Rigidty of random graphs in high dimensions

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joint works with Niv Peleg ; Alan Lew, Eran Nevo and Orit Raz

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Erdős-Rényi Random Graphs

- A $G(n, p)$ random graph:
  - Consists of n vertices.
  - Each edge appears independently with probability $p = p(n)$.
- Study of properties that occur asymptotically almost surely.

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Theorem (Erdős-Rényi)

Let $G \sim G\left(n,\ p = \frac{\log n + \omega(n)}{n}\right)$. *Then,*

$$\mathbb{P}(G \textit{ is connected}) \to \begin{cases} 0 & \textit{if } \omega(n) \to -\infty \\ 1 & \textit{if } \omega(n) \to +\infty. \end{cases}$$

- $G(n,p)$ is connected if and only if it has no isolated vertices.

Rigidity of $G(n,p)$ for fixed d

Theorem (Lew, Nevo, P., Raz)

***Fix** $d \geq 1$, let $G \sim G\left(n,\ p = \frac{\log n + (d-1)\log\log n + \omega(n)}{n}\right)$. Then,*

$$\mathbb{P}(G \textit{ is } d\textit{-rigid}) \to \begin{cases} 0 & \textit{if } \omega(n) \to -\infty \\ 1 & \textit{if } \omega(n) \to +\infty. \end{cases}$$

- $G(n,p)$ is d-rigid iff its minimum degree is at least d.
- $d = 1$: Erdős-Rényi,
  $d = 2$: Jackson-Servatius-Servatius,
  $d > 2$: Király-Theran, Jordán-Tanigawa.

Rigidity of $G(n, p)$ for large d

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Minimum-degree bottleneck: $d \leq \delta(G)$.
Edge-number bottleneck: $p\binom{n}{2} \geq dn - \binom{d+1}{2}$, equivalently

$$d \leq (1+o(1))(1-\sqrt{1-p})n.$$

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Conjecture (Krivelevich, Lew, Michaeli)

If $p \gg \log n / n$ then G is a.a.s. $(1 - o(1))(1 - \sqrt{1 - p})n$-rigid.

- **Theorem [KLM:]** G is a.a.s. $c \cdot np/\log(np)$-rigid (using strong rigidity partitions.)

Two rigidity regimes

Suppose $p = c \log n / n$, where $c > 0$ is fixed.

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  where $a(c)$ is the smallest non-negative root of $1 - c + a - a\log(a/c) = 0$.

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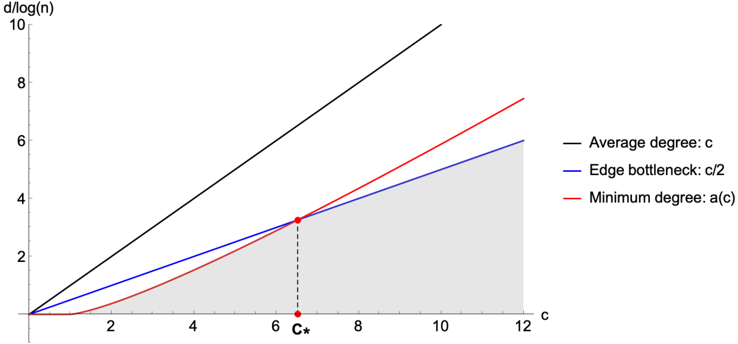
- The critical point is $C_* = 2/(1 - \log 2) = 6.518...$



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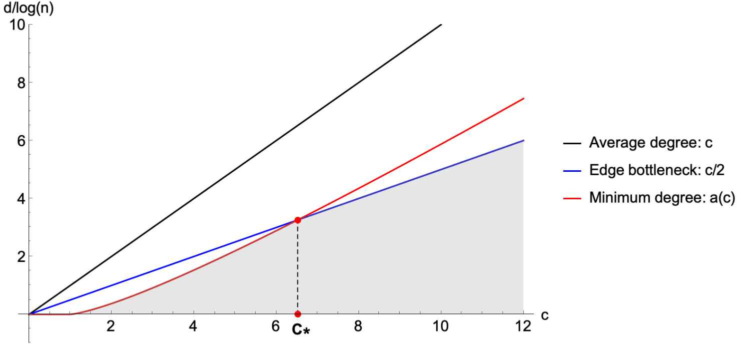


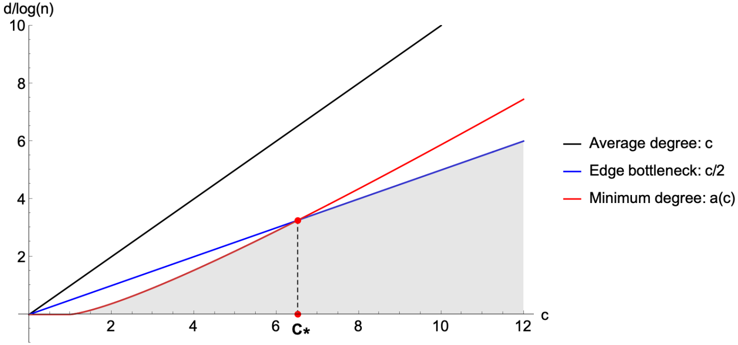
Remark: min-degree and edge-number bottlenceks also appear in [KLM]'s minimum degree conditions for rigidity.

Main results

Theorem (P., Peleg)

Let $G \sim G(n,p)$. *Then, for every* $\varepsilon > 0$,

1. *If* $p < (1-\varepsilon)C_* \log n/n$, *then* G *is a.a.s.* $\delta(G)$*-rigid.*
2. *If* $C_* \log n/n \le p \ll n^{-1/2}$, *then* G *is a.a.s.* $(1-\varepsilon)\frac{np}{2}$*-rigid.*



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- Proves KLM's conjecture in the regime $p \ll n^{-1/2}$.
- Both items are sharp: $1-\varepsilon$ cannot be replaced by $1+\varepsilon$ due to the Edge-number bottleneck.

Proof ideas: (I) large closure

Suppose $\mathcal{M}$ is an m-element matroid of rank r. Let M_p be a random subset of $\mathcal{M}$ where each element appears independently with probability p.

Lemma

If $r \leq (1-\varepsilon)pm$ *then* $\mathbb{P}\left(|cl(M_p)| \geq \frac{\varepsilon}{2}m\right) \geq 1 - e^{-K\cdot\varepsilon^2\cdot pm}$

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- $\mathcal{M}_d$ - d-rigidity matroid
- $C_d(G)$ - closure of G in $\mathcal{M}_d$.

Corollary

Let $G \sim G(n,p)$*. If* $d \leq (1-\varepsilon)\frac{np}{2}$ *then a.a.s.* $|C_d(G)| \geq \frac{\varepsilon}{4}n^2$,

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Let $G \sim G(n,p)$. *If* $d \leq (1-\varepsilon)\frac{np}{2}$ *then a.a.s.* $|C_d(G)| \geq \frac{\varepsilon}{4}n^2$, *hence,* $\exists$ *induced* $C \subseteq C_d(G)$ *with* $\delta(C) \geq \frac{\varepsilon}{4}n$.

Proof ideas: (II) large clique in closure

- $\exists$ induced $C \subseteq C_d(G)$ with $\delta(C) \geq \frac{\varepsilon}{4}n \geq d(d+1)$.

Lemma (Villányi)

Suppose that a graph C is closed in $\mathcal{M}_d$ and $\delta(C) \geq d(d+1)$, then there exists a vertex $v \in C$ whose neighbors induce a clique.

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- Challenge: rule out the possibility that $C_d(G) = G \cup K_A$ for some $|A| = \varepsilon n/4$.
- Idea: apply the "matroid closure lemma" for the matroid contraction $\mathcal{M}_d/\binom{A}{2}$, (Rigidity-wise: "pin" the vertices of A.).
- ... $\implies C_d(G)$ contains a clique with $0.99n$ vertices.

Open Problems

- KLM's conjecture remains open for $p \gg n^{-1/2}$.
  An interesting special case:

  $$G(n,1/2) \text{ is a.a.s. } (1-\sqrt{1/2}-\varepsilon)n\text{-rigid.}$$

  We do not even know that $G(n,0.999)$ is $(0.001 \cdot n)$-rigid.
- Computer simulations suggest that if $p > C_* \log n/n$ then $G(n,p)$ is a.a.s d_E-rigid where

  $$d_E := \max\left\{d \ : \ |E(G(n,p)| \geq dn - \binom{d+1}{2}\right\},$$

  while our theorem only gives $(1-\varepsilon)d_E$-rigidity.