Higher order rigidity and higher order derivative tests

•with Holmes-Cerfon and Theran

Rigidity

•we will study $(G, \mathbf{p})$ bar and joint frameworks in R^d.

•$\mathbf{p}$ is a configuration of n points in R^d, G is a graph with n vertices (draw 4 chain)

•rigid: i cannot move the points without changing the euclidean length of at least one edge (draw rigid prism)
• • we pin out Euclidean isometries for simplicity.

•otherwise flexible (draw two examples)

Motivation

•we want sufficiency tests for rigidity

• •   show figures.

•we also want natural notion of “how rigid” a rigid framework is.

•we will explore this using critical point derivative tests of an energy function.

(j,k) flex for $(G,\mathbf{p})$!

•Let $\mathbf{m}(\mathbf{q})$ be the map from a configuration $\mathbf{q}$ to the vector of squared edge lengths.

•• $m_{uv}(\mathbf{q}) := ||\mathbf{q}_u - \mathbf{q}_v||^2$.

•a (j,k) flex ($j \leq k$) is an analytic trajectory

$$\mathbf{p}(t) := \mathbf{p} + \mathbf{p}^{(j)}t^j + ...\mathbf{p}^{(k)}t^k + ...$$

with $\mathbf{p}^{(j)} \neq 0$, such that for $i \in [1..k]$, we have

$$\frac{d^i}{dt^i}\mathbf{m}(\mathbf{p}(t))|_0 = 0$$

•def is based off ideas explored by Sabitov, Stachel and Nawratil

•a $(1,1)$ flex is aka a non-trivial infinitesimal flex

•a $(1,2)$ flex is aka a non-trivial second order flex

•analyticity: if $(G,\mathbf{p})$ is flexible, then there exists a j so that for any k there is a (j,k) flex.

Known stuff

•thm 1: If $(G, \mathbf{p})$ has no $(1,1)$ flex , then it is rigid. (inf rig)

•thm 2: If $(G, \mathbf{p})$ has no $(1,2)$ flex, then it is rigid. [C80] (2or)

•can we continue these theorems?

•No, there exists a cusp mechanism that is not rigid, it has a $(2,\infty)$ flex, but it has no $(1,3)$ flex! [CS94]

single inf flex

•when the space of infinitesimal flex coefficients $\mathbf{p}'$ is one dimensional, things get better, due to V. Alexandrov.

•thm 3: Suppose dimInfFlex=1, if there is a k so that $(G, \mathbf{p})$ has no $(1, k)$ flex then it is rigid. [A01]

•proofs of 2 and 3 are kind of magical.

•we reprove thms 2 and 3 using an energy and critical point analysis.

energy

•Koiter 45/67, Solerno 92, Garcia et al 05. CW96

•given $(G, \mathbf{p})$, with squared edge lengths d_{ij}^2, we create a stiff bar energy

$$E(\mathbf{q}) := \sum_{ij \in G} E_{ij}(m_{ij}(\mathbf{q}))$$

•Each E_{ij} is analytic. with

$$\begin{aligned} \frac{d}{dl} E_{ij}|_{d_{ij}^2} &= 0 \\ \frac{d^2}{dl^2} E_{ij}|_{d_{ij}^2} &> 0 \end{aligned}$$

• • The edges want to be at their lengths in $\mathbf{p}$.

•$\mathbf{p}$ is a crticial point and local min of E.

•$(G, \mathbf{p})$ is rigid iff $\mathbf{p}$ is a strict local min of E.

•will study this critical point using derivative tests.

energy and growth!

•suppose that $\mathbf{p}$ is an slm of E. we can quantify the speed of growth as we leave $\mathbf{p}$. Let $s > 0$ be a rational number.

•We say that E grows *always-s-quickly* if there is some $c > 0$ and a ball B around $\mathbf{p}$, so that for all $\mathbf{q}$ in B, we have

$$E(\mathbf{q}) - E(\mathbf{p}) \;\geq\; c|\mathbf{q} - \mathbf{p}|^s$$

• • lower bound on growth

• • smaller s means faster growth

•We say that E grows *sometimes-s-slowly* if there exists an analytic trajectory at $\mathbf{p}$, $\mathbf{p}(t)$, and a $c > 0$ and an ϵ so that for $t \in [0, \epsilon]$

$$E(\mathbf{p}(t)) - E(\mathbf{p}) \;\leq\; c|\mathbf{p}(t) - \mathbf{p}|^s$$

• • upper bound on the above lower bound

...

•We say that E grows *s-tightly* if it grows sometimes-s-slowly and always-s-quickly.

•thm [B-N et al 96]: For E analytic, at an slm, there exists a tight value for s.
• • need not be integer, even if E is a polynomial.

order of rigidity

•the following def seems natural

•Let $(G, \mathbf{p})$ be a framework and let E be a stiff bar energy for $(G, \mathbf{p})$. Suppose that E grows s-tightly for some (rational) value of s. Then we say that the *rigidity order* of $(G, \mathbf{p}, E)$ is $s/2$.

• • the $1/2$ is for convenience

M and E derivs!

•ME lemma (generalizing Solerno): Let E be any stiff bar-energy for $(G, \mathbf{p})$. A trajectory $\mathbf{p}(t)$ satisfies for $i \in [1..2k+1]$

$$\frac{d^i}{dt^i} E(\mathbf{p}(t))|_0 = 0$$

iff for $i \in [1..k]$

$$\frac{d^i}{dt^i} \mathbf{m}(\mathbf{p}(t))|_0 = 0$$

•proof: taylor series and chain rule

•this connects flex-based and energy-based analysis

•and shows that all stiff bar energies are equivalent

flexes and E growth

•easy Lemma: Let E be any stiff bar-energy for $(G, \mathbf{p})$ Suppose that there EXISTS a (j, k) flex for $(G, \mathbf{p})$. Then E grows sometimes-s-slowly where $s = \frac{2k+2}{j}$.

•• a flex provides some slowly energy growing trajectory!

•easy thm: the order of rigidity equals the maximal value of $\frac{k+1}{j}$ over all (j, k) flexes.

previous definitions

•in agreement with notions from Garcia and also Tachi

•definition is quite different than Nawratil.

revist the theorems

•NONEXISTENCE of flexes of specific orders does not typically give us info about always-growth.

•but using derivative tests (discussed next) in some cases can.

•thm 1+: If $(G, \mathbf{p})$ has no $(1, 1)$ flex, then its order of rigidity is 1.

•thm 2+: Else, if $(G, \mathbf{p})$ has no $(1, 2)$ flex, then its order of rigidity is 2.

•thm 3+: Suppose dimInfFlex=1. If there exits some k such that $(G, \mathbf{p})$ has a $(1, k-1)$ flex but no $(1, k)$ flex, then its order of rigidity is k.

Part II, derivative tests

•let $f(x, y)$ be any sufficiently smooth bivariate function, with a critical point at the origin, and with $f(0, 0) = 0$.

•wish to determine slm, wlm sdl, wlM, slM, using taylor expansion of f

2dt I

•second derivative test (for slm).

•write multivariate second order approximation (at the origin) $f = f_2 + hot$.

• • f_2 must be homogeneous, as we are at a critical point.

•if f_2 has a slm at the origin (Hessian is PD), then f_2 grows always 2 quickly.

•in a small enough ball, the h.o.t. are dominated. certifies slm and certifies always 2-quick growth

•But if f_2 has a wlm at the origin (Hessian is only PSD), then the h.o.t. can have an influcence, so the test is indeterminate.

naive 4dt

•what to do next is surprisingly subtle.

•use fourth order approximation $f = f_4 + hot$.
•• In multivariate setting, f_4 need not be homogeneous

•suppose f_4 has a slm at the origin

•this does not mean that f_4 grows always 4 quickly.

•so h.o.t. can still dominate and f_4 will give the wrong answer!

example

•Let $f(x,y) = (x - y^2)^2 + x^2y^2 - y^6$.

•we have $f_2(x,y) = x^2$. (wlm, zero on y axis)

•take $f_4(x,y) = (x - y^2)^2 + x^2y^2$.

•this has slm: first term is postive except on the parabola $x = y^2$. Second term is positive except on axes.

•but f_4 grows sometimes 6-slowly.
•• let $x(t) = t^2$ and $y(t) = t$.
•• then $g(t) := f(x(t), y(t)) = t^6$. (and radius grows with first order in t).

•so h.o.t. can be relevant

•in fact f has a saddle at the origin!

•Q: can you use 4th or higher partial derivatives of $f(x,y)$ or t-derivatives of $f(x(t), y(t))$ for some set of trajectories $(x(t), y(t))$, to classify a critical point?

old thm new thm and lost thm

•thm [ancient]: there exists an efficient general multivariate 2nd derivative test

• • it can certify always 2-growth

•thm [GHT]: there exists a general multivariate 4th derivative test

• • it can certify always 4-growth

• • no efficiency claim

•thm [Cushing '75] when the Hessian has nullity one, then there exists an efficient $2k$th derivatvie test for any k.

• • it can certify always 2k-growth

• • for f analytic, the sequence of tests will eventually halt for slm, sdl, and slM.

•feel free to talk about the details later

back to rigidity

•[thm folklore] If there is no $(1, 1)$ flex then the 2dt will certify an slm of E.

•[thm GHT] If there is no $(1, 2)$ flex then our 4dt will certify an slm of E.

•[thm GHT] When $dimInfFlex = 1$, if there is no $(1, k)$ flex, then Cushing's 2kdt will certifying an slm of E.

•QED 123.

complexity

•in general determining rigid/flexible is NP-HARD

•in general ruling out a $(1,1)$ flex can be done using linear algebra

•in general ruling out a $(1,2)$ flex has no known efficient algorithm

•But when dimInfFlex=1, then for any k, ruling out a $(1,k)$ flex can be done by solving linear systems!
•• only one choice for $\mathbf{p}'$. Fix it and then search for $\mathbf{p}''$ using a lin sys. (Essentially) only once choice for $\mathbf{p}''$. ...

examples

prestress

•all stiff bar energies are interchangable.

•if one allows for energies that are not bar-like then things get more complicated

•but one can try to analyze the order of growth of $(G, \mathbf{p}, E)$.

•this leads to the notion of (first order) prestress rigidity of $(G, \mathbf{p})$, as well as a notion of higher order prestress rigidity.