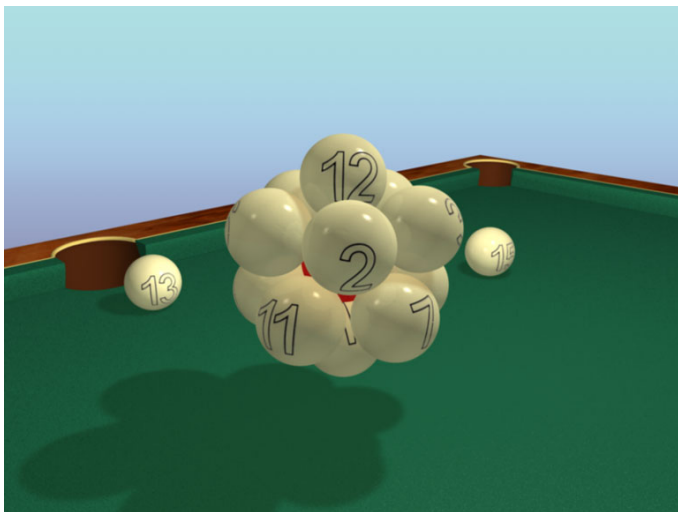
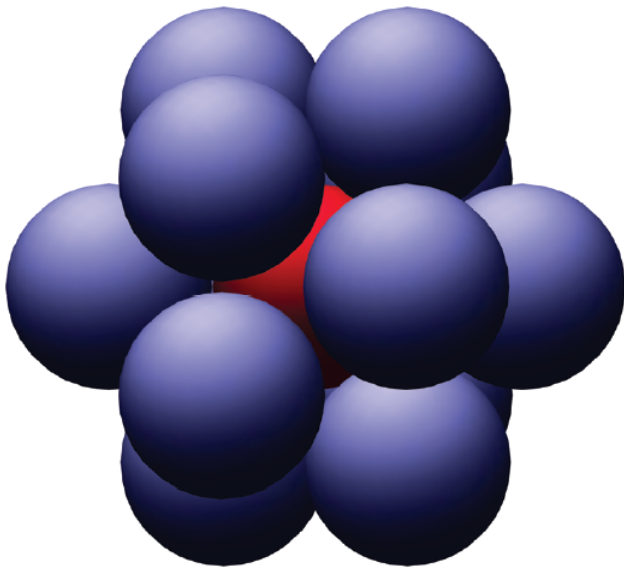


Optimal circle and sphere packings in low dimensions using distance distribution

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University of Texas Rio Grande Valley

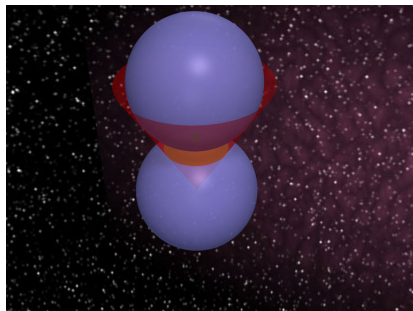




(Graphics: Detlev Stalling, ZIB Berlin)

Packing by spherical caps

If unit spheres kiss the unit sphere S , then the set of kissing points is the arrangement on S such that the angular distance between any two points is at least 60° . Thus, the kissing number is the maximal number of nonoverlapping spherical caps of radius 30° on S .



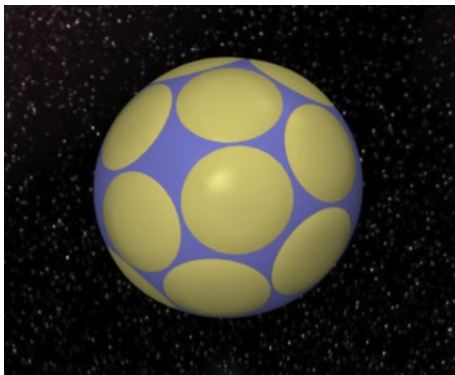
Status 2025: Kissing numbers

The only exact values of kissing numbers known:

n	lattice	regular polytope
$k(1) = 2$	A_1	
$k(2) = 6$	A_2	hexagon
$k(3) = 12$	H_3	icosahedron
$k(4) = 24$	$?D_4$	?24-cell
$k(8) = 240$	E_8	
$k(24) = 196,560$	Λ_{24}	

Tammes' problem

Tammes' problem. How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible



- I. **Area inequalities.** L. Fejes Tóth (1943); for $d > 3$ Coxeter (1963) and Böröczky (1978)
- II. **Contact and distance graphs.** Schütte, and van der Waerden (1951); Danzer (1963); Leech (1956);...
- III. **LP and SDP.** Delsarte et al (1977); Kabatiansky and Levenshtein (1978); Odlyzko & Sloane (1979); Bachoc and Vallentin (2008); ...
- IV. **Other methods...**

The Tammes problem

L. Fejes Tóth (1943): $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951): $N = 5, 7, 8, 9$

L. Danzer (1963): $N = 10, 11$

R. M. Robinson (1961): $N = 24$

M. & Tarasov: $N = 13$ and $N = 14$

N	d_N
4	109.4712206
5	90.0000000
6	90.0000000
7	77.8695421
8	74.8584922
9	70.5287794
10	66.1468220
11	63.4349488
12	63.4349488
13	57.1367031
14	55.6705700
.....	
15	53.6578501
16	52.2443957
17	51.0903285

Contact graphs

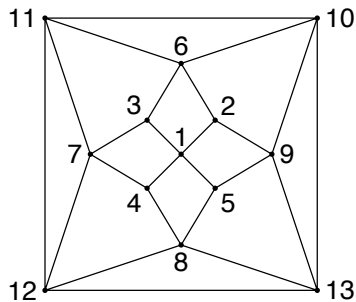
Let X be a finite subset of a metric space M ,

$$\psi(X) := \min_{x,y \in X} \{\text{dist}(x,y)\}, \text{ where } x \neq y.$$

The *contact graph* $\text{CG}(X)$ is the graph with vertices in X and edges (x,y) , $x,y \in X$ such that

$$\text{dist}(x,y) = \psi(X)$$

The contact graph $\Gamma_{13} := \text{CG}(P_{13})$ with $\psi(P_{13}) \approx 57.1367^\circ$



Tammes' problem for $N = 13$

The value $d = \psi(P_{13})$ can be found analytically.

$$2 \tan \left(\frac{3\pi}{8} - \frac{a}{4} \right) = \frac{1 - 2 \cos a}{\cos^2 a}$$

$$d = \cos^{-1} \left(\frac{\cos a}{1 - \cos a} \right).$$

Thus, we have $a \approx 69.4051^\circ$ and $d \approx 57.1367^\circ$.

Tammes' problem for $N = 13$

Theorem (M. & Tarasov'12). The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Tammes' problem for $N = 14$

Theorem (M. & Tarasov'15) The arrangement of 14 points P_{14} in $\mathbb{S}^2$ is the best possible and the maximal arrangement is unique up to isometry.

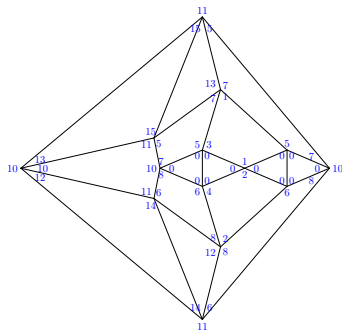
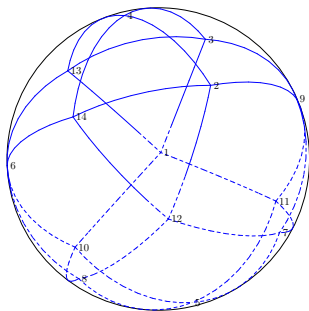


Figure: An arrangement of 14 points P_{14} and its contact graph Γ_{14} with $\psi(P_{14}) \approx 55.67057^\circ$.

Packings of circles on a square flat torus.

The problem is: for a given number $N \geq 1$ of points, find the maximal $r \in \mathbb{R}^+$ such that N circles of radius r could be put on the square flat torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ without overlapping, or, equivalently, to find the maximal $d \in \mathbb{R}^+$ such that there are N points on the torus with pairwise distances not less than d (where $d = 2r$).

$$N = 7, 8$$

Theorem (M. & Nikitenko'16). *There are three, up to isometry or up to a move of a free disk, optimal arrangements of 7 points in $\mathbb{T}^2$ which are shown in Figures 7.1 - 7.3 where $d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.*

Corollary (M. & Nikitenko). *There is one unique, up to isometry, optimal arrangement of 8 points in $\mathbb{T}^2$, which is shown in Figure 8 where $d(8) = d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.*

Optimal packings of circles on a square flat torus: 7.1

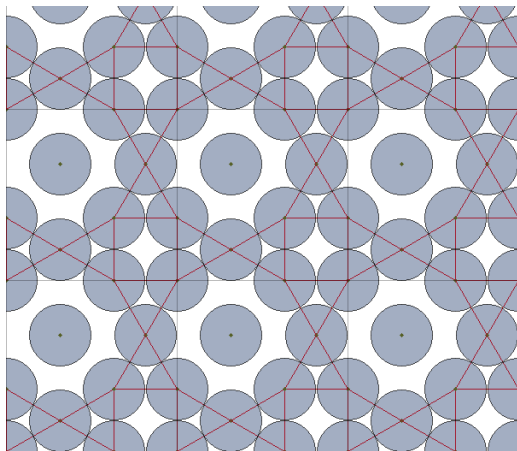


Figure: The first optimal configurations for $N=7$

Optimal packings of circles on a square flat torus: 7.2

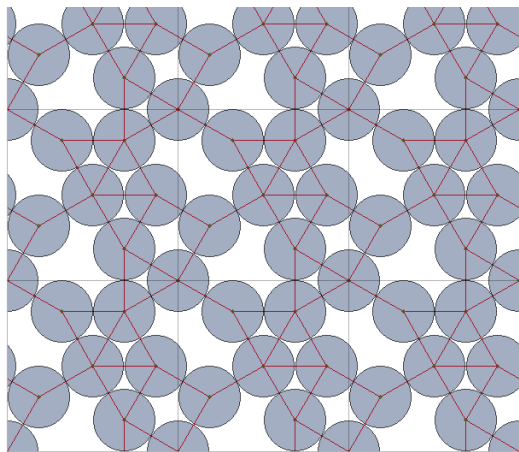


Figure: The second optimal configurations for $N=7$

Optimal packings of circles on a square flat torus: 7.3

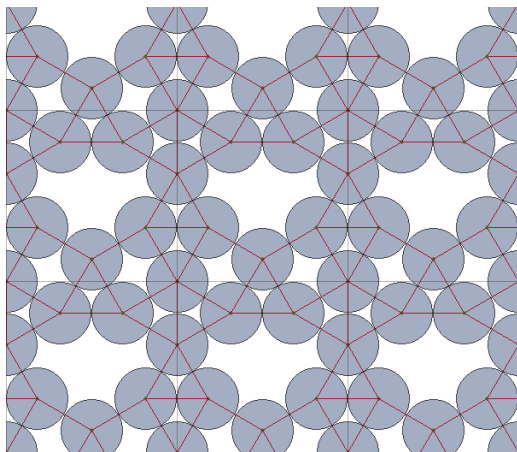


Figure: The third optimal configurations for $N=7$

Figure 8

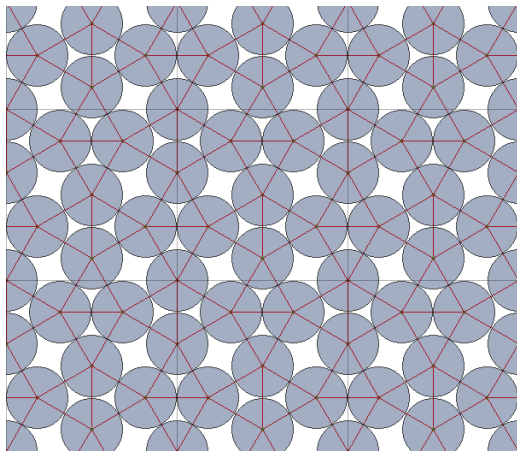
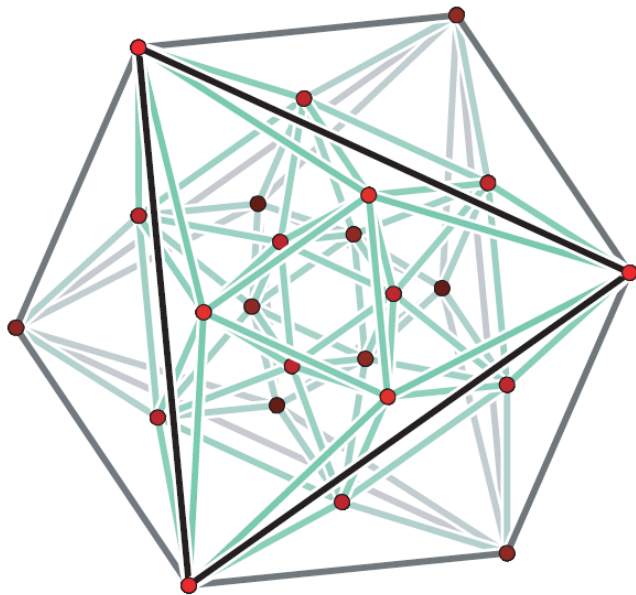


Figure: The optimal configuration for $N=8$

The 24-cell

$n = 4$: There are 24 vectors with two zero components and two components equal to ± 1 ; they all have length $\sqrt{2}$ and a minimum distance of $\sqrt{2}$. Properly rescaled (that is, multiplied by $\sqrt{2}$), they yield the centers for a kissing configuration of unit spheres and imply that $k(4) \geq 24$. The convex hull of the 24 points yields a famous 4-dimensional regular polytope, the “24-cell”, discovered in 1842 by Ludwig Schläfli. Its facets are 24 regular octahedra.



(Graphics: Michael Joswig/polymake [13])



$$N \leq \frac{f(1)}{f_0}$$

$$N \leq \frac{f(1) + \hat{h}(n, T, f)}{f_0}$$

$$N^2 \leq \frac{F(1, 1, 1) + 3(N - 1)B}{f_0}$$

$$N^2 \leq \frac{F(1, 1, 1) + 3(N - 1)B + 3N\hat{h}(n, T, g)}{f_0}$$

Delsarte's method

Ph. **Delsarte** (1972); **V. M. Sidelnikov** (1974)
Delsarte, Goethals and **Seidel** (1975, 1977)
G.A. Kabatiansky and **V.I. Levenshtein** (1978)

Theorem (Delsarte et al)

If

$$f(t) = \sum_{k=0}^d c_k G_k^{(n)}(t)$$

is nonnegative combination of Gegenbauer polynomials, with $c_k \geq 0$ and $c_0 > 0$, and if $f(t) \leq 0$ holds for all $t \in [-1, \frac{1}{2}]$, then the kissing number in n dimensions is bounded by

$$k(n) \leq \frac{f(1)}{c_0}$$

$$K(8)=240; k(24)=196560$$

G.A. Kabatiansky and **V.I. Levenshtein** (1978):

$$2^{0.2075n(1+o(1))} \leq k(n) \leq 2^{0.401n(1+o(1))}$$

In 1979: **V. I. Levenshtein** and independently **A. Odlyzko** and **N.J.A. Sloane** using Delsarte's method have proved that $k(8) = 240$, and $k(24) = 196560$.

Odlyzko & Sloane: upper bounds on $k(n)$ for $n = 4, 5, 6, 7$, and 8 are 25, 46, 82, 140, and 240, respectively.

The bound for kissing numbers (M., 2003, 2008)

Let $f(x) = \sum f_k G_k^{(n)}(x)$, $f_k \geq 0$, $f_0 > 0$
 $f(x) \leq 0$ for all $x \in [-1, 1/2] \setminus T$.

Then

$$N \leq \frac{f(1) + \hat{h}(n, T, f)}{f_0}.$$

$$n = 4; \quad T = [-1, -0.6058], \quad f_0 = 1, \quad N < 24.865$$

The kissing problem in four dimensions

$$f_4(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.38t^4 - 9.83t^3 - 4.12t^2 + 0.434t - 0.016$$

Lemma

Let $P = \{p_1, \dots, p_m\}$ be unit vectors in $\mathbb{R}^4$ (i.e. points on the unit sphere S^3). Then

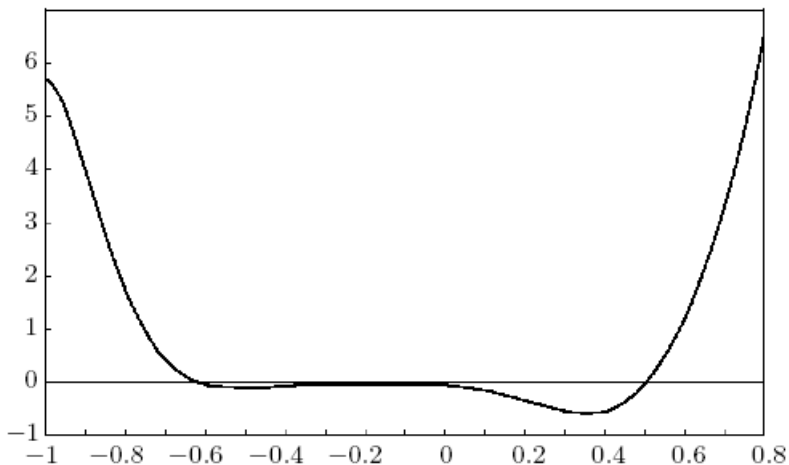
$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) \geq m^2.$$

Lemma

Let $P = \{p_1, \dots, p_m\}$ be a kissing arrangement on the unit sphere S^3 (i.e. $p_k \cdot p_\ell \leq \frac{1}{2}$). Then

$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) < 25m.$$

The graph of $y = f_4(t)$



$$D := \{(x, y, z) : -1 \leq x, y, z \leq 1/2, 1 + 2xyz - x^2 - y^2 - z^2 \geq 0\}.$$

$$\text{BV}(n, f_0) := \{F : F(x, y, z) = \sum \langle F_k, S_k^n(x, y, z) \rangle, F_k \succeq 0, F_0 - f_0 E_0 \succeq 0\}$$

Suppose

- (1) $F \in \text{BV}(n, f_0)$,
- (2) $F(x, x, 1) \leq B$ for all $x \in [-1, 1/2]$,
- (3) $F(x, y, z) \leq 0$ for all $(x, y, z) \in D$,

Then

$$N^2 \leq \frac{F(1, 1, 1) + 3(N-1)B}{f_0}$$

dim=4: uniqueness of the maximal kissing arrangement

LP bound [Odlyzko & Sloane; Arestov & Babenko] = 25.558...

M. (2003): $k(4) < 24.865$

C. Bachoc & F. Vallentin (2008): $S_7(4) = 24.5797...$

H. D. Mittelmann & F. Vallentin (2010)

$S_{11}(4) = 24.10550859...$

$S_{12}(4) = 24.09098111...$

$S_{13}(4) = 24.07519774...$

$S_{14}(4) = 24.06628391...$

F.C. Machado & F.M. de Oliveira Filho (2018)

$S_{15}(4) = 24.062758...$

$S_{16}(4) = 24.056903...$

Theorem: 4-point SDP bound. (M.'14)

$$N^3 \leq \frac{F(\mathbf{1}) + 4(N-1)B_{3,1} + 3(N-1)B_{2,2} + 6(N-1)(N-2)B_{2,1,1}}{f_0}.$$

D. de Laat, N. M. Leijenhorst, W. H. H. de Muinck Keizer,
Optimality and uniqueness of the D_4 root system, preprint, arXiv:
2404.18794

In this paper authors apply the 4-point SDP bound theorem to prove the uniqueness of kissing configuration in $\mathbb{R}^4$.

$g_T(x) = g(x)$ for $x \in T$ and $g_T(x) = 0$ otherwise.

Definition

For given $n, f_0, T \subset [-1, 1)$, $g : T \rightarrow \mathbb{R}$, B and θ denote by $\mathcal{F}(n, f_0, T, g, B, \theta)$ the class of symmetric polynomials $F(x, y, z)$ that satisfy the following properties:

- ① $F \in \text{BV}(n, f_0)$,
- ② $F(x, x, 1) \leq B + 6g_T(x)$ for all $x \in [-1, \cos \theta]$,
- ③ $F(x, y, z) \leq g_T(x) + g_T(y) + g_T(z)$ for all $(x, y, z) \in D(\theta)$.

Theorem (M., 2019)

Let $F \in \mathcal{F}(n, f_0, T, g, B, \theta)$. Then an (N, n, θ) spherical code satisfies the following inequality

$$N^2 \leq \frac{F(1, 1, 1) + 3(N-1)B + 3N\hat{h}(n, \theta, T, g)}{f_0}.$$

Distance distribution

Let C be an (N, n, θ) spherical code. The *distance distribution* of C is the system of numbers $\{A_t : -1 \leq t \leq 1\}$.

$$A_t(u) := |\{v \in C : v \cdot u = t\}|, \quad A_t := \frac{1}{N} \sum_{u \in C} A_t(u).$$

$$A_t = 0 \text{ for } s := \cos \theta < t < 1, \quad \sum_{-1 \leq t \leq s} A_t = N - 1.$$

$$A(T) := \sum_{t \in T: A_t > 0} A_t, \quad T \subset [-1, 1].$$

SDP bounds for distance distribution

Theorem (M., 2019)

Let $F \in \mathcal{F}(n, f_0, T, g, B, \theta)$. Suppose $T \subset [-1, \cos \theta]$ and $g(t) \leq -a < 0$ for all $t \in T$. Then for every (N, n, θ) spherical code C we have

$$A(T) \leq \frac{2}{N} \lfloor Q \rfloor, \quad Q := \frac{F(1, 1, 1) + 3(N-1)B - f_0 N^2}{6a}.$$

Theorem (M., 2019)

Let $F \in \mathcal{F}(n, f_0, T, g, B, \theta)$. Let $a > 0$. Suppose $T \subset [-1, \cos \theta]$ and $g(t) \leq a$ for all $t \in T$. Then for every (N, n, θ) spherical code C we have

$$A(T) \geq \frac{2}{N} \lceil R \rceil, \quad R := \frac{f_0 N^2 - F(1, 1, 1) - 3(N-1)B}{6a}.$$

Kissing arrangement in four dimensions:

$$A(\{-1\}) = 1, \quad A(\{-1/2\}) = 8, \quad A(\{0\}) = 6, \quad A(\{1/2\}) = 8$$

$$A_t = 0 \text{ for all } t \neq \{-1, -1/2, 0, 1/2, 1\}$$

SDP bounds on $(24, 4, \pi/3)$ – spherical code

Theorem (Dostert–Kolpakov–Moustrou–M.)

Let C be a $(24, 4, \pi/3)$ – spherical code. Then

$$A([-1, -0.45]) \leq 9; \quad A([-1, 0.35]) \leq 15,$$

$$A([-0.73, 0.35]) \leq 14, \quad A([-0.05, 0.5]) \leq 14,$$

$$A([-1, -0.73]) \geq 1, \quad A([0.35, 0.5]) \geq 8$$

General bounds for spherical codes

Let C be an N -element subset of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

$$I(C) := \{t = x \cdot y \mid x, y \in C \text{ \& } x \neq y\}.$$

Let $T \subset [-1, 1)$. We say that C is an (N, n, T) *spherical code* if $I(C) \subset T$.

Let g be a real function on $I(C)$. Define

$$E_g(C) := \sum_{(x,y) \in C^2, x \neq y} g(x \cdot y)$$

$$S_g(C) := \sum_{(x,y) \in C^2} g(x \cdot y) = E_g(C) + Ng(1)$$

General 2-point (Delsarte – Yudin) bound

Theorem

Let C be an (N, n, T) spherical code. Suppose $g : T \rightarrow \mathbb{R}$, $f : [-1, 1] \rightarrow \mathbb{R}$ and $f_0 \in \mathbb{R}$ are such that

- 1 $f(t) \leq g(t)$ for all $t \in T$.
- 2 $S_f(C) \geq f_0 N^2$.

Then

$$f_0 N^2 \leq Nf(1) + E_g(C)$$

General 3-point bound

$$D_3(T) := \{(t, u, v) : t, u, v \in T \text{ \& } 1 + 2tuv - t^2 - u^2 - v^2 \geq 0\}.$$

Theorem

Let C be an (N, n, T) spherical code and $F : [-1, 1]^3 \rightarrow \mathbb{R}$ be a symmetric function. Suppose $f : T \rightarrow \mathbb{R}$ and $g : T \rightarrow \mathbb{R}$, are such that

- ① $F(1, t, t) \leq f(t)$ for all $t \in T$,
- ② $F(t, u, v) \leq g(t) + g(u) + g(v)$ for all $(t, u, v) \in D_3(T)$.

If

$$S_F(C) := \sum_{(x,y,z) \in C^3} F(x \cdot y, x \cdot z, y \cdot z) \geq F_0 N^3,$$

where $F_0 \in \mathbb{R}$, then

$$F_0 N^3 \leq NF(1, 1, 1) + 3E_f(C) + (3N - 6)E_g(C).$$

General 3-point bound

Corollary

Under the assumptions of Theorem let $f(t) = B + 2g(t) - q(t)$ with $q : [-1, 1] \rightarrow \mathbb{R}$. If $q(C) \geq 0$, then

$$E_g(C) = \sum_{t \in T} A_t g(t) \geq r_g,$$

$$r_g = \frac{1}{3}(F_0 N^2 - F(1, 1, 1)) - q(1) - (N - 1)B.$$

Let p be a Gegenbauer polynomial G_k^n . Then $S_p(C) \geq 0$, i.e.

$$E_p(C) = \sum_{t \in T} A_t p(t) \geq r_p, \quad r_p = -Np(1)$$

General 3-point bound: SDP program

Corollary

Let $a_0, \dots, a_d \geq 0$, $F_0, \dots, F_d \succeq 0$, $T \subset [-1, 1)$, $g : T \rightarrow \mathbb{R}$, $M \in \mathbb{R}$, $F(u, v, t) = \sum_k \langle F_k, S_k^n(u, v, t) \rangle$. If

$$a_0 + \dots + a_d + F(1, 1, 1) \leq M - 1, \quad (1)$$

$$\sum_{k=0}^d a_k G_k^n(u) + 3F(u, u, 1) \leq -1 + 6g(u) \text{ for } u \in T, \quad (2)$$

$$F(u, v, t) \leq g(u) + g(v) + g(t) \text{ for } (u, v, t) \in D_3(T). \quad (3)$$

Then for every (N, n, T) spherical code, we have

$$\sum_{t \in T} A_t g(t) \geq \frac{N - M}{3N}$$

Polynomial g_1

The following polynomial gives a sharp lower bound in $[-1, -0.73]$, $a = 1/50$ and $M = 22.645212490128051$

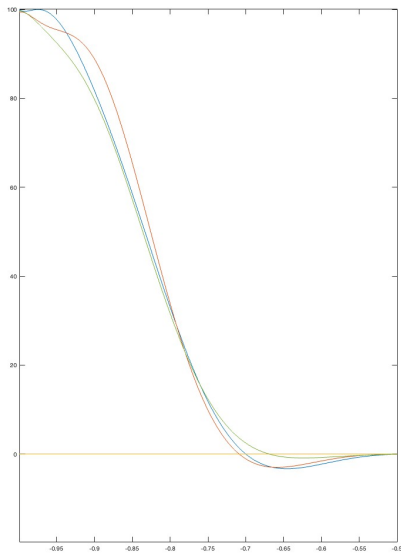
$$p_1(x) = \sum_{k=0}^{22} G_k^{(4)}(x),$$

where $[c_0, \dots, c_{22}] = [0.222, 0.8648, 1.8875, 3.1425, 4.5059, 5.7052, 6.5739, 6.9286, 6.7119, 6.0157, 4.9575, 3.7767, 2.6446, 1.6914, 0.9947, 0.5249, 0.2524, 0.1097, 0.0409, 0.0153, 0.0042, 0.001, 0.0002]$.

Let $g_1 = k p_2$, where $k = 100/p_1(-1)$. Then $g_1(-1) = 100$. Let $T = [-1, 0.5]$. The Corollary yields

$$\sum_{t \in T} A_t g_1(t) \geq B_1(24) \approx 95 > -52.2431 = LP(24).$$

Example: $gSDP$



Example

Since

$$\sum_{t \in [-1, 0.5]} A_t g_1(t) \geq 95$$

we have

$$A([-1, -0.73]) \geq 1$$

$$p_2(x) = \sum_{k=0}^{22} G_k^{(4)}(x),$$

$[c_0, \dots, c_{22}] = [-0.5438, -2.0024, -3.8887, -5.6414, -6.7025, -6.8508, -6.0698, -4.6566, -3.0047, -1.4686, -0.3226, 0.3704, 0.6521, 0.6486, 0.5104, 0.3361, 0.1911, 0.0963, 0.0411, 0.0157, 0.0056, 0.001, 0.0004]$.

The SDP bound in the Corollary gives $M = M_2 := 22.5689$, $B_2(25) = 0.0324$ and $B_2(24) = 0.0199$.

Short proof: $k(4) = 24$

Theorem 1 (M., 2008) yields that

$$k(4) \leq \frac{1}{c_0} \max\{h_0, h_1, \dots, h_\mu\}$$

Let C be an $(25, 4, \pi/3)$ spherical code,

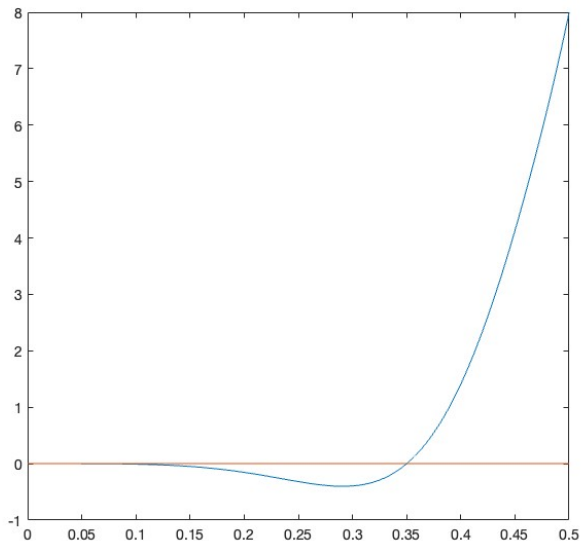
$$R_2 := \sum_t A_t g_1(t)$$

Since $g_2(t) \leq 0$ for all $t \in [-\sqrt{2}/2, 1/2]$, we have $t_0 = -\sqrt{2}/2$ and $\mu = 4$. Then we consider $\mu = 0, 1, 2, 3, 4$ to find the maximum of R_2 . This maximum is achieved at $\mu = 2$ and is 0.0266, i.e. $R_2 < 0.0266$. On the other side we have $R_2 > B_2(25) = 0.0324$, a contradiction.

$c(0 : 17) = [3.2313 \ 34.6000 \ 97.5893 \ 137.3081 \ 119.9142 \ 74.1812 \ 36.5605$
 $15.0065 \ 4.2796 \ 0.0946 \ -0.7087 \ -0.4672 \ -0.1755 \ -0.0283 \ 0.0097 \ 0.0070$
 $0.0016 \ 0.0001]$

$$\sum_{t \in [-1, 0.5]} A_t g_3(t) > R_3 = 7.93$$

Polynomial g_3



Thank you