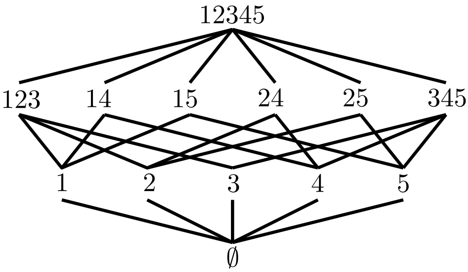


$\mathcal{L}_G$

The Bergman fan of a graph

For $e \in E$, write $\chi_e \in \mathbb{R}^E$ for corresponding unit vector,
For $A \subseteq E$, write $\chi_A = \sum_{e \in A} \chi_e \in \mathbb{R}^E$.

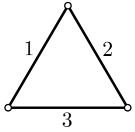
Definition

Given a chain of flats $\mathcal{F}$: $\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E$, define

$$\sigma_{\mathcal{F}} = \text{cone}(\chi_{F_i} \, : \, F_i \in \mathcal{F}) + \mathbb{R} \cdot \chi_E \, .$$

The *Bergman fan* of G is the $(n-1)$-dimensional fan in $\mathbb{R}^E$

$$\mathcal{B}_G = \bigcup_{\mathcal{F} \subseteq \mathcal{L}_G} \sigma_{\mathcal{F}} \subseteq \mathbb{R}^E, \qquad \overline{\mathcal{B}}_G := \mathcal{B}_G / (\mathbb{R} \cdot \chi_E) \subseteq \mathbb{R}^E / (\mathbb{R} \cdot \chi_E) \cong \mathbb{R}^{|E|-1}$$



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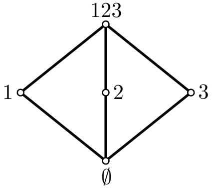
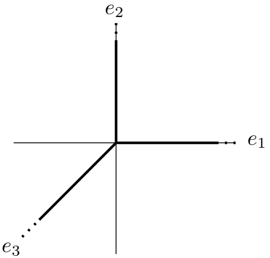


$\mathcal{L}_G$



$\mathcal{B}_G \subseteq \mathbb{R}^3$ $\overline{\mathcal{B}}_G \subseteq \mathbb{R}^2$

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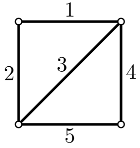

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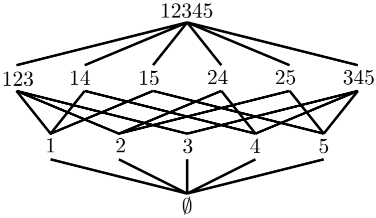
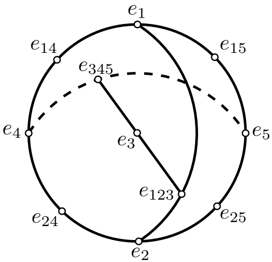


$\mathcal{L}_G$



$\mathcal{B}_G \subseteq \mathbb{R}^5 \quad \overline{\mathcal{B}}_G \subseteq \mathbb{R}^4$

Realisation numbers from Bergman fans

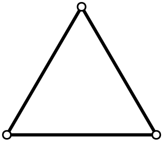
When G a minimally 2-rigid graph, $|E| = 2n - 3$

$\Rightarrow \mathcal{B}_G \subseteq \mathbb{R}^{2n-3}$, $\overline{\mathcal{B}}_G \subseteq \mathbb{R}^{2n-4}$ of codimension $n - 2$.

Theorem (Clarke et al. 25+)

G a minimally 2-rigid graph and $\alpha \in \mathbb{R}^{|E|-1}$ generic,

$$c_2(G) = \frac{1}{2}\left|\overline{\mathcal{B}}_G \cap (\alpha - \overline{\mathcal{B}}_G)\right| .$$



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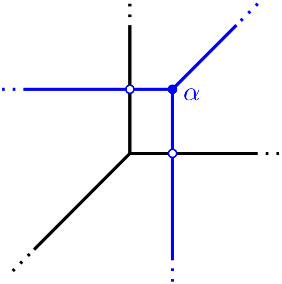
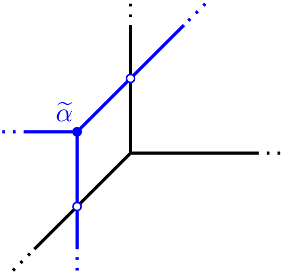


$\mathcal{B}_G \cap (\alpha - \mathcal{B}_G)$



$\mathcal{B}_G \cap (\widetilde{\alpha} - \mathcal{B}_G)$

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Tropical geometry

Motto

Tropical varieties are the combinatorial/polyhedral shadow of algebraic varieties.

Given $f = \sum_{a \in S} c_a X^a \in \mathbb{C}[X_1^{\pm}, \ldots, X_n^{\pm}]$ with support $S \subseteq \mathbb{Z}^n$,

$$\begin{aligned} \mathrm{trop}(f)\colon \mathbb{R}^n &\to \mathbb{R} \\ p &\mapsto \min_{a \in S}(a \cdot p) \end{aligned}$$

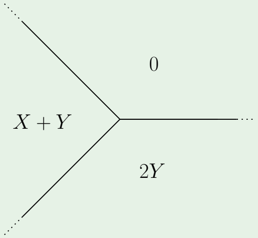
The *tropical hypersurface* defined by f is

$$\mathcal{T}(f) = \{\, p \in \mathbb{R}^n \mid \mathrm{trop}(f)(p) \text{ attains minimum at least twice } \}$$

Example

$$f = Y^2 + XY + 1 \in \mathbb{C}[X^{\pm}, Y^{\pm}]$$

$$\mathrm{trop}(f) = \min(2Y, X + Y, 0)$$



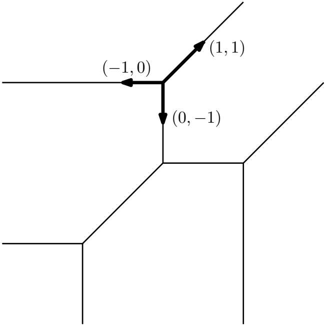
Tropical geometry

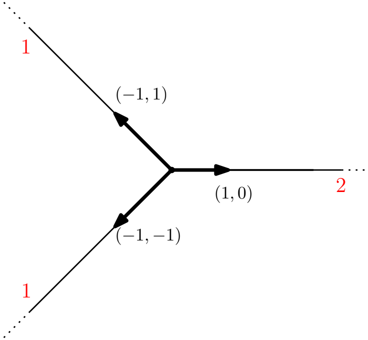
Given $I \subseteq \mathbb{C}[X_1^{\pm}, \dots, X_n^{\pm}]$ ideal, its *tropical variety* is

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f)\,.$$

Structure Theorem (abridged)

$\mathcal{T}(I)$ is a balanced polyhedral complex with multiplicities, of the same dimension as $V(I)$.





$$1 \cdot (-1,1) + 1 \cdot (-1,-1) + 2 \cdot (1,0) = 0$$

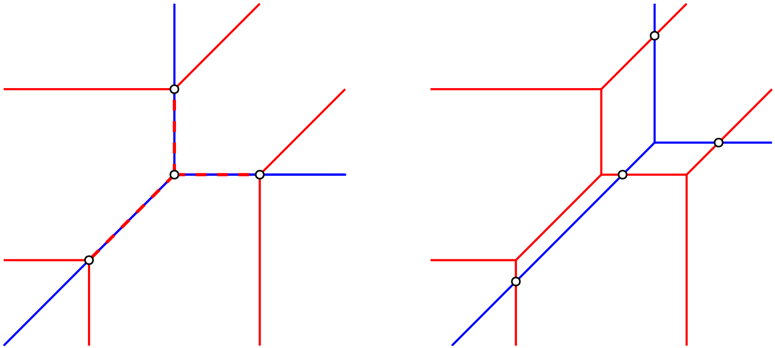
Stable intersection

Non-generic intersections of tropical varieties may not be tropical varieties.

Definition

Let Σ_1, Σ_2 be polyhedral complexes in $\mathbb{R}^n$. Their *stable intersection* is

$$\Sigma_1 \wedge \Sigma_2 := \lim_{\epsilon \to 0} (\Sigma_1 \cap (\Sigma_2 + \epsilon v)), \quad \epsilon > 0, \ v \in \mathbb{R}^n \text{ generic} \ .$$



Tropical root counts

Recall $|V(I)| = 4 \cdot c_2(G)$ where $I = H_1 + H_2 + J$:

$$H_1 := \langle \sum_{(i,j)\in C} y_{ij} \mid C \text{ cycle of } G \rangle \qquad \mathcal{T}(H_1) = \mathcal{B}_G \,,$$

$$H_2 := \langle \sum_{(i,j)\in C} a_{ij} y_{ij}^{-1} \mid C \text{ cycle of } G \rangle \qquad \mathcal{T}(H_2) = -\mathcal{B}_G \,,$$

$$J := \langle c_{1,1} y_{12}^2 + c_{1,2} \rangle \qquad \mathcal{T}(J) = 2 \cdot \{y_{12} = 0\}.$$

Theorem (Folklore?)

The Bergman fan can equivalently be characterised as

$$\mathcal{B}_G = \{y \in \mathbb{R}^E \mid \min_{e\in C}(y_e) \text{ attains minimum at least twice for each cycle } C\}$$

Sketch of proof

Theorem (Helminck, Ren '22)

Let $I_1, \ldots, I_k \subseteq \mathbb{C}[X_1^{\pm}, \cdots, X_d^{\pm}]$ be ideals, with $I := I_1 + \cdots + I_k$ zero-dimensional. If [technical conditions], then

$$|V(I)| = |\mathcal{T}(I_1) \wedge \ldots \wedge \mathcal{T}(I_k)|$$

$$\begin{aligned} 4 \cdot c_2(G) &= |V(I)| \\ &= |\mathcal{B}_G \wedge (-\mathcal{B}_G) \wedge 2 \cdot \{y_{12} = 0\}| \\ 2 \cdot c_2(G) &= |\mathcal{B}_G \wedge (-\mathcal{B}_G) \wedge \{y_{12} = 0\}| \\ &= |\overline{\mathcal{B}}_G \wedge (-\overline{\mathcal{B}}_G)| \end{aligned}$$

Theorem (Clarke et al. 25+)

G a minimally 2-rigid graph then

$$c_2(G) = \frac{1}{2} \left| \overline{\mathcal{B}}_G \wedge (-\overline{\mathcal{B}}_G) \right| .$$

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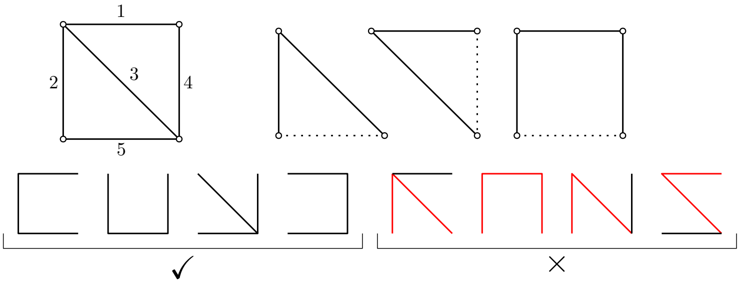
Nbc-bases

Definition

G graph with total order $\prec$ on E.

- A *broken circuit* is $C - \min_{i \in C}(i)$ for cycle C of G.
- A *nbc-basis* is a spanning tree T containing no broken circuits.

We write $\mathrm{nbc}(G) := \#\text{nbc-bases}$.



- The number of nbc-bases is invariant under changing the order $\prec$,
- $\mathrm{nbc}(G) = T_G(1, 0)$, an evaluation of the Tutte polynomial,
- $\mathrm{nbc}(G) =$ linear term of chromatic polynomial of G.

A combinatorial upper bound

Theorem (Clarke et al. 25+)

G a minimally 2-rigid graph,

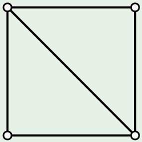
$$c_2(G) \leq \frac{1}{2}\,\mathrm{nbc}(G)\,.$$

- $\mathrm{nbc}(G) = |\overline{\mathcal{B}}_{U_{n-1,E}} \wedge (-\overline{\mathcal{B}}_G)|$ (Adiprasito, Huh, Katz '18)
- Deform $\overline{\mathcal{B}}_G$ into $\overline{\mathcal{B}}_{U_{n-1,E}}$, can only gain intersection points:

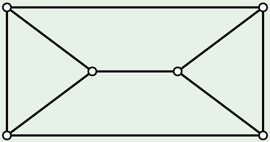
$$2 \cdot c_2(G) = |\overline{\mathcal{B}}_G \wedge (-\overline{\mathcal{B}}_G)| \leq |\overline{\mathcal{B}}_{U_{n-1,E}} \wedge (-\overline{\mathcal{B}}_G)| = \mathrm{nbc}(G)$$

Example

$$c_2(G) = 2$$
$$\frac{1}{2}\,\mathrm{nbc}(G) = 2$$



$$c_2(G') = 12$$
$$\frac{1}{2}\,\mathrm{nbc}(G') = 13$$



Heuristic

Nbc bases are faster to compute than mixed volumes, and generally give better bounds.

Open questions

Question 1

Is there a combinatorial formula for $\overline{\mathcal{B}}_G \wedge (-\overline{\mathcal{B}}_G)$ purely in terms of G (or M_G)?

$$|\overline{\mathcal{B}}_M \wedge (-\overline{\mathcal{B}}_{U_{n-r,n}})| = \mathrm{nbc}(M) \qquad \text{(Adiprasito, Huh, Katz '18)}$$

$$|\overline{\mathcal{B}}_{M\setminus e} \wedge (-\overline{\mathcal{B}}_{(M/e)^*})| = \beta(M) \qquad \text{(Ardila-Mantilla, Eur, Penaguiao '22)}$$

Conjecture (Jackson, Owen '19)

Every minimally 2-rigid graph G with n vertices has $c_2(G) \geq 2^{n-3}$.

Question 2

Can we use this machinery to get lower bounds?

Thank you for listening!