Multitriangulations and rigidity

Francisco Santos

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria, Spain
http://personales.unican.es/santosf

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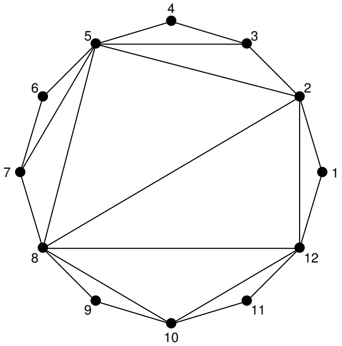
Multitriangulations

Triangulations

Let **p** be n points in convex position in the plane, labeled $\{1, \ldots, n\}$ in cyclical order.
A triangulation of the n-gon is a maximal straightline graph on **p** with no crossings.



Triangulations

Many nice properties:

- All triangulations have the same number of edges ($2n - 3$) and triangles ($n - 2$).
- They are counted by Catalan numbers.
- They can all be constructed iteratively adding “ears” to a triangle.
- They can be connected by flips, forming (the graph of) a polytope (the associahedron).

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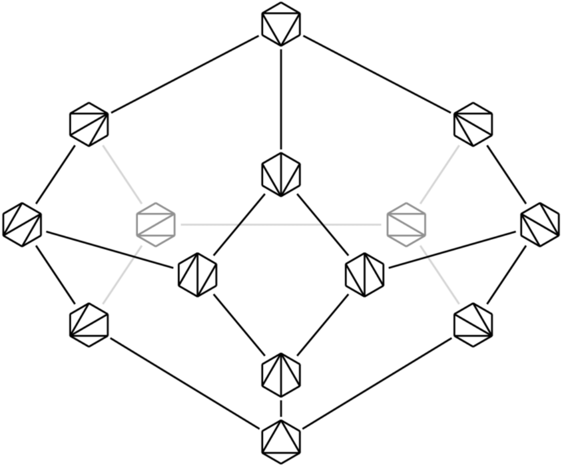
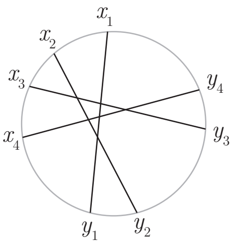


The 3-associahedron

k-crossings

Definition

A *k-crossing* is a set of k edges in $\binom{[n]}{2}$ that mutually cross.



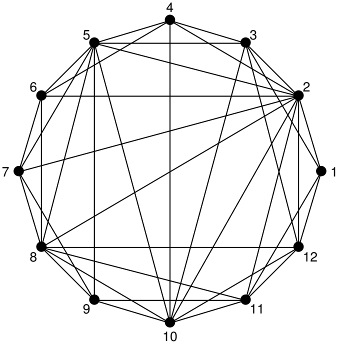
A 4-CROSSING

Remark: The definition is purely combinatorial. A k-crossing is a set $\{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \subset \binom{[n]}{2}$ of k edges with

$$i_1 < i_2 < \cdots i_k < j_1 < \cdots < j_k < i_1 \text{ (cyclically)}.$$

k-triangulations

A *k-triangulation* is a maximal graph on $\mathbf{p}$ with no $(k+1)$-crossings.

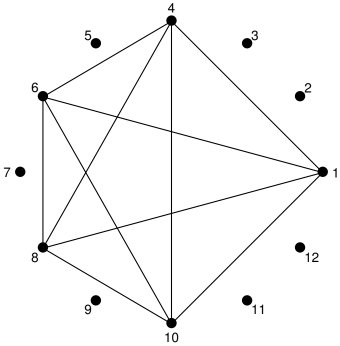


A 2-TRIANGULATION OF THE 12-GON

k-triangulations

Two easy constructions

- Start with a complete graph on $2k+1$ of the points and iteratively add "k-ears"



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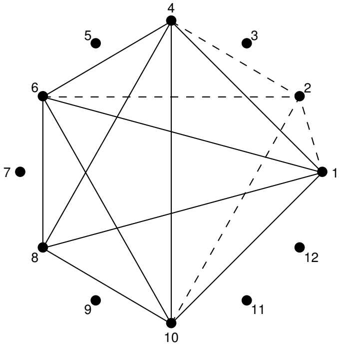

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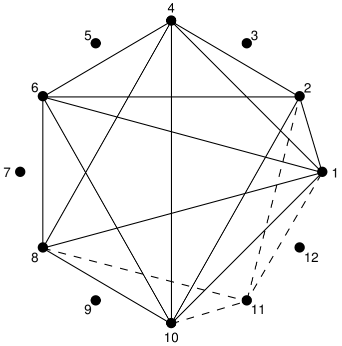

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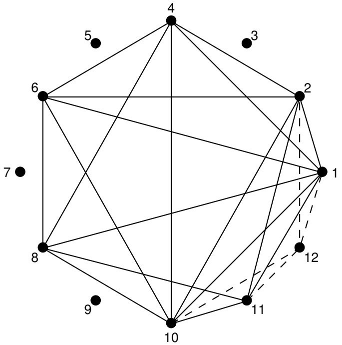

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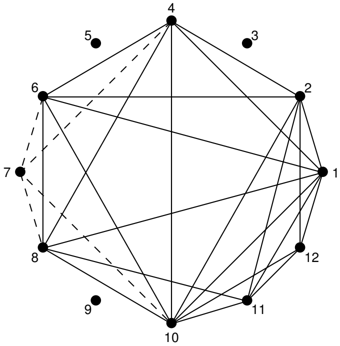

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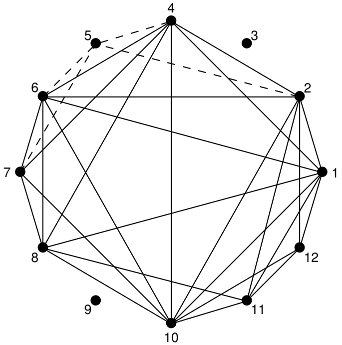

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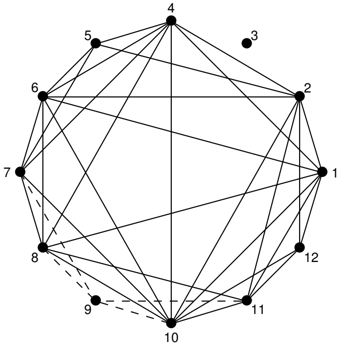

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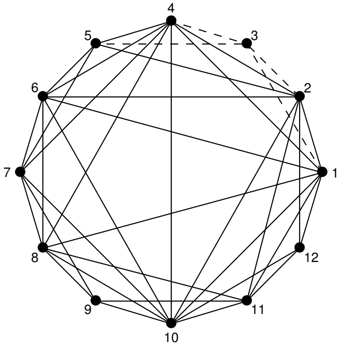

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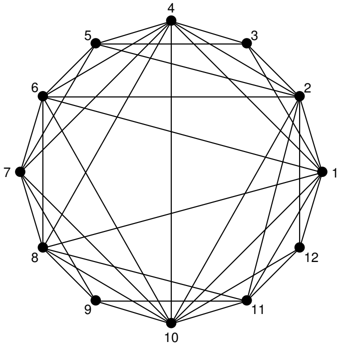

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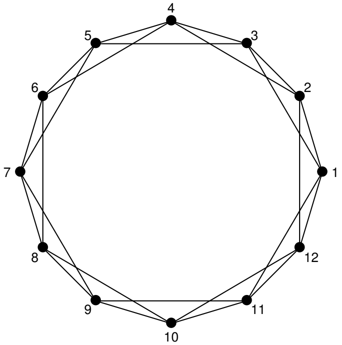
Two easy constructions

- Start with the edges of length at most k and add all edges from nodes $1, \ldots, k$.



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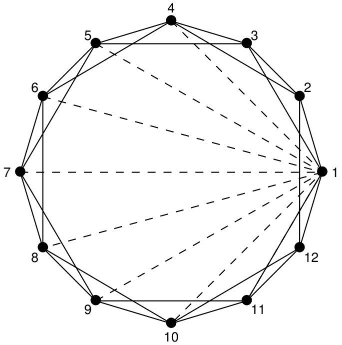

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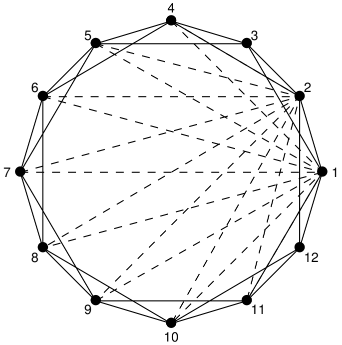

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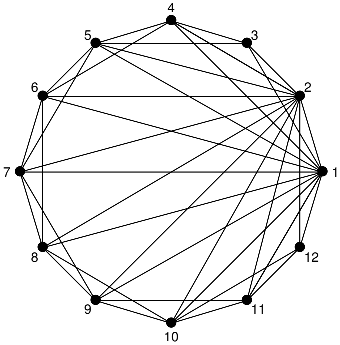

k-triangulations

Theorem (Capoyleas-Pach 1992, Nakamigawa 2000, Dress-Moulton-Koolen 2002)

All k-triangulations of the n-gon have the same number of edges, equal to $2kn - \binom{2k+1}{2}$.
Moreover, they are connected by "flips" (operations that remove an edge and insert another).

k-associahedron

Is there a "polytope of k-triangulations of the n-gon?

The associahedron as a simplicial complex

Asso(n) = the simplicial complex with vertices the $\binom{n}{2}$ diagonals of the n-gon and having as faces the the crossing-free sets of diagonals.
= clique complex of the crossing relation among the $\binom{n}{2}$ diagonals.

Vertices $= \binom{[n]}{2} = \{\{i, j\} : 1 \leq i < j \leq n\}$

Maximal faces ("facets") = triangulations of the n-gon.

Minimal non-faces = crossings.

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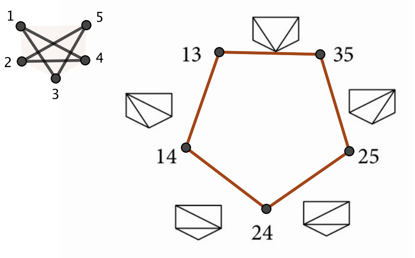
The associahedron as a simplicial complex

Example: Asso(5).



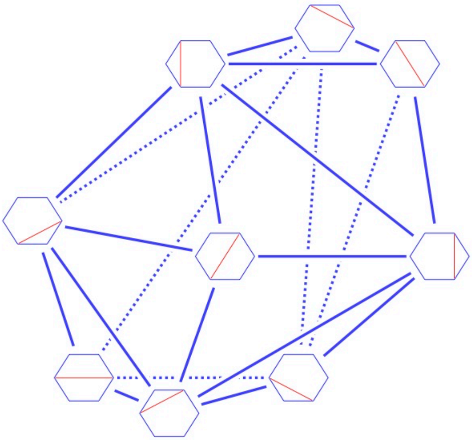
Remark: the "irrelevant edges" $\{i, i+1\}$ are not shown in the complex. Formally, we distinguish between Asso(n), with $\binom{n}{2}$ vertices and dimension $2n-4$, and $\overline{Asso}(n)$, with $\binom{n}{2} - n$ vertices and dimension $n-4$.

Theorem (Tamari-Stasheff-Milnor-Haiman, Lee 1989)

$\overline{Asso}(n)$ is a polytopal $(n-4)$-sphere. That is, there is a simplicial $(n-3)$-polytope with face poset isomorphic to it.

The 3-dimensional (simplicial) associahedron

$n = 6$: $\overline{Asso}(6)$ is a 2-sphere with 9-vertices, 21 edges, and 14 triangles.



The k-associahedron

DEFINITION: $\text{Asso}_k(n)$ = the simplicial complex with vertices the $\binom{n}{2}$ diagonals of the n-gon and whose faces are the sets of diagonals containing no $(k+1)$-crossing.

$\overline{Asso}_k(n)$ =the subcomplex induced by the relevant edges (edges of length greater than k).

Maximal faces = k-triangulations of the n-gon.
Minimal non-faces = $(k+1)$-crossings.

Theorem (Jonsson 2003)

$\overline{Asso}_k(n)$ is a shellable sphere of dimension $k(n-2k-1)-1$

The main conjecture

Conjecture 1 (Folklore?, Jonsson?)

The shellable sphere $\overline{\text{Asso}}_k(n)$ is polytopal.

That is, there is a simplicial polytope of dimension $k(n-2k-1)$ with face poset isomorphic to the inclusion poset of subsets of diagonals of the n-gon not containing a $k+1$-crossing.

- True for $n \leq 2k+3$ (see below).
- True for $(k,n) = (2,8)$ (Bokowski and Pilaud, 2009)
- True for $(2,9), (2,10), (3,10)$ (Crespo-S. 2024+).

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A weaker conjecture

Conjecture 1'

The shellable sphere $\overline{\text{Asso}}_k(n)$ is geodesic (a.k.a. star-convex).

That is, there is a **complete simplicial fan** of dimension $k(n-2k-1)$ with face poset isomorphic to the inclusion poset of subsets of diagonals of the n-gon not containing a $k+1$-crossing.

The weaker conjecture holds for

- $n \leq 2k+4$ (Bergeron-Ceballos-Labbé, 2015)
- $k=2$ and $n \leq 13$ (Manneville 2017).
- $(3,11)$ and $(4,13)$ (Crespo-S. 2024+).

This includes every (k,n) with $n \leq 13$ except $(3,12)$ and $(3,13)$

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Remarks & examples

$n = 2k + 1$

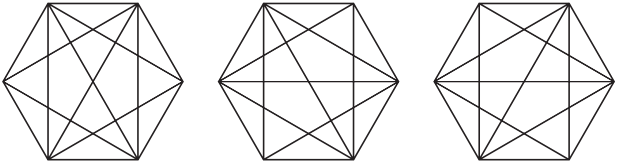
The complete graph K_{2k+1} is the unique k-triangulation of the $(2k+1)$-gon.

$Asso_k(2k+1)$ is a point.

$n = 2k + 2$

k-triangulations of the $(2k+2)$-gon are obtained by removing any long diagonal from the complete graph K_{2k+2}.

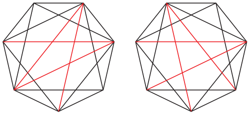
$Asso_k(2k+2)$ is the boundary of a k-simplex.



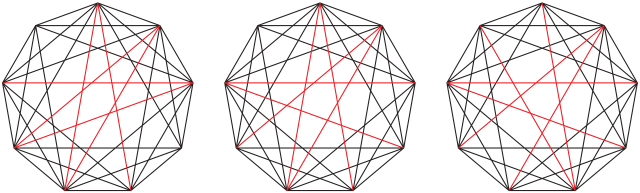
Remarks & examples

$n = 2k + 3$

There are fourteen 2-triangulations of the heptagon:



There are thirty 3-triangulations of the nonagon:



In general, $\overline{Asso_k(2k+3)}$ is (combinatorially) the boundary of a cyclic $2k$-polytope with $2k + 3$ vertices.

Relation to subword complexes

Let Q be a word of length N in the Coxeter group A_n and assume that Q contains a reduced expresson for the longest element w. The subword complex $sub_w(Q)$ is the simplicial complex on N whose facets are the complements of reduced expressions for w contained in Q. Then:

- $\text{sub}_w(Q)$ is a shellable sphere of dimension $\text{length}(Q)\text{-length}(w) - 1$ (Knutson and Miller, 2004)
- There is a certain word for which $\text{sub}_w(Q) \cong \overline{\text{Asso}}_k(n)$ (Stump 2011, Pilaud-Pocchiola 2010)
- Every $\text{sub}_w(Q)$ is a link in some $\overline{\text{Asso}}_k(n)$ (Pilaud-S. 2011)

Hence, Conjecture 1 is equivalent to

Conjecture 1"

The shellable sphere $\text{sub}_w(Q)$ is polytopal, for every word containing w.

Rigidity

Bar-and-joint (infinitesimal) rigidity

Let $\mathbf{p} = \{p_1, \ldots, p_n\} \in \mathbb{R}^d$ be points and let $G = ([n], E)$ be a graph. We call the pair $(G, \mathbf{p})$ a framework.

The framework is (infinitesimally) flexible if there is a non-trivial assignment of velocities $v_1, \ldots, v_n \in \mathbb{R}^d$ to the points that maintains all distances in the graph. That is,

$$\langle v_i - v_j, p_i - p_j \rangle = 0 \quad \text{for every } \{i, j\} \in E.$$

If this does not happen, we say $(G, \mathbf{p})$ is (infinitesimally) rigid.

Theorem (Maxwell?)

Suppose that $\mathbf{p}$ affinely spans $\mathbb{R}^d$. Then rigid frameworks on $\mathbf{p}$ are the spanning sets of rows of a matrix of size $\binom{n}{2} \times nd$ and rank $nd - \binom{d+1}{2}$.

The rigidity matrix

$$R(\mathbf{p}) := \begin{pmatrix} p_1 - p_2 & p_2 - p_1 & 0 & \dots & 0 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_1 - p_n & 0 & 0 & \dots & 0 & p_n - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} - p_n & p_n - p_{n-1} \end{pmatrix}.$$

This in particular defines the rigidity matroid $\mathcal{R}(\mathbf{p})$ of $\mathbf{p}$, with $\binom{n}{2}$ elements and rank $nd - \binom{d+1}{2}$.

A numerical coincidence

If we let $d = 2k$ then the rank of the rigidity matrix equals

$$2nk - \binom{2k+1}{2} = \text{size of every } k\text{-triangulation.}$$

This led us to conjecture

Conjecture 2 (Pilaud-S. 2009)

k-triangulations are bases in the rigidity matroid for some (hence for any generic) choice of points $\mathbf{p} \subset \mathbb{R}^{2k}$.

Relation btw. Conjectures 1 and 2

If Conjecture 2 holds then the rows of $R(\mathbf{p})$ (for a valid $\mathbf{p}$) provide a vector configuration in which every k-triangulation is a linear basis.

This configuration might be the set of normal vectors of a simplicial fan realizing $\overline{\text{Asso}_k(n)}$ ($\Rightarrow$ Conjecture 1').

Hopefully, the fan is polytopal ($\Rightarrow$ Conjecture 1).

Status of Conjecture 2

- It holds for $k = 2$ (Pilaud-S., 2009).
- In all cases where Conjecture 1 is known, Conjecture 2 is known too.
- For every $k \geq 3$ and $n \geq 2k + 3$ there is a $\mathbf{p}$ along the moment curve that is not valid: it makes some k-triangulation dependent. (Crespo-S. 2024+).

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Two alternative forms of rigidity

As before, let $\mathbf{p} = \{p_1, \dots, p_n\} \in \mathbb{R}^d$ be points, and consider the following modified rigidity matrix:

$$H(\mathbf{p}) := \begin{pmatrix} p_2 & -p_1 & 0 & \dots & 0 & 0 \\ p_3 & 0 & -p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_n & 0 & 0 & \dots & 0 & -p_1 \\ 0 & p_3 & -p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_n & -p_{n-1} \end{pmatrix}.$$

Kalai's hyperconnectivity matrix / matroid

Two alternative forms of rigidity

Let now $\mathbf{q} = \{(x_1, y_1), \dots, (x_n, y_n)\} \in \mathbb{R}^2$ be points, choose a “degree” $d \in \mathbb{N}$, and consider the following modified rigidity matrix:

$$C_d(\mathbf{q}) := \begin{pmatrix} c_{1,2} & -c_{2,1} & 0 & \dots & 0 & 0 \\ c_{1,3} & 0 & -c_{3,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n,2} & 0 & 0 & \dots & 0 & -c_{n,1} \\ 0 & c_{2,3} & -c_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{n-1,n} & -c_{n-1,n} \end{pmatrix},$$

with $c_{ij} := \left(x_{ij}^{d-1}, y_{ij}x_{ij}^{d-2}, \dots, y_{ij}^{d-1}\right), \quad x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j.$

Whiteley’s cofactor matrix / matroid

Two alternative forms of rigidity

Theorem (Kalai 1985, Whitely 1990)

For $\mathbf{p}$ *or* $\mathbf{q}$ *in general position, the (row vectors of) matrices* $H(\mathbf{p})$ *and* $C_d(\mathbf{q})$ *share the following properties with* $R(\mathbf{p})$*:*

1. *Their rank equals* $nd - \binom{d+1}{2}$*.*
2. *Every* K_{d+2} *is a circuit.*

Matroids in $\binom{[n]}{2}$ with these properties are precisely the abstract rigidity matroids of Graver 1991 (as proved by Nguyen 2010).

Relation between the three

For each d, n, in each of the three theories there is a most free matroid that is obtained for generic points.

We denote them $\mathcal{R}_d(n)$, $\mathcal{H}_d(n)$, $\mathcal{C}_d(n)$.

We have:

- $d = 1, 2$: $\mathcal{H}_d(n) = \mathcal{R}_d(n) = \mathcal{C}_d(n)$.
- $d = 3$: $\mathcal{C}_3(n)$ is the most generic rigidity matroid (Clinch-Jackson-Tanigawa 2022).
  Conjecture: $\mathcal{C}_3(n) = \mathcal{R}_3(n)$ (Whiteley).
- $d \geq 4$: Known that $\mathcal{H}_d(n) \not\geq \mathcal{R}_d(n) \not\geq \mathcal{C}_d(n)$.
  Conjecture: that $\mathcal{H}_d(n) \leq \mathcal{R}_d(n) \leq \mathcal{C}_d(n)$ (Kalai, Whiteley).

An important common case; the moment curve

Let $t = (t_1, \ldots, t_n)$ be real parameters, and consider the configurations $\mathbf{p}(t) \subset \mathbb{R}^d$ with $p_i = (t_1, \ldots, t_i^d)$ along the moment curve and $\mathbf{q}(t) \subset \mathbb{R}^2$ with $q_i = (t_1, t_i^2)$ along the parabola. Then

Theorem (Crespo-Santos 2023)

The matrices $R(\mathbf{p}(t))$, $H(\mathbf{q}(t))$ and $C_d(\mathbf{q}(t))$ are equivalent under multiplication on the left by a nonsingular matrix. In particular, the associated oriented matroids coincide.

We denote this common (oriented) matroid $\mathcal{P}_d(t)$, and call $\mathcal{P}_d(n)$ the generic one.

Conjecture 2' (Stronger than Conjecture 2)

k-triangulations of the n-gon are bases in $\mathcal{P}_{2k}(n)$.

Status: same as Conjecture 2 (Crespo-S. 2024+).

Two results

Theorem (Crespo-S. 2024)

k- triangulations of the n-gon are bases in $\mathcal{H}_{2k}(n)$.

Proof is via Gröbner bases of the Pfaffian ideal. Based on previous work of Pachter-Sturmfels (2005) and Jonsson-Welker (2007) showing that $\mathcal{H}_{2k}(n)$ is the algebraic matroid of Pfaffians and relating Pfaffians to $\overline{Asso}_k(n)$.

Theorem (Crespo-S. 2023)

Restricted to bipartite graphs, $\mathcal{H}_d(n) \leq \mathcal{R}_d(n)$.

A bright idea

Cofactor rigidity of degree d shares most of the properties of bar-and-joint rigidity in dimension d, yet it is about points in the plane.

Maybe this is the right tool to embed the multiassociahedron.

Conjecture 3 (S., $\simeq$ 2021)

For every choice of points $\mathbf{q} = \{q_1, \ldots, q_n\}$ in convex position, the rows of $C_{2k}(\mathbf{q})$ embed $\overline{\text{Asso}_k(n)}$ as a polytopal fan.

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For every choice of points $\mathbf{q} = \{q_1, \dots, q_n\}$ in convex position, the rows of $C_{2k}(\mathbf{q})$ embed $\overline{\text{Asso}}_k(n)$ as a polytopal fan.

Status of Conjecture 3:

- True for $k = 1$ (Rote-S.-Streinu 2003)
- FALSE for $k = 3, n \geq 9$ (Crespo-S., 2024+)

A bright idea

Cofactor rigidity of degree d shares most of the properties of bar-and-joint rigidity in dimension d, yet it is about points in the plane.

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~~Conjecture 3'~~

For some choice of points $\mathbf{q} = \{q_1, \ldots, q_n\}$ in convex position, the rows of $C_{2k}(\mathbf{q})$ embed $\overline{\text{Asso}}_k(n)$ as a polytopal fan.

Status of Conjecture 3':

- True for $k = 1$ (Rote-S.-Streinu 2003)
- FALSE for $k = 3, n \geq 12$ (Crespo-S., 2024+)

Multitriangulations and Rigidity

Polytopality via vector configurations

Our heuristics for politopality: Given a simplicial $(d-1)$-sphere Δ with vertex set $[n]$ and a vector configuration $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ we check three things (each stronger than the previous one):

1. Are all faces of Δ linearly independent in V? (compute ranks)
2. Is Δ a "triangulation of V" (a.k.a. simplicial fan)? (compute orientations)
3. Is Δ a "regular triangulation of V" (a.k.a. projective fan; a.k.a. the normal fan of a simplicial polytope)? (linear feasibility)

If successful, these three computations answer Conjectures 2, 1' and 1 in the positive, respectively.

Our experiments

We have implemented this with $\Delta = \overline{\text{Asso}}_k(n)$ and with $V =$"rows of the cofactor matrix of n points along the parabola" (equivalently, "bar-and-joint with points along the moment curve").

There are two "natural" choices of points:

- $t_i = i$, that is, $q_i = (i, i^2)$ ("equispaced along the parabola")
- Vertices of a regular n-gon, sent to the parabola via projective transformation ("equispaced along the circle")

Remark: projective transformation preserves the three forms of rigidity.

Our experiments; $k = 2$

- With $k = 2$ all positions we have tried realize the complete fan, but not always the polytope. (We have been able to compute up to $n = 13$).
- Equispaced positions along the parabola give a polytopal fan for $n \leq 9$.
- Positions $t = (2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ give a polytopal fan for $n = 10$.
- We have not found positions giving a polytopal fan for $n > 10$ (but our experiments are not conclusive).

Conjecture 3''' (S.-Crespo 2023)

For $k = 2$ and any n, all positions along the parabola / moment curve realize $\overline{Asso}_2(n)$ as a complete simplicial fan.

Our experiments; $k > 2$

- With $k = 3$ and $n \geq 9$ there are positions where some k-triangulations are not bases.
- With $k = 3$ and $n \leq 11$ (and $k = 4$ and $n \leq 13$) equispaced positions on the circle realize the fan.
- With $k = 3$ and $n \leq 10$ the positions $t = (2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ realize the polytope.
- With $k = 3$ and $n \geq 12$ (and $k > 3$ and $n \geq 2k + 6$) no positions realize the fan.

An obstruction

The last point is not an experiment, but a theorem:

Theorem (Crespo-S. 2023)

For any choice $\mathbf{q} = \{q_1, \ldots, q_{12}\} \subset \mathbb{R}^2$ *of points in convex position there is a* 3*-triangulation that does not get the right orientation as a cone in the row-vectors of cofactor rigidity* $C_3(\mathbf{q})$*.*

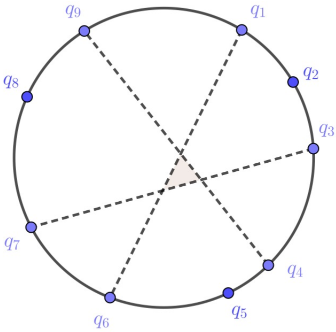
Idea of proof

- Let $T_9 := K_9 \setminus \{ 16, 37, 49 \}$. It is a 3-triangulation, and is also a triple cone over the graph of an octahedron.
- The graph of an octahedron is a circuit or a basis or in $C_3(6)$ depending on whether the three missing edges are concurrent or not ("Morgan-Scott obstruction", 1975)
- Rigidity (both cofactor and bar-and-joint) behaves well with coning. T_9 is independent in $C_6(9)$ if and only if deleting the three cone points the octahedral graph is independent in $C_3(6)$.

Idea of proof

Corollary

T_9 gets the correct orientation in $C_6(q_1, \ldots, q_9)$ if, and only if, the "inner half-planes" defined by the three missing edges $16, 37$, and 49 have non-empty intersection.



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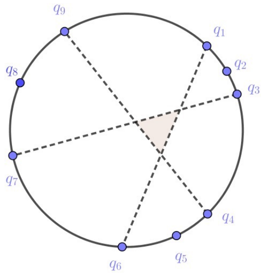
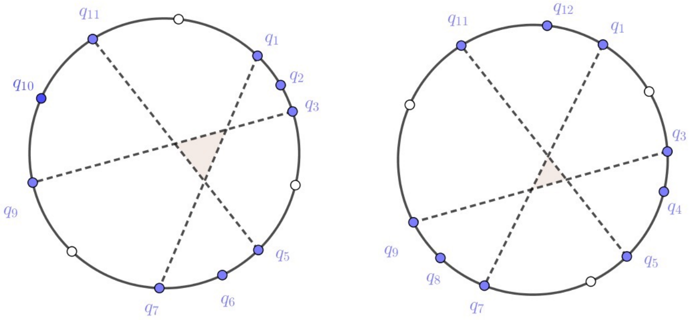


bad

Idea of proof

Corollary

For any 12 points $\mathbf{q} = \{q_1, \ldots, q_{12}\} \subset \mathbb{R}^2$ in convex position either the 3 triangulation containing T_9 on $\mathbf{q} \setminus \{q_2, q_6, q_{10}\}$ or the one on on $\mathbf{q} \setminus \{q_4, q_8, q_{12}\}$ gets the wrong orientation on $C_6(12)$.



bad bad

Summing up

- Rigidity seemed a bright idea to realize the multiassociahedron. . . but it is proven not to work.
- Maybe the polytopality conjecture is false . . . This would be the first (?) family of "naturally defined" shellable simplicial spheres that turn out not to be polytopal.
- The case $k = 2$ of the polytopality conjecture may still be true.

A computational challenge

Is $\overline{\text{Asso}}_3(12)$ polytopal? (42 vertices, 379 236 facets, dimension 14)

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The end

Thank you