Volume Rigidity of Simplicial Manifolds

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Volume Rigidity of Frameworks

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- The **volume** of a hyperedge $\Delta \in E$ in (G, p), $\mathrm{Vol}(p(\Delta))$, is equal to the volume of the convex hull of $p(\Delta)$ in the affine subspace of $\mathbb{R}^d$ spanned by $p(\Delta)$ when $|\Delta| \leq d + 1$ and $p(\Delta)$ is in general position in $\mathbb{R}^d$; otherwise $\mathrm{Vol}(p(\Delta)) = 0$.

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- Note that, if G is a graph, then the volume rigidity of (G, p) is equivalent to the standard definition of 'bar-joint' rigidity.

Example

Consider a generic realisation of the 3-simplex in $\mathbb{R}^2$. It is well known that the framework containing its six edges is (volume) rigid. On the other hand, the framework containing its four 2-faces cannot be volume rigid since a simple dimension counting argument tells us we need $2 \times 4 - 3 = 5$ constraints to become rigid. A more careful analysis tells us that the constraints determined by the four 2-faces are dependent. It is not obvious which combinations of 5 edges and faces give rigidity. More generally, it is a difficult open problem to decide when a given hypergraph is generically rigid in $\mathbb{R}^2$.

Volume Rigidity Matrix

- The d-**dimensional volume rigidity map** of a hypergraph $G = (V, E)$ is the function $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ given by

$$f_G(p) = (f_p(\Delta_1), f_p(\Delta_2), \ldots, f_p(\Delta_m))$$

where $E = \{\Delta_1, \Delta_2, \ldots, \Delta_m\}$ and $f_p(\Delta_h) = \mathrm{Vol}(p(\Delta_h))^2$.

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- The **volume rigidity matrix** $R(G, p)$ of a realisation p of G in $\mathbb{R}^d$ is given by the Jacobean matrix of f_G evaluated at p.
- For $v_i \in \Delta \in E$ and $p(v_i) = p_i$ we have

$$\mathrm{Vol}(p(\Delta))^2 = \begin{cases} \frac{1}{k^2}\|p_i - p_i^\Delta\|^2\,\mathrm{Vol}(p(\Delta - v_i))^2 & (|\Delta| = k \geq 3) \\ \|p_i - p_j\|^2 & (\Delta = \{v_i, v_j\}) \end{cases}$$

  where p_i^Δ is the projection of p_i onto the affine subspace of $\mathbb{R}^d$ spanned by $p(\Delta - v_i)$.

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- Hence, by applying an appropriate rescaling, we can take the entry in row Δ and columns v_i of $R(G,p)$ to be:

$$\begin{aligned} (p_i - p_i^\Delta)\,\mathrm{Vol}(p(\Delta - v_i))^2 &\text{ when } v_i \in \Delta \text{ and } |\Delta| \geq 3; \\ p_i - p_j &\text{ when } \Delta = \{v_i, v_j\}; \\ \mathbf{0} &\text{ when } v_i \notin \Delta. \end{aligned}$$

Infinitesimal Volume Rigidity

The elements of the left kernel of $R(G,p)$ are the **infinitesimal motions** of (G,p). Each infinitesimal isometry of $\mathbb{R}^d$ will give rise to an infinitesimal motion of (G,p) and hence, when $p(V)$ affinely spans $\mathbb{R}^d$, rank $R(G,p) \leq d|V| - \binom{d+1}{2}$. More generally,

$$\text{rank } R(G,p) \leq d|V| - \binom{d+1}{2} + \binom{d-t}{2}$$

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Lemma

The infinitesimal volume rigidity of (G,p) is a sufficient condition for its volume rigidity. And the two conditions are equivalent when p is generic.

Volume Rigidity Matroid

The d-**dimensional volume rigidity matroid** $\mathcal{R}_d(G)$ of a hypergraph $G = (V, E)$ is the matroid on E given by the row matroid of $R(G, p)$ for any generic realisation p in $\mathbb{R}^d$.

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The hypergraph G is said to be **volume rigid in** $\mathbb{R}^d$ if

$$\text{rank } \mathcal{R}_d(G) = \begin{cases} d|V| - \binom{d+1}{2} & \text{when } |V| \geq d+1, \\ \binom{|V|}{2} & \text{when } |V| \leq d. \end{cases}$$

All generic realisation of such hypergraphs in $\mathbb{R}^d$ will be volume rigid.

Previous results

Previous work has concentrated on the d-dimensional volume rigidity matroid of $(d+1)$-uniform hypergraphs $G = (V, E)$.

- Borcea and Streinu observed that rank $\mathcal{R}_d(G)$ can be at most $d|V| - d^2 - d + 1$ (since the map $p(v) \mapsto Ap(v) + t$ will be an infinitesimal motion of (G, p) for any orthogonal $A \in \mathbb{R}^{d \times d}$ and $t \in \mathbb{R}^d$).

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- Borcea and Streinu, and subsequently Southgate, obtained bounds on the number of distinct realisations of G in $\mathbb{R}^d$ which are volume equivalent to a given generic realisation when rank $\mathcal{R}_d(G)=d|V|-d^2-d+1$.

Complete Uniform Hypergraphs

Let K_n^k denote the complete k-uniform hypergraph on n vertices.

Theorem: Cruickshank, Jackson and Tanigawa; Lew, Nevo, Peled, and Raz; 2025+

K_n^k is volume rigid in $\mathbb{R}^d$ for all $2 \leq k \leq d-1$ and all $n \geq d+1$.

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These results tell us there exist k-uniform hypergraphs which are volume rigid in $\mathbb{R}^d$ whenever $2 \leq k \leq d$ and encourage us to find larger families of such hypergraphs.

Simplicial manifolds

A **simplicial** d**-manifold** $\mathcal{S}$ is an abstract simplicial d-complex whose underlying topological space is a connected d-dimensional manifold. In particular, a simplicial 2-manifold is a simplicial 2-complex whose facets are the triangles of a triangulated surface. We will see that the k-skeleton hypergraph of a simplicial $(d-1)$-manifold is volume rigid in $\mathbb{R}^d$ whenever $1 \leq k \leq d-2$.

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The case $k = 1$ and $\mathcal{S}$ is the boundary complex of a convex simplicial d-polytope P i.e. the convex hull of a finite number number of points in $\mathbb{R}^d$ in which each $(d-1)$-dimensional face is a simplex, follows from classical results of Dehn and Whiteley, which tell us that the 1-skeleton of P is an infinitesimally rigid framework in $\mathbb{R}^d$ (without any assumption of genericity).

Rigidity of 1-skeletons of simplicial manifolds

Theorem (Dehn, 1916)

The 1-skeleton of every convex simplicial polyhedron is an infinitesimally rigid framework in $\mathbb{R}^3$.

Theorem (Whiteley, 1984)

The 1-skeleton of every convex simplicial d-polytope is an infinitesimally rigid framework in $\mathbb{R}^d$ for all $d \geq 3$.

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Theorem (Kalai, 1987)

The graph of every simplicial $(d-1)$-manifold is (generically) rigid in $\mathbb{R}^d$ for all $d \geq 4$.

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Theorem

Let G be the k-uniform hypergraph consisting of the $(k-1)$-faces of a connected simplicial $(d-1)$-manifold. Then G is volume rigid in $\mathbb{R}^d$ for all $1 \leq k \leq d-2$ when $d \geq 4$, and also for $k = d-1$ when $d = 3, 4, 5, 6$.

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Conjecture

The $(d-1)$-uniform hypergraph consisting of the $(d-2)$-faces of a connected simplicial $(d-1)$-manifold is volume rigid in $\mathbb{R}^d$ for all $d \geq 3$.

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Question

Is the framework defined by the k-faces of a convex d-polytope infinitesimally volume rigid in $\mathbb{R}^d$ when $1 \leq k \leq d-2$ and $d \geq 3$?

Three Combinatorial Lemmas

Gluing Lemma

If G_1, G_2 are volume rigid in $\mathbb{R}^d$ and $|V(G_1) \cap V(G_2)| \geq d$, then $G = G_1 \cup G_2$ is volume rigid in $\mathbb{R}^d$.

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Given a hypergraph $G = (V, E)$ and $w \in V$ let:

$$G - w = (V - w, \{\Delta \in E : w \notin \Delta\});$$

$$G_w = (V - w, \{\Delta - w : w \in \Delta \in E\}).$$

Coning Lemma

Let d be an integer, $G = (V, E)$ be a k-uniform hypergraph with $3 \leq k \leq d$ and $w \in V$. Suppose that $G - w$ is volume rigid in $\mathbb{R}^d$ and each of the strongly connected components of G_w contains a copy of K_k^{k-1}. Then G is volume rigid in $\mathbb{R}^{d+1}$.

Three Combinatorial Lemmas

Given a hypergraph $G = (V, E)$ and $u, v \in V$, let:

$$E_{uv} = \{\Delta \in E : \{u, v\} \subseteq \Delta\};$$

$$E_v^u = \{\Delta \in E : v \in \Delta,\ u \notin \Delta,\ \Delta - v + u \in E\};$$

$$G/uv = (G - u) \cup \{\Delta - u + v : u \in \Delta,\ v \notin \Delta\}).$$

For $p : V - u \to \mathbb{R}^d$, $\mathbf{d} \in \mathbb{R}^d$, and $\Delta \in E_{uv}$ let $\mathbf{d}_\Delta$ denote the projection of $\mathbf{d}$ onto the orthogonal complement of the linear subspace of $\mathbb{R}^d$ spanned by $\{p(x) - p(v) : x \in \Delta - u\}$ and put

$$A_{uv}(G, p, \mathbf{d}) = \{\mathbf{d}_\Delta : \Delta \in E_{uv}\} \cup \{p(v) - p(v)^\Delta : \Delta \in E_v^u\}.$$

Vertex Splitting Lemma

Let $G = (V, E)$ be a hypergraph, u, v be distinct vertices of G, p be an infinitesimally volume rigid realisation of G/uv in $\mathbb{R}^d$ and $\mathbf{d} \in \mathbb{R}^d$. Suppose $A_{uv}(G, p, \mathbf{d})$ spans $\mathbb{R}^d$. Then G is volume rigid in $\mathbb{R}^d$.