

Reconstructing simplicial polytopes from affine stresses

Isabella Novik

joint work with

Satoshi Murai (Waseda University) and

Hailun Zheng (University of Hawai'i)

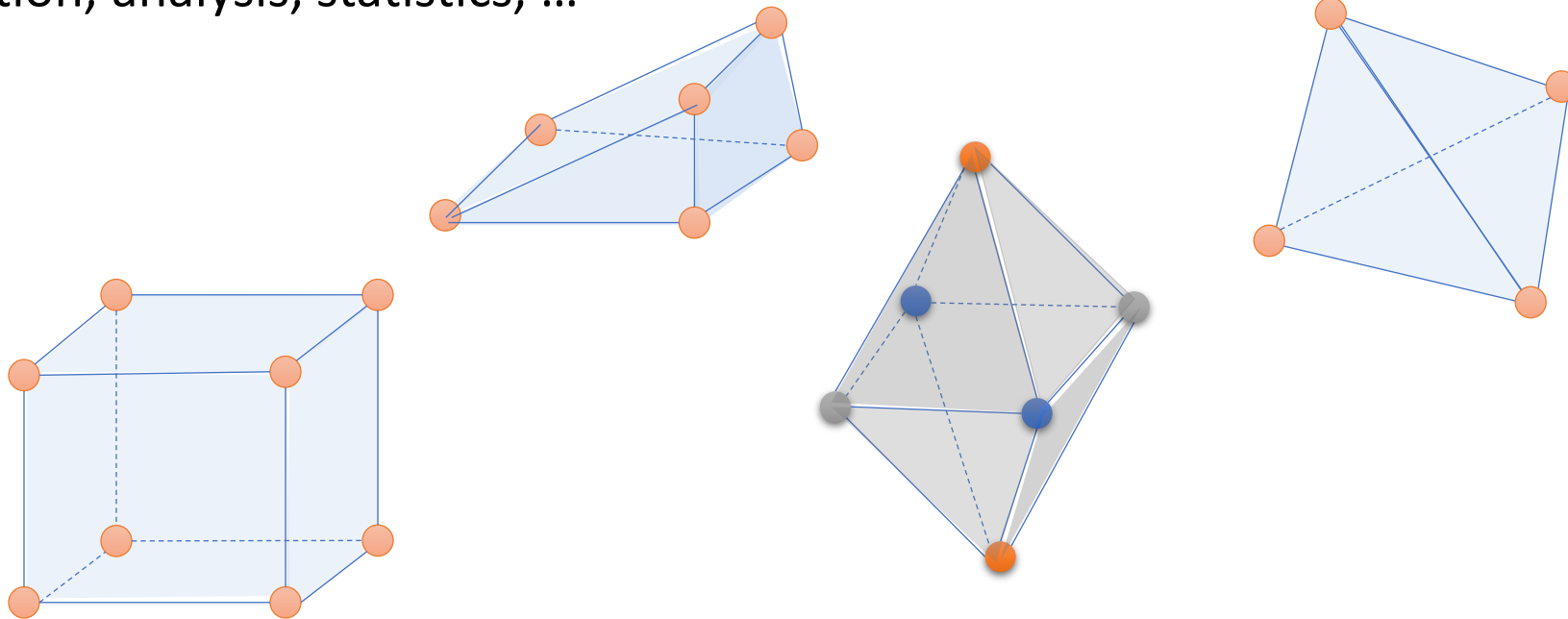
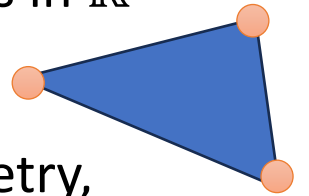
Basics on Polytopes

- A **polytope** is the convex hull of *finitely many* points in \mathbb{R}^d

Equivalently, it is the bounded intersection of *finitely many* closed half-spaces in \mathbb{R}^d

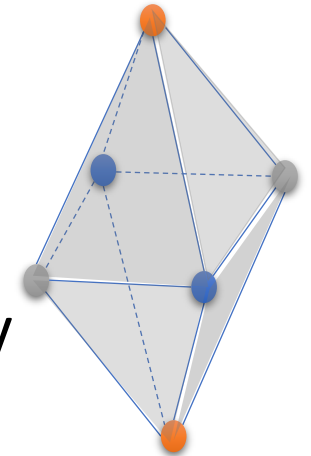
- **Example:** a **simplex** is the convex hull of affinely independent points

Polytopes are studied and have applications in combinatorics, discrete geometry, optimization, analysis, statistics, ...



Faces and face lattices

- The **dimension** of a polytope P is the dimension of its affine hull
- A (proper) **face** of P is the intersection of P with a supporting hyperplane
- A face F of P is itself a polytope; it is an **i -face** if $\dim F = i$
- Any polytope P has finitely many faces; the set of faces of P (together with the empty face and P itself) ordered by inclusion forms a graded lattice --- **the face lattice of P**
- A polytope P is **simplicial** if all proper faces of P are simplices
- We usually label the vertices of a simplicial polytope $P \subset \mathbb{R}^d$ by $1, 2, \dots, n$, and their position vectors by $p(1), p(2), \dots, p(n)$

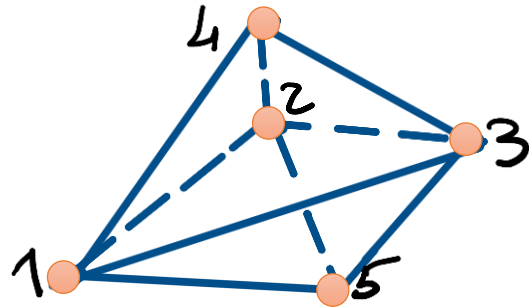


The boundary complex of P

- Recall: a face F is a k -face if $\dim F = k$; 0-faces are **vertices**, 1-faces are **edges**
- If P is a *simplicial d -polytope*, we define the *boundary complex* of P
 $\Delta = \partial P := \{\text{vertex sets of proper faces of } P\}$

This is a(n abstract) simplicial complex of dimension $d - 1$.

Example:



$$\partial P = \{124, 234, 134, 125, 235, 135, \text{ and all their subsets}\}$$

Note: 123 is not a face of ∂P but all its subsets are.
 We say that 123 is a *missing face* of P

- For $k < d$, the k -th *skeleton* of ∂P is $\text{Skel}_k(\partial P) := \{F \in \partial P : |F| \leq k + 1\}$
 For instance, $\text{Skel}_1(\partial P)$ is the graph of P , and will be also denoted by $G(P)$

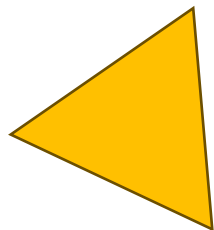
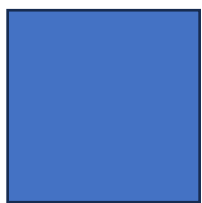
Combinatorial and affine types of polytopes

- Two simplicial polytopes P and Q are *combinatorially equivalent* if their boundary complexes are isomorphic

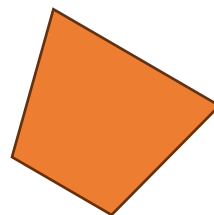
Two (general) polytopes are combinatorially equivalent if their face lattices are isomorphic

- Two d -polytopes P and Q in \mathbb{R}^d are *affinely equivalent* if there is an invertible affine transformation $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T(P) = Q$

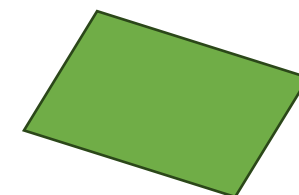
- Examples:



Not combinatorially equivalent



Combinatorially equivalent
but not affinely equivalent



Affinely equivalent (and hence
also combinatorially equivalent)

Rough outline of the problem

What partial information about a d -polytope P determines

1. the combinatorial type of P ?
 2. the affine type of P ?
- **Whitney, '32**: combinatorial types of 3-polytopes are determined by their graphs
 - **Grünbaum, '67**: to determine the combinatorial type of a general d -polytope P one needs to know the $(d - 2)$ -skeleton of P
 - **Blind and Mani, '87**: combinatorial types of **simple** d -polytopes are determined by their graphs

Rough outline of the problem continued

What partial information about a **simplicial** d -polytope P determines

1. the combinatorial type of P ?
2. the affine type of P ?

For $d \geq 4$, neither the graph nor even the $\left(\left\lfloor \frac{d}{2} \right\rfloor - 1\right)$ -skeleton provides enough info:

Theorem (Shemer, 1982, Padrol 2013): For $n \gg 0$ and $d \geq 4$, there exist many (on the order of $2^{\Omega(n \log n)}$) simplicial d -polytopes with n vertices that have **complete** $\left(\left\lfloor \frac{d}{2} \right\rfloor - 1\right)$ -skeleta but distinct combinatorial types

Such polytopes are called $\left\lfloor \frac{d}{2} \right\rfloor$ -neighborly

Two results of Perles

Theorem 1 [Perles; Dancis, 1980s]

The boundary complex of a simplicial d -polytope P is determined by the $\lfloor \frac{d}{2} \rfloor$ -skeleton of ∂P . In other words, $\text{Skel}_{\lfloor \frac{d}{2} \rfloor}(\partial P)$ determines the combinatorial type of P

Proof: suffices to determine missing faces; this can be done using Alexander duality for spheres

Theorem 2 [part of the theory of Gale diagrams developed by Perles in 1960s]

The space of affine dependencies of position vectors of vertices of P determines the affine type of P , and hence also the combinatorial type of P .

This is the null space of $\begin{bmatrix} p(1) & | & p(2) & | & \cdots & | & p(n) \\ 1 & | & 1 & | & \cdots & | & 1 \end{bmatrix}$ (n is the number of vertices)

Kalai's conjecture interpolating between these two results

Conjecture (Kalai, 1994)

Let P be a simplicial d -polytope and let $1 \leq k \leq \lfloor \frac{d}{2} \rfloor + 1$. Then the

$(k - 1)$ -skeleton of ∂P and the space of affine k -stresses of P determine the combinatorial type of P

- $k = 1$: affine 1-stresses are affine dependencies of vertices, and the 0-skeleton only tells us the number of vertices. Thus, the $k = 1$ case of this conjecture is Theorem 2
- $k = \lfloor \frac{d}{2} \rfloor + 1$: Stanley's g -theorem (1980) implies that the space of affine $\lfloor \frac{d}{2} \rfloor$ -stresses of P is (0) . Thus, the $k = \lfloor \frac{d}{2} \rfloor + 1$ case of this conjecture is Theorem 1

What are affine stresses? Geometric definition

Let P be a simplicial d -polytope with edge set E and position vectors of vertices $p(1), \dots, p(n) \in \mathbb{R}^d$

Affine stresses are higher-dimensional analogs of affine dependencies

Definition: an affine k -stress on P is an assignment of weights $\omega: (k-1)\text{-faces of } \partial P \rightarrow \mathbb{R}$ that satisfies a certain balancing condition at every $(k-2)$ -face. The set of affine k -stresses is a vector space denoted by $S_k(P)$

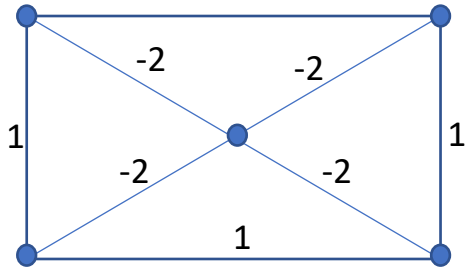
- When $k = 1$, ω assigns weights to *vertices* and the balancing condition at the empty face is $\sum_{i=1}^n \omega_i \binom{p(i)}{1} = 0$, so ω is an affine dependence of vertices
- When $k = 2$, ω assigns weights to *edges* so that for every vertex i ,

$$\sum_{j: ij \in E} \omega_{ij} (p(j) - p(i)) = 0$$

Affine stresses continued

Affine 2-stresses can be defined for any graph $G = (V, E)$ with a given embedding $p: V \rightarrow \mathbb{R}^d$

Example:



In general, an affine k -stress on P is a vector $\omega \in \mathbb{R}^{f_{k-1}(P)}$ whose coordinates are labeled by $(k - 1)$ -faces of ∂P .

Here $f_{k-1}(P)$ is the number of $(k - 1)$ -faces of P

Stresses and face numbers

- Dehn's lemma (1916): if P is a simplicial 3-polytope, then $\dim S_2(P) = 0$, that is, P has no non-trivial affine 2-stresses
- Theorem (Stanley, 1980; Whiteley, 1984; Lee, 1996)

Let P be a simplicial d -polytope. Then for $k \leq \lfloor d/2 \rfloor$, $\dim S_k(P) = g_k(P)$ --- a certain function of the face numbers of P , while for $k > \lfloor d/2 \rfloor$, $S_k(P) = (0)$

Main applications of stresses to simplicial polytopes (and even triangulations of manifolds): can use them to produce *inequalities on face numbers!*

First results on Kalai's conjecture

Theorem [N-Zheng, 2023]

Let $d \geq 4$. The graph of a simplicial d -polytope P and the space of affine 2-stresses of P determine the combinatorial type of P

In fact, we can do better:

for $a \in \mathbb{R}$, let $\text{sign}(a) = +, -, \text{ or } 0$ depending on whether $a > 0, < 0, = 0$

Definition: Let P be a simplicial d -polytope. Define

$$\mathcal{V}_2(P) := \{(\text{sign } \omega_e)_{e \text{ is an edge}} : \omega \in S_2(P)\}$$

to be the *set of sign vectors of affine 2-stresses on P*

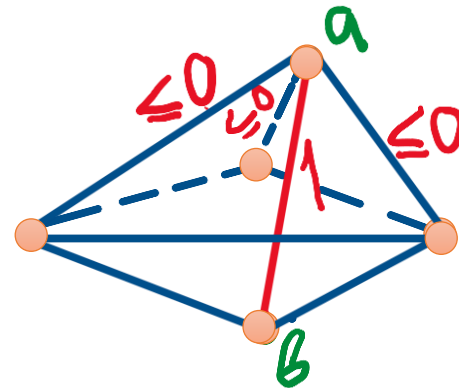
Theorem [N-Zheng, 2023]

Let $d \geq 4$ and let P be a simplicial d -polytope. The graph of P and $\mathcal{V}_2(P)$ determine the combinatorial type of P

The key proposition

Proposition [N-Zheng, 2023]

Let $d \geq 3$. Let Q be a simplicial d -polytope. If ab is a **missing edge** of Q , then there is an affine 2-stress ω supported on $G(Q) \cup ab$ such that $\omega_{ab} = 1$ and $\omega_{ac} \leq 0$ for all edges ac of Q incident with a



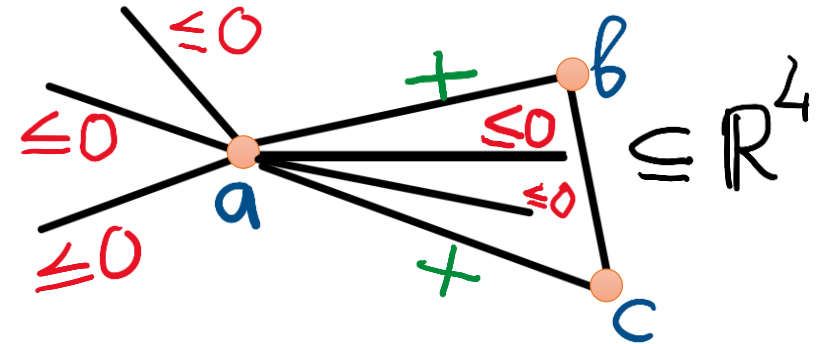
Remark: when $d = 3$, by Dehn's lemma, a stress with $\omega_{ab} = 1$ is unique. By symmetry, it satisfies $\omega_e \leq 0$ for every edge e incident with a or b

Using the key proposition

- Let P be a simplicial 4-polytope. We are given $G(P)$ and $\mathcal{V}_2(P)$ --- the graph and the set of sign vectors of affine 2-stresses
- To reconstruct the 2-skeleton, for every 3-clique abc , need to decide if it is a face or a missing face. The Key Proposition applied to vertex figures together with the Cone Lemma implies

Theorem:

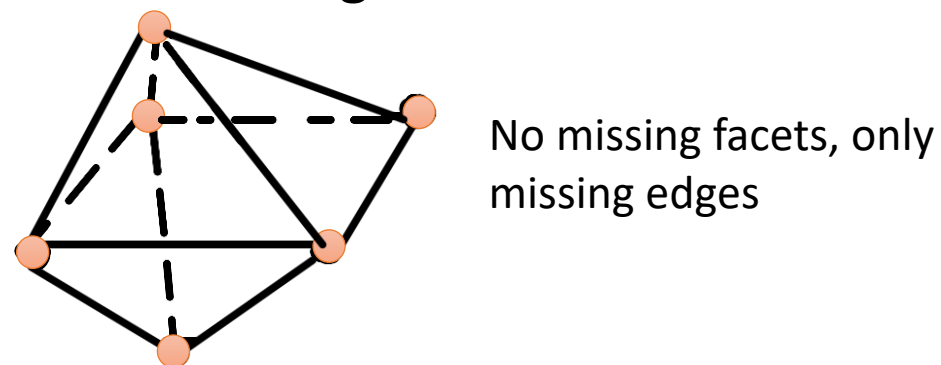
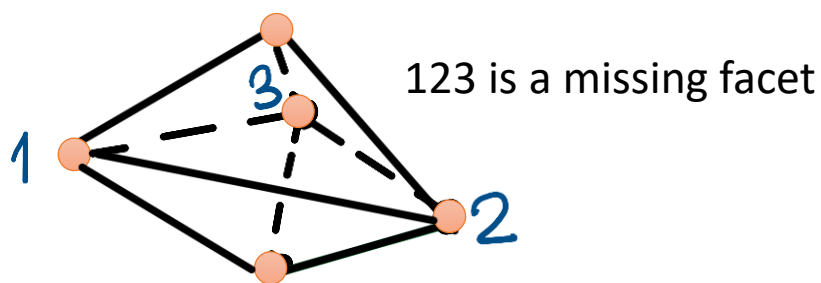
abc is a missing face of P if and only if there is an affine 2-stress λ on P such that $\lambda_{ab} > 0$, $\lambda_{ac} > 0$, and $\lambda_{ax} \leq 0$ for every edge ax of P with $x \neq b, c$



- Now use the Perles--Dancis theorem to reconstruct the entire complex ∂P

Another conjecture of Kalai

A *missing facet* of a simplicial d -polytope P is a missing face with d vertices



Conjecture (Kalai): Let $d \geq 4$ and let P be a simplicial d -polytope that has no missing facets. Then the **graph of P** and the **space of affine 2-stresses of P** determine P up to **affine equivalence**

The requirement that there are no missing facets is necessary: consider stacked polytopes

Theorem (Cruickshank, Jackson, Tanigawa 2024): Kalai's conjecture holds for polytopes with "generic" coordinates

More results

Theorem (Murai-N-Zheng, 2024)

Kalai's conjecture holds for $d \geq 5$: If $d \geq 5$ and P is a simplicial d -polytope that has no missing facets, then the space of affine 2-stresses of P determines the **affine type** of P

Comments:

- The proof uses a lot of commutative algebra
- The $d = 4$ case is still open

What are affine k -stresses? Algebraic definition

The theory of higher stresses was developed in 1990s by [Tay, White, and Whiteley](#), and also by [Lee](#). (It relates to [Macaulay's](#) inverse systems.) We follow the definition of [Carl Lee](#)

Set-up: P is a simplicial d -polytope with vertices $1, 2, \dots, n$, and their position vectors $p(1), p(2), \dots, p(n) \in \mathbb{R}^d$

M is the $(d + 1) \times n$ matrix

$$\underbrace{\left[\begin{array}{cccc} p(1) & p(2) & \dots & p(n) \\ 1 & 1 & \dots & 1 \end{array} \right]}_n \left. \begin{array}{l} \} d \\ \} 1 \end{array} \right.$$

$\mathbb{R}[X]$ is the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$

Rows of M give rise to $d + 1$ linear forms in $\mathbb{R}[X]$:

$$\theta_i := \sum_{j=1}^n p(j)_i x_j \quad \text{for } i = 1, \dots, d, \text{ and}$$

$$\ell = x_1 + \dots + x_n$$

What are affine k -stresses? Algebraic definition

For a polynomial $q(X) \in \mathbb{R}[X]$ and a linear form $a(X) = a_1x_1 + \cdots + a_nx_n$, define

$$\partial_a q := \sum_{j=1}^n a_j \partial_{x_j}(q)$$

Definition: An (algebraic) *affine k -stress* λ on P is a homogeneous polynomial $\lambda(X) \in \mathbb{R}[X]$ of degree k such that

- All terms of λ are **supported on faces**: if $\lambda_{\nu} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$ is a summand of λ with $\lambda_{\nu} \neq 0$, then $\{j : \nu_j > 0\} \in \partial P$
- $\partial_{\theta_j}(\lambda) = 0$ for all $j = 1, \dots, d$, and
- $\partial_{\ell}(\lambda) = 0$

Denote the set of all algebraic affine k -stresses by $S_k^a(P)$

Algebraic definition vs geometric definition

An **algebraic** affine k -stress $\lambda(X)$ on P is a homogeneous polynomial of degree k in $\mathbb{R}[X]$ that usually contains **non-squarefree** terms.

A **geometric** affine k -stress on P assigns weights to $(k - 1)$ -faces of ∂P , so can be thought as a homogeneous **squarefree** polynomial of degree k in $\mathbb{R}[X]$

Let $\varphi: \mathbb{R}[X] \rightarrow$ **squarefree polynomials of $\mathbb{R}[X]$** be the map that **deletes** non-squarefree terms

$$\text{Example: } \varphi(-x_1^2 + 4x_1x_3 + 3x_2^2 + 7x_2x_5) = 4x_1x_3 + 7x_2x_5$$

Theorem (Lee, 1996) φ is an isomorphism between $S_k^a(P)$ and $S_k(P)$: for every **algebraic** affine k -stress λ on P , $\varphi(\lambda)$ is a **geometric** affine k -stress on P , and for every **geometric** affine k -stress $\bar{\lambda}$, **there is a unique algebraic** affine k -stress λ s.t. $\varphi(\lambda) = \bar{\lambda}$

More results

Theorem (Murai-N-Zheng, 2024)

If $d \geq 2k + 1$, $k \geq 2$, and P is a simplicial d -polytope that has no missing faces of size $\geq d - k + 2$, then the space of algebraic affine k -stresses of P determines the affine type of P

Proof uses a lot of commutative algebra: Stanley-Reisner rings, Matlis dual, algebraic graded Betti numbers, socles,...

Conjecture: The same holds if $d = 2k$ for all $k \geq 2$

Known in some special cases (e.g., P has no missing faces except missing edges)

The general case is open even for $d = 4, k = 2$

Proof idea

- **Easy:** if λ is an affine k -stress and $a \in \mathbb{R}[X]$ is a linear form, then $\partial_a \lambda$ is an affine $(k - 1)$ -stress
- We need the *converse* statement: if $d \geq 2k + 1$, P has no large missing faces, and ω is an affine $(k - 1)$ -stress, then there exists an affine k -stress λ on P and a linear form $a(X) \in \mathbb{R}[X]$ such that $\partial_a \lambda = \omega$. In particular,

$$S_{k-1}^a(P) = \{\partial_a \lambda : \lambda \in S_k^a(P), a \in \mathbb{R}[X]_1\}$$

Thus, $S_k^a(P) \mapsto S_{k-1}^a(P) \mapsto S_{k-2}^a(P) \mapsto \cdots \mapsto S_1^a(P)$, and this allows us to determine the affine type of P !

In fact, we prove that if $k < \frac{d}{2}$ and P has $m_{d-k+1}(P)$ missing faces of dimension $d - k + 1$, then $\{\partial_a \lambda : \lambda \in S_k^a(P), a \in \mathbb{R}[X]_1\}$ is a subspace of $S_{k-1}^a(P)$ of codimension $m_{d-k+1}(P)$

What's next?

- Understanding the **middle stress space** $S_{d/2}^a(P)$ for even d
- The famous **g -theorem** (Stanley, Billera-Lee, 1980) provides a complete characterization of face vectors of simplicial polytopes (now also simplicial spheres: Adiprasito, Papadakis-Petrotou, Karu-Xiao, 2018-2023)
- However, face numbers of simplicial polytopes **without large missing faces** are very poorly understood (but see works of Migliore-Nagel, Nagel, Nevo). Improving this understanding would be a fantastic next step!

THANK YOU!