From Ising, dimers, and UST in 2d statistical physics to discrete surfaces in Minkowski spaces

Dmitry Chelkak, University of Michigan, Ann Arbor
[joint works w/ M.Basok, B.Laslier, R.Mahfouf, S.C.Park, S.Ramassamy, M.Russkikh]

Circle Packings, Minimal Surfaces, and Discrete Differential Geometry

ICERM, February 14, 2025

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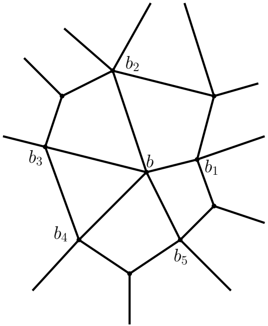
Simple random walks on a planar graph: Tutte's barycentric embeddings



▷ Question: given a (large) weighted planar graph (B, c), how can we describe the (limit of) random walks on B?

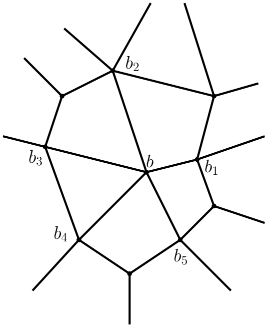
[Above, c_e, $e \in E(G)$, are positive *conductances*; the random walk jumps from b to its neighbors with probabilities $c_{v'v} / \sum_{v' \sim v} c_{v'v}$ proportional to $c_{vv'}$. The case $c_e = 1$ is already general enough.]

[!] We should first decide **how to draw/embed B in $\mathbb{C}$.**

[if G is already embedded into the complex plane, we can consider *re-embedding* it in a 'nicer' way]

▷ **Modern motivation:** 2d lattice models of statistical physics (universality & conformal invariance; random planar maps)

Simple random walks on a planar graph: Tutte's barycentric embeddings



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[!] We should first decide **how to draw/embed B in $\mathbb{C}$.**

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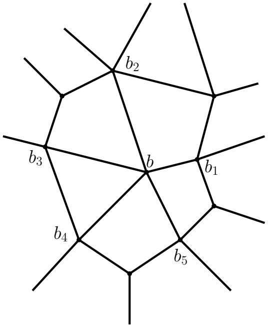
▷ Tutte's barycentric embeddings $\mathcal{H}$: each vertex $b \in B$ is positioned at the (weighted) barycenter of its neighbors $\Leftrightarrow \mathcal{H} = \mathcal{H}_1 + \mathrm{i}\mathcal{H}_2 : B \to \mathbb{C}$ is a discrete harmonic function.

▷ Theorem (Tutte, 'How to draw a graph', 1963): Let B be a *finite* planar graph with a marked outer face and $\mathcal{H}$ its barycentric embedding. If the (images of) boundary vertices form a convex polygon, then $\mathcal{H}$ is a *proper* embedding.

[$\rightsquigarrow$ one can check the planarity of a given graph in this way]

Simple random walks on a planar graph: Tutte's barycentric embeddings

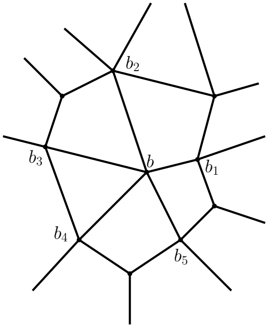


- ▷ Let $\mathcal{H}^\delta : B^\delta \to \mathbb{C}$ be a sequence of Tutte's barycentric embeddings with 'mesh sizes' $\delta \to 0$.
- ▷ By construction, both coordinates $\mathrm{Re}\,\mathcal{H}^\delta(X_n^\delta)$, $\mathrm{Im}\,\mathcal{H}^\delta(X_n^\delta)$ are *martingales* if X_n^δ is the random walk on B^δ. Should we expect that these RWs converge to the *Brownian motion?*

[How do we understand the convergence? Possible ways:

- ∘ convergence of discrete harmonic functions: harmonic measures = hitting probabilities, Green functions etc;
- ∘ $\rightsquigarrow$ convergence of trajectories w/o time-parametrization.]

Simple random walks on a planar graph: Tutte's barycentric embeddings



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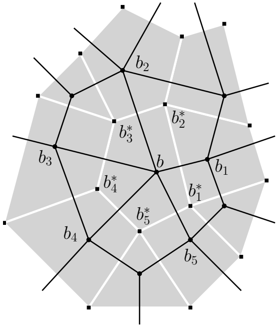
[How do we understand the convergence? Possible ways:
- ∘ convergence of discrete harmonic functions: harmonic measures = hitting probabilities, Green functions etc;
- ∘ $\rightsquigarrow$ convergence of trajectories w/o time-parametrization.]

- ▷ Certainly, **not to the BM:** the best we can hope for is a subsequential convergence to a *driftless diffusion*

$$Lh = -\alpha(z)h_{zz} + 2\beta(z)h_{z\bar{z}} - \overline{\alpha(z)}h_{\bar{z}\bar{z}} = 0, \quad |\alpha(z)| < \beta(z).$$

- ▷ **Question:** what kind of additional information do we need to determine $\alpha(z)/\beta(z)$?

Simple random walks on a planar graph: Tutte's barycentric embeddings



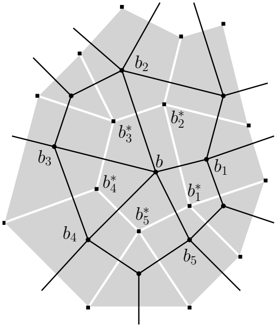
- ▷ We may be lucky and have the convergence to the BM if $\mathcal{H}^\delta$ can be extended to *orthodiagonal embeddings.*

- ▷ Given $\mathcal{H}$, one can construct a *dual harmonic embedding* $\mathcal{H}^* : B^* \to \mathbb{C}$ (with conductances $c_{e^*} = 1/c_e$) by setting

$$\mathcal{H}^*(b_2^*) - \mathcal{H}^*(b_1^*) := \mathrm{i}c_{b_1 b_2} \cdot (\mathcal{H}(b_2) - \mathcal{H}(b_1)).$$

If we are lucky and all quads $\mathcal{H}^\delta(b_1)\mathcal{H}^{\delta*}(b_1^*)\mathcal{H}^\delta(b_2)\mathcal{H}^{\delta*}(b_2^*)$ are non-self-intersecting and have $\mathrm{diam} \leqslant \delta$, then $(\mathcal{H}^\delta, \mathcal{H}^{\delta*})$ is called an *orthodiagonal embedding of mesh size δ.*

Simple random walks on a planar graph: Tutte's barycentric embeddings



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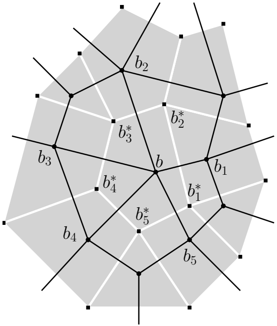
[......... , Gurel-Gurevich–Jerison–Nachmias'19, Binder–Pechersky'24, Bou-Rabee–Gwynne'24]

▷ **Theorem:** Discrete harmonic functions on orthodiagonal embeddings converge to harmonic functions ($L = \Delta$). Trajectories of the random walks converge to the BM.

- No[!] 'technical' assumptions on angles/edge lengths are needed.
- We have natural <u>linear</u> discrete Cauchy–Riemann equations

Simple random walks on a planar graph: Tutte's barycentric embeddings



▷ **Informal theorem** [C.–Basok–Laslier–Russkikh'25]:

Assume that *"the mappings $\Phi^\delta : \mathcal{H}^\delta \mapsto \mathcal{H}^{*\delta}$"* converge, as $\delta \to 0$, to a Lipshitz mapping $\phi : \Omega \to \mathbb{C}$ such that

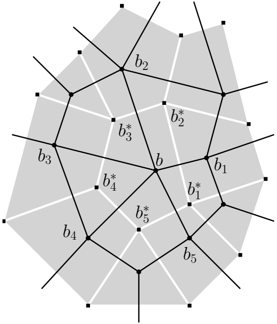
$$|\phi_{\bar{z}}| \leqslant k \operatorname{Re} \phi_z \text{ a.e. for some } k < 1 .$$

Then, discrete harmonic measures and Green functions of RWs on $\mathcal{H}^\delta$ converge to those of the elliptic operator

$$Lh := 2 \operatorname{Re} \phi_z \cdot h_{z\bar{z}} - \phi_{\bar{z}} \cdot h_{zz} - \overline{\phi}_z \cdot h_{\bar{z}\bar{z}} .$$

[Requires a very weak 'non-degeneracy' assumption EXP-FAT.]

Simple random walks on a planar graph: Tutte's barycentric embeddings



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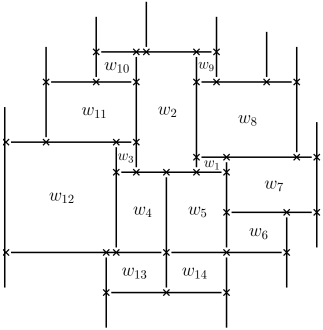
[Requires a very weak 'non-degeneracy' assumption EXP-FAT.]

▷ **Corollary:** If $L\phi \neq 0$, then the limits of random walks on G and on G^* are different: in a common parametrization ζ the martingale parts are equal but the drifts are not.

If $L\phi = 0$, then there exists a parametrization ζ such that $Lh = 0 \Leftrightarrow h_{\zeta\bar\zeta} = 0$, where ζ is a conformal parametrization of a maximal space-like 2-surface

$$\Theta := (z + \phi(z); z - \phi(z))_{z \in \Omega} \subset \mathbb{R}^{2,2}.$$

Discrete harmonic functions and square/rectangular tilings.

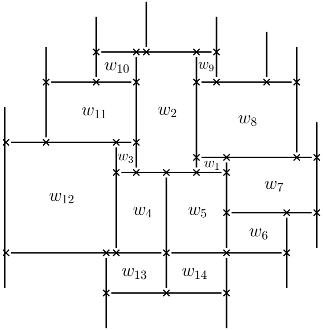


▷ Brooks–Smith–Stone–Tutte(1940): a *real* harmonic function $H_1 : B \to \mathbb{R}$ and its harmonic conjugate $H_1^* : B^* \to i\mathbb{R}$ define a *square/rectangular* tiling $\mathcal{R}$:

- vertical segments have x-coordinate $H_1(b)$;
- horizontal segments have y-coordinate $H_1^*(b^*)$.
- harmonic functions are linear on segments of $\mathcal{R}$;
- gradients satisfy Cauchy–Riemann on $B \cup B^*$.

[more generally: RWs on T-graphs (Kenyon–Sheffield'03): arrive at a T-intersection $\rightsquigarrow$ proceed left/right till the end]

Discrete harmonic functions and square/rectangular tilings.



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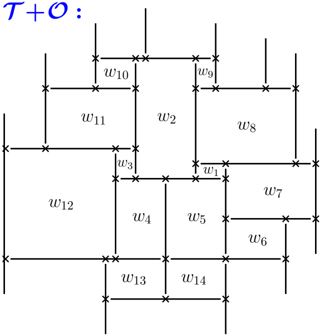
[more generally: RWs on T-graphs (Kenyon–Sheffield'03): arrive at a T-intersection $\rightsquigarrow$ proceed left/right till the end]

▷ Question: which additional input do we need to describe the limit of these RWs?
[Or maybe they should always converge to the BM because of Cauchy–Riemann equations?]

▷ Answer: another harmonic function $H_2 : B \to i\mathbb{R}$ (and its conjugate $H_2^* : B^* \to \mathbb{R}$), which gives a 'dual' tiling $\mathcal{R}^*$. [in fact, fixing $H_2 \Leftrightarrow$ choosing an invariant measure on $\mathcal{R}$]

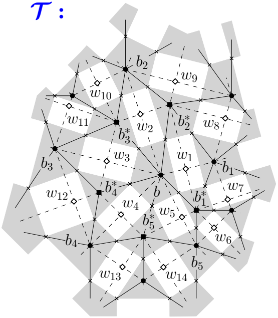
Square/rectangular tilings ← t-embeddings/t-surfaces → Tutte's embeddings

$\mathcal{T}+\mathcal{O}$:



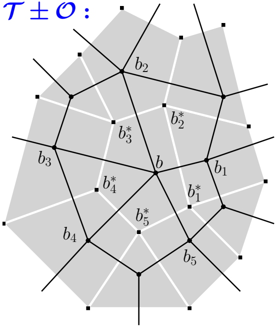
←

$\mathcal{T}$:



→

$\mathcal{T}\pm\overline{\mathcal{O}}$:



t-embedding $\mathcal{T}(v) := \frac{1}{2}(\mathcal{H}(b)+\mathcal{H}(b^*))$ can be viewed as a crease pattern

origami map $\overline{\mathcal{O}}(v) := \frac{1}{2}(\mathcal{H}(b)-\mathcal{H}(b^*))$: **"fold $\mathcal{T}$-plane over all segments"** [KLRR18]

- dark faces corresponding to $b \in B$ are translated keeping orientation;
- dark faces corresponding to $b^* \in B^*$ are rotated by π and translated;
- light rectangular faces are folded over by $z \mapsto \overline{\eta}_w^2 \overline{z} +$ translations.

[!!!] t-surface

$(\mathcal{T};\mathcal{O}) : \mathcal{G}^* \to \mathbb{R}^{2,2}$

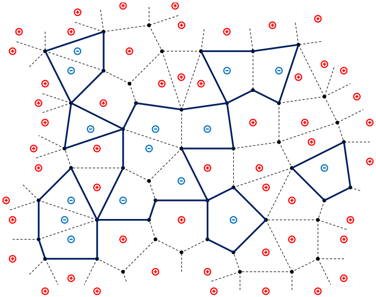
Planar Ising model (no magnetic field) and s-embeddings [arXiv:2006.14559v5]

Given a (large) weighted graph (G°, x), one assigns random spins $\sigma \in \{\pm 1\}^{V(G^\circ)}$ to its vertices (or faces) so that the probability to get $(\sigma_u)_{u \in V(G^\circ)}$ equals

$$\mathcal{Z}^{-1} \exp\left[\beta \sum_{\langle uv \rangle} J_{uv} \sigma_u \sigma_v\right] = \mathcal{Z}^{-1} \prod_{\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv},$$

where $J_{uv} > 0$ are called *interaction constants*, $\beta = 1/kT$ is the *inverse temperature*, and $x_{uv} := \exp[-2\beta J_{uv}] \in (0, 1)$.

[The normalizing factor $\mathcal{Z}$ is called the *partition function*.]



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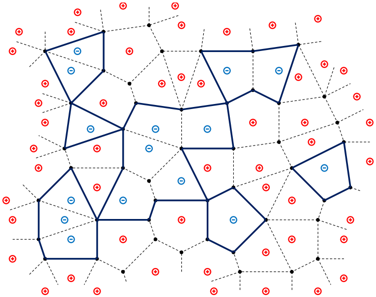
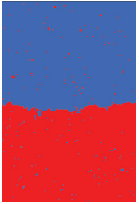
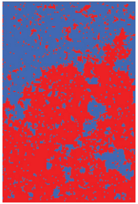


Phase transition in the homogeneous model on regular grids (e.g., $\mathbb{Z}^2$):



$x < x_{\mathrm{crit}}$



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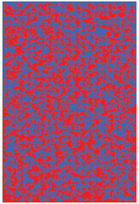

$x > x_{\mathrm{crit}}$

Conformal invariance at x_{crit}:

- correlations of lattice fields (e.g., spins) $\to$ CFT predictions
- interfaces/loop ensembles $\to$ SLE(3) curves/CLE(3)
- **Near-critical:** $x = x_{\mathrm{crit}} + m\delta$

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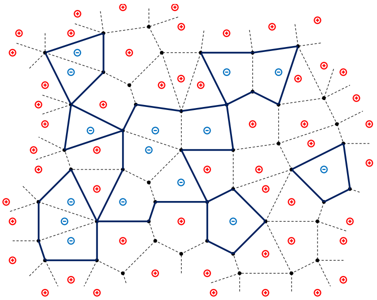

- ▷ This is a *free fermion* model: $\mathcal{Z} = \mathrm{Pf}[\mathcal{A}_{(G,x)}]$
  fermions = entries of $\mathcal{A}_{(G,x)}^{-1}$ [cf. Green function]

- ▷ On isoradial/rhombic grids
  the matrix $\mathcal{A}_{(G,x)}$ can be thought of as a *discrete Dirac operator* $\begin{bmatrix} im & \partial_z \\ -\partial_{\bar z} & -im \end{bmatrix}$

- ▷ **Question:** what if G is not a 'regular grid'?

Conformal invariance at x_{crit}:

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- ∘ interfaces/loop ensembles → SLE(3) curves/CLE(3)
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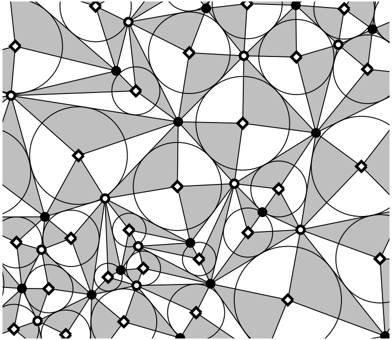

- ▷ *s-embeddings* = tilings by *tangential quads*
- ▷ should be viewed as *surfaces in* $\mathbb{R}^{2,1}$*:* define $\mathcal{Q}(v^\bullet)-\mathcal{Q}(v^\circ) := |\mathcal{S}(v^\bullet)-\mathcal{S}(v^\circ)|$ for $v^\bullet \sim v^\circ$
  [this is nothing but the corresponding t-surface]
- ▷ $x_e = \tan\frac{1}{2}\theta_e$ if $\tan\theta_e$ = the ratio of the $\mathbb{R}^{2,1}$-lengths of the diagonals of the non-planar quad $(\mathcal{S};\mathcal{Q})(v_1^\circ v_1^\bullet v_2^\circ v_2^\bullet)$
- ▷ greatly generalize 'regular grids' = tilings by rhombi
  [⇝ Baxter's Z-invariant weights on isoradial graphs]

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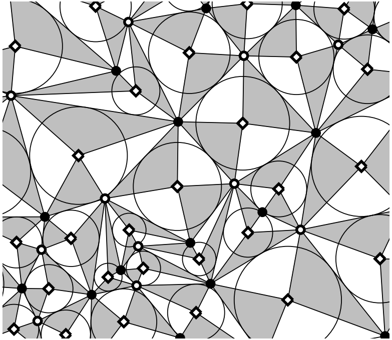

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- ▷ greatly generalize 'regular grids' = tilings by rhombi
  [$\rightsquigarrow$ Baxter's Z-invariant weights on isoradial graphs]
- ▷ **Theorem** (C.'20, conformal invariance on 'flat' s-embeddings):
  Ising interfaces on s-embeddings S^δ converge to SLE provided that $\mathcal{Q}^\delta = O(\delta)$ and $\mathcal{S}^\delta$ satisfy $\textsc{Unif}(\delta)$: all edges are comparable to δ, all angles are bounded.
- ▷ All critical doubly periodic (G°, x) admit a unique 'flat' s-embedding [see also KLRR18]

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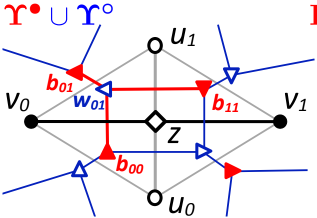
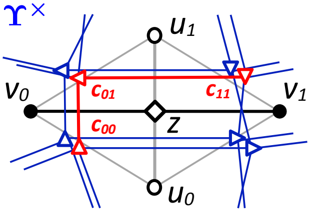


$\mathrm{Ker}\,\mathcal{A}$: functions on $\Upsilon^\bullet$ satisfying the equation

$$\mathbf{X}(\mathbf{b}_{01}) = \pm\mathbf{X}(\mathbf{b}_{00})\cos\theta_z \pm \mathbf{X}(\mathbf{b}_{11})\sin\theta_z$$

for $b_{00}, b_{01}, b_{11} \sim w_{01}$



$\Longleftrightarrow$ spinors on $\Upsilon^\times$ satisfying the equation

$$\mathbf{X}(\mathbf{c}_{01}) = \mathbf{X}(\mathbf{c}_{00})\cos\theta_z + \mathbf{X}(\mathbf{c}_{11})\sin\theta_z$$

for $c_{00} \sim c_{01} \sim c_{11}$

▷ $\Upsilon^\times$ branches over all $z \in \diamondsuit$, $v \in G^\bullet$ and $u \in G^\circ$

▷ bosonization:

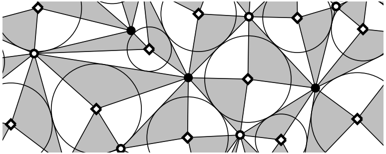
Ising $\leftrightarrow$ dimers on $\Upsilon^\bullet \cup \Upsilon^\circ$

[Wu–Lin'75, Dubédat'11]

▷ $\mathbb{C}$-solution of the propagation equation $\mathcal{X}$ $\rightsquigarrow$ s-embedding:

$$\mathcal{S}_\mathcal{X}(v_p^\bullet) - \mathcal{S}_\mathcal{X}(u_q^\circ) := (\mathcal{X}(c_{pq}))^2$$

$$\mathcal{Q}_\mathcal{X}(v_p^\bullet) - \mathcal{Q}_\mathcal{X}(u_q^\circ) := |\mathcal{X}(c_{pq})|^2$$



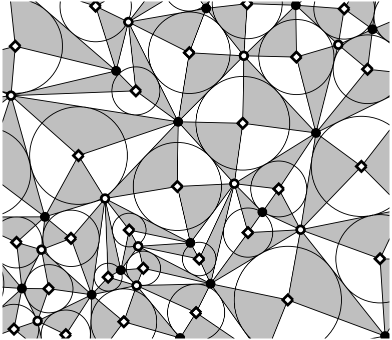
Planar Ising model (no magnetic field) and s-embeddings [arXiv:2006.14559v5]

Mass as the mean curvature of a surface in $\mathbb{R}^{2,1}$

- Let $\Theta^\delta = (\mathcal{S}^\delta; \mathcal{Q}^\delta) \to$ smooth $\Theta = (z, \vartheta(z))_{z\in\Omega}$. Let ζ be a conformal parametrization of Θ.
- Fermionic observables $\rightsquigarrow$ closed $F^\delta d\mathcal{S}^\delta + \overline{F}^\delta d\mathcal{Q}^\delta$
- If $F^\delta \to_{\delta\to 0} f$ and $\phi := z_\zeta^{1/2}\cdot f + \overline{z}_\zeta^{1/2}\cdot\overline{f}$, then

$$\boldsymbol{\partial_{\overline{\zeta}}\phi + im(\zeta)\overline{\phi} = 0,}$$

where $m(\zeta)$ is the mean curvature of Θ multiplied by its metric element in the parametrization ζ.



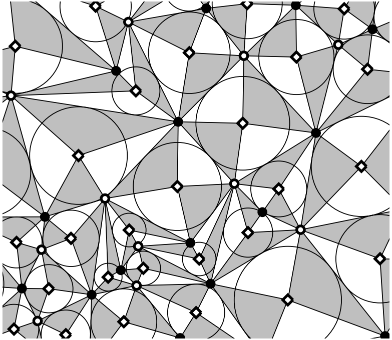
Planar Ising model (no magnetic field) and s-embeddings [arXiv:2006.14559v5]

Mass as the mean curvature of a surface in $\mathbb{R}^{2,1}$

- Let $\Theta^\delta = (\mathcal{S}^\delta; \mathcal{Q}^\delta) \to$ smooth $\Theta = (z, \vartheta(z))_{z\in\Omega}$. Let ζ be a conformal parametrization of Θ.
- Fermionic observables $\rightsquigarrow$ closed $F^\delta d\mathcal{S}^\delta + \overline{F}^\delta d\mathcal{Q}^\delta$
- If $F^\delta \to_{\delta\to 0} f$ and $\phi := z_\zeta^{1/2}\cdot f + \overline{z}_\zeta^{1/2}\cdot \overline{f}$, then

$$\partial_{\overline{\zeta}}\phi + im(\zeta)\overline{\phi} = 0,$$

where $m(\zeta)$ is the mean curvature of Θ multiplied by its metric element in the parametrization ζ.



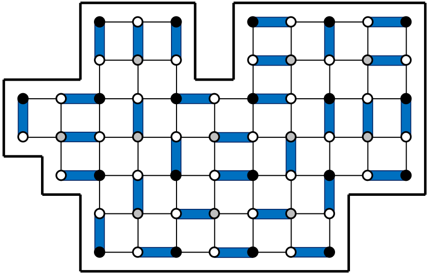
- Theorem (w/ R. Mahfouf and S.C. Park '25): Let $\mathcal{S}^\delta$ satisfy $\textsc{Unif}(\delta)$ and discrete surfaces Θ^δ are $O(\delta)$-close to a C^2-smooth surface $\Theta \subset \mathbb{R}^{2,1}$. Then, as $\delta \to 0$:
- Fermionic observables in arbitrarily rough domains $\to$ massive holomorphic spinors.
- If Θ is maximal, then Ising interfaces converge to SLE [in conformal ζ] curves on Θ.
- [UNIF(δ) is used near *rough boundaries:* analysis of b.c. is much heavier than for RW's...]

Bipartite dimer model: basics

- $(\mathcal{G}, \nu_{bw})$ – finite weighted bipartite planar graph (w/ marked outer face);
- *Dimer configuration* = perfect matching $\mathcal{D} \subset E(\mathcal{G})$: subset of edges such that each vertex is covered exactly once;
- *Probability* $\mathbb{P}(\mathcal{D}) \propto \nu(\mathcal{D}) = \prod_{e\in\mathcal{D}} \nu_e$;
- *Partition function* $\mathcal{Z}_\nu(\mathcal{G}) = \sum_{\mathcal{D}} \nu(\mathcal{D})$.

(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



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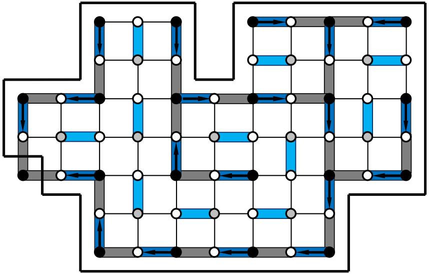


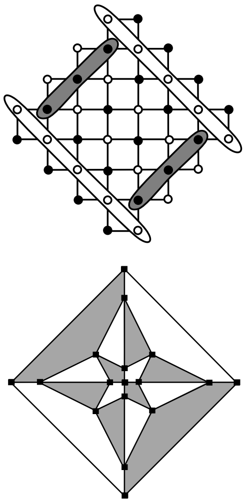
▷ Bipartite dimer model includes both ▷ the UST (via Temperley bijection)
▷ the planar Ising model (via bosonization)

▷ Theorem (Kasteleyn, 1961): given a planar graph $(\mathcal{G}, \nu)$, one can orient its edges so that $\mathcal{Z}_\nu(G) = |\mathrm{Pf}\, K_\nu| = |\det K_\nu|^{1/2}$, where $K_\nu = -K_\nu^\top$ is the signed adjacency matrix of G.
If $\mathcal{G}$ is bipartite, then K_ν is anti-block-diagonal and $|\,\mathrm{Pf}\, K_\nu| = |\det K_\nu|^{1/2} = \det |K_\nu^{\circ\to\bullet}|$.

T-embeddings of weighted bipartite planar graphs carrying the dimer model

[Kenyon–Lam–Ramassamy–Russkikh'18]
[C.–Laslier–Russkikh'20+'21] , [C.–Ramassamy'20]

- ▷ given $(\mathcal{G},\nu)$, find $\mathcal{T}:\mathcal{G}^*\to\mathbb{C}$ [$\mathcal{G}^*$ – *augmented dual*] s.t.
  - ◦ weights ν_{bw} are *gauge equivalent* to $\chi_{bw}:=|\mathcal{T}(v')-\mathcal{T}(v)|$ (i.e., $\nu_{bw}=g_b\chi_{bw}g_w$ for some $g:B\cup W\to\mathbb{R}_+$);
  - ◦ $\mathcal{T}$ is proper: tiles do not overlap;
  - ◦ at each *inner* vertex $\mathcal{T}(v)$, the sum of black angles $=\pi$.



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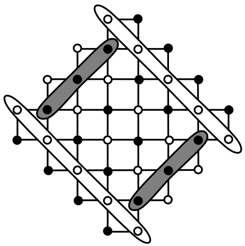

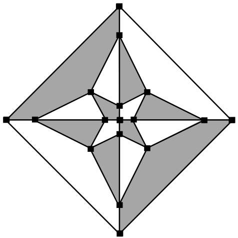

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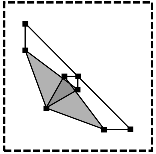
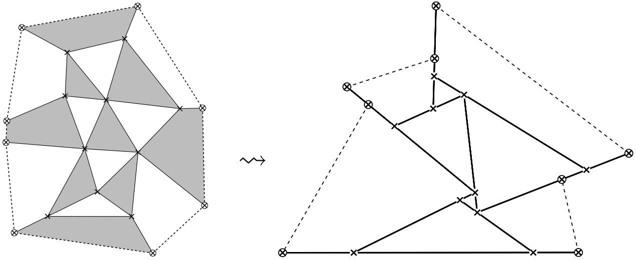
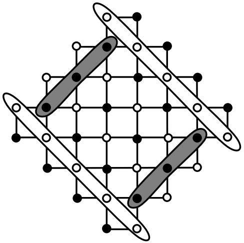
- ▷ *origami* $\mathcal{O}:\mathcal{G}^*\to\mathbb{C}$ *"fold $\mathcal{T}$ along all segments"*
- ▷ *t-surface* $(\mathcal{T},\mathcal{O}):\mathcal{G}^*\to\mathbb{R}^{2,2}$ can be thought of as a piece-wise linear surface with light-like faces
- ▷ isometries of $\mathbb{R}^{2,2}\rightsquigarrow$ gauge equivalent weights

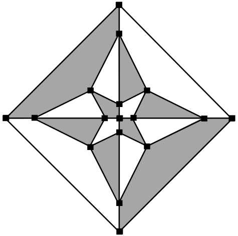


T-embeddings of weighted bipartite planar graphs carrying the dimer model

▷ t-surface $(\mathcal{T},\mathcal{O})$ $\rightsquigarrow$ family of T-graphs $\mathcal{T}+\alpha^2\mathcal{O}$, $|\alpha|=1$







▷ *t-holomorphic functions* $F^\circ : W \to \mathbb{C}$ are

$\overline{\alpha}\cdot$(gradients of harmonic on $\mathcal{T}+\alpha^2\mathcal{O}$)

$\rightsquigarrow$ closed forms $F^\circ d\mathcal{T} + \overline{F}^\circ d\overline{\mathcal{O}}$ [and similarly $F^\bullet d\mathcal{T} + \overline{F}^\bullet d\mathcal{O}$]

▷ this (a) does not[!] depend on α; *(b) respects isometries of* $\mathbb{R}^{2,2}$

T-embeddings of weighted bipartite planar graphs carrying the dimer model

▷ **A priori regularity theory** under two assumptions [CLR20]:

∘ $\text{Lip}(\kappa,\delta)$, $\kappa<1$ (quantitatively space-like above scale δ):

$$|\mathcal{T}^\delta(v')-\mathcal{T}^\delta(v)|<\delta \;\Rightarrow\; |\mathcal{O}^\delta(v')-\mathcal{O}^\delta(v)|\leqslant\kappa\cdot|\mathcal{T}^\delta(v')-\mathcal{T}^\delta(v)|$$

$\Rightarrow$ Hölder-type regularity of t-holomorphic functions

∘ $\text{Exp-Fat}(\delta,\delta')$, $\delta\leqslant\delta'\to 0$ (exponential non-degeneracy):

if one removes all $\delta\exp(-\delta'\delta^{-1})$-fat triangles from $\mathcal{T}^\delta$, then the diameter of remaining vertex-connected components $\leqslant\delta'$

$\Rightarrow$ Lipschitz-type regularity of harm. functions on $\mathcal{T}+\alpha^2\mathcal{O}$

▷ *t-holomorphic functions* $F^\circ : W\to\mathbb{C}$ are

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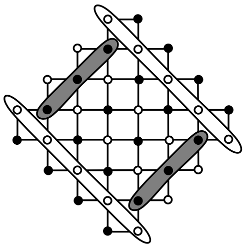
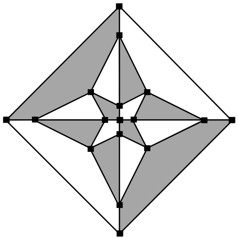




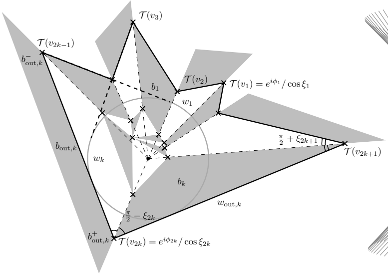
T-embeddings of weighted bipartite planar graphs carrying the dimer model

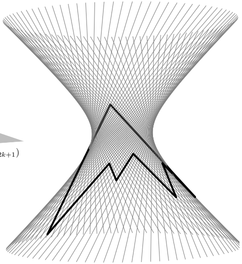
Perfect t-embeddings [CLR21]: outer vertices belong to the intersection of a light-cone in $\mathbb{R}^{2,2}$ and a (2,1)-hyperplane.

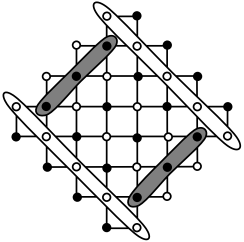
Q: Why should we care? **A:** They give a correct gauge!

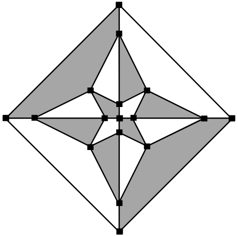
Theorem: under Lip and Exp-Fat on compacts, one has $(\mathcal{K}^\delta)^{-1} = O(1)$ in the bulk as $\delta \to 0$.

▷ one can move $\mathbb{R}^{2,2}$ such that this (2,1)-space is $\{\mathrm{Im}\,\theta = 1\}$:









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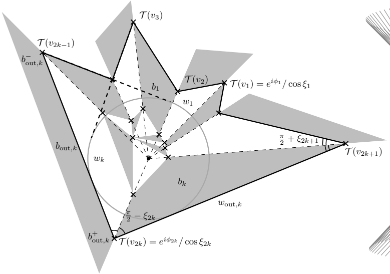
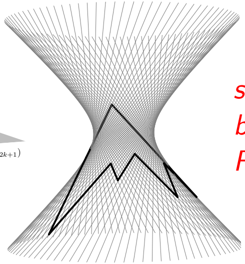
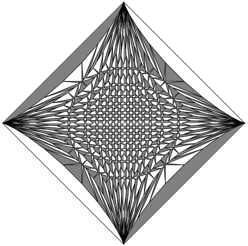


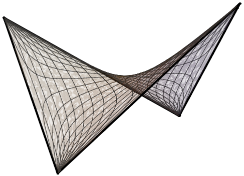


see next talk by Marianna Russkikh [!]

Aztec: [deg $v_{\mathrm{out}} = 4$ $\Rightarrow$ all t-emb are perfect]



$\downarrow$ (as $N \to \infty$)



Open question #1:

Is there an intrinsic argument that guarantees the convergence of $\textsc{Unif}(\delta)$ t-embeddings with periodic dimer weights to *maximal* 2-surfaces $\Theta \subset \mathbb{R}^{2,2}$?

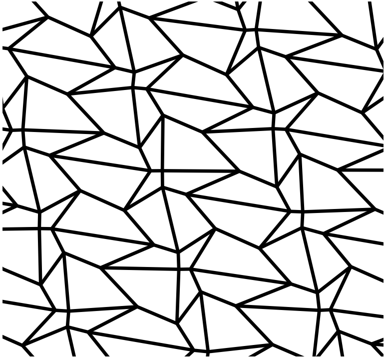
Simplest version: Let $\mathcal{T}^\delta$ be tilings of a fixed region $\Omega \subset \mathbb{C}$ such that

- ▷ $\mathcal{T}^\delta$ have combinatorics of $\mathbb{Z}^2$, all tiles are of size $\asymp \delta$, angles are uniformly bounded;
- ▷ angle condition holds $\rightsquigarrow \Theta^\delta = (\mathcal{T}^\delta; \mathcal{O}^\delta)$;
- ▷ at each vertex (n, m) one has

$$\|\Theta^\delta(n+1, m) - \Theta^\delta(n-1, m)\|_{2,2} = \|\Theta^\delta(n, m+1) - \Theta^\delta(n, m-1)\|_{2,2}.$$

Assume that $\Theta^\delta = (\mathcal{T}^\delta; \mathcal{O}^\delta) \to \Theta \subset \mathbb{R}^{2,2}$

Prove[?!] that Θ is a maximal 2-surface.



[Aztec $N = 1600$ near $(0.4; 0.25)$]

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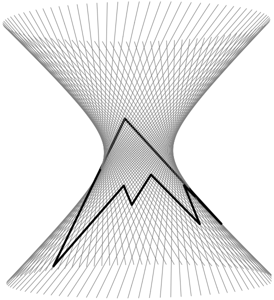
Open question #2:

Prove/disprove the existence of perfect t-embeddings for generic finite bipartite weighted graphs $(\mathcal{G}; \nu_{bw})$.

Particular case (Ising): perfect s-embeddings

Given a collection of quads in $\mathbb{R}^{2,1}$ with light-like sides and fixed ratios of diagonals, can one always scale them and assemble together in a prescribed way so that *the boundary belongs to the hyperboloid?*



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Open question #3:

Zeroes of $\mathcal{Z}_{\text{Ising}}[x] = 0 \rightsquigarrow$ discrete surfaces **in [??]**

- ▷ $\mathcal{Z}_{\text{Ising}}[(x_e)_{e\in E}] = \sum_{\text{even subgraphs } G'\subset G} \prod_{e\in G'} x_e$
- ▷ s-embeddings(immersions) into $\mathbb{R}^{2,1} \rightsquigarrow x \in \mathbb{R}$
- ▷ Livine–Bonzom [triangulations]: $\dim_{\mathbb{R}} = \#E - 1$ half-dim set of solutions $x \in \mathbb{C}$ via polyhedra in $\mathbb{R}^3$

Motivation for #3:

Recent work of Livine–Bonzom [2405.01253; Phys. Rev. D'25] on 3d quantum gravity:

- ▷ $x_e = e^{\frac{i}{2}\theta} \cdot (\tan \frac{1}{2}\phi_1 \tan \frac{1}{2}\phi_2)^{\frac{1}{2}}$ half-dim set of Ising zeroes from triangulations in $\mathbb{R}^3$
- ▷ Proved by Lis [2409.19639]

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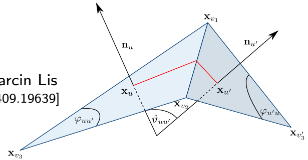

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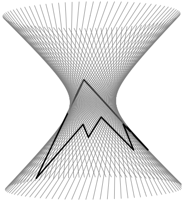
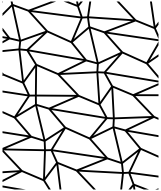
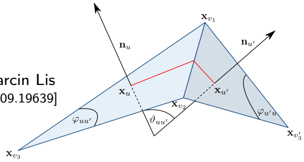
Thank you!







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