Finite subdivision rules

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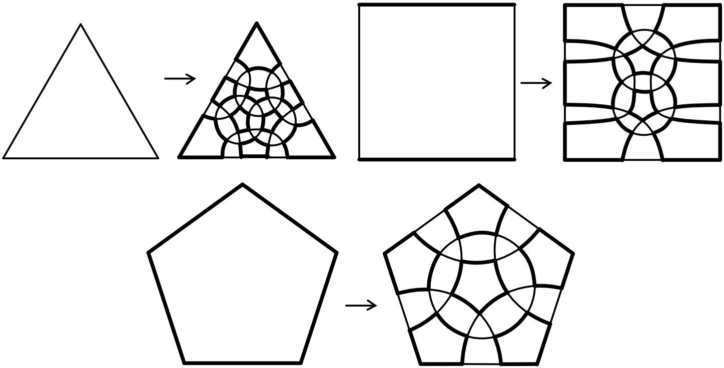
Finite subdivision rules

- A finite subdivision rule is essentially a finite combinatorial scheme for recursively subdividing planar complexes. The data consists of finitely many polygonal tiles, called *tile types*, together with instructions on how to compatibly subdive each tile type into tiles that correspond to tile types.
- There is a model subdivision complex $S_{\mathcal{R}}$ which is the union of its closed 2-cells, a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and a cellular subdivision map $\sigma_{\mathcal{R}}\colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ which takes each open cell of $\mathcal{R}(S_{\mathcal{R}})$ homeomorphically to an open cell of $S_{\mathcal{R}}$.
- By iterating the subdivision map, one can recursively subdivide the model subdivision complexes and complexes that are modeled on it.
- C, F, andPa, Finite subdivision rules, *Conform. Geom. Dyn.* **5** (2001), 153–196 (electronic).

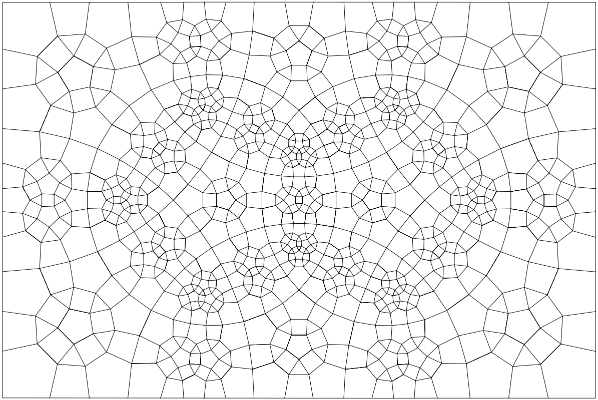
Example 1. The dodecahedral subdivision rule

The model subdivision complex has one vertex and two edges, and is too hard to draw. Here are the subdivisions of the three tile types.



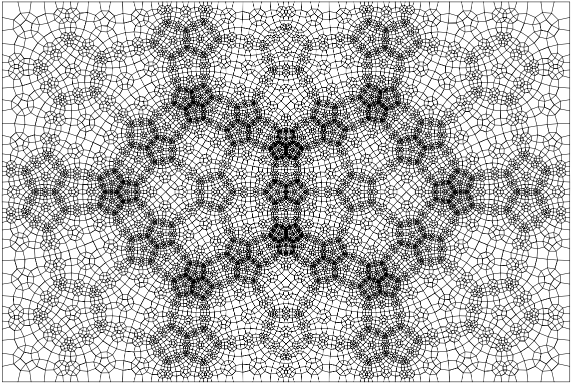
The second subdivision of the quadrilateral tile type

This is drawn using CirclePack.

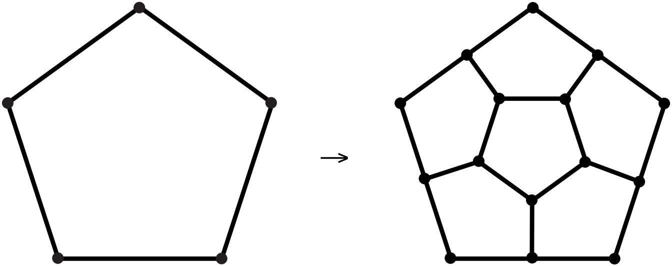


The third subdivision of the quadrilateral tile type

This is drawn using CirclePack.



The model subdivision complex has one vertex and one edge. Here is the subdivision of the tile type.



What are finite subdivision rules good for?

- For the first ten years or so, they were mostly used for a (hard) toy problem in our approach to Cannon's conjecture. They were used to model the recursion at infinity for a Gromov-hyperbolic group with boundary a 2-sphere.
- In the last twenty years or so, they've been mostly used in complex dynamics. If a finite subdivision rule has model subdivision complex a 2-sphere, then the subdivision map is postcritically finite.

Conjecture: If G is a Gromov-hyperbolic discrete group whose space at infinity is S^2, then G acts properly discontinuously, cocompactly, and isometrically on $\mathbb{H}^3$.

- Suppose G is a group and Γ is a locally finite Cayley graph. G is *Gromov-hyperbolic* if Γ has *thin triangles*.
- Points in the space at infinity are equivalence classes of geodesic rays; $R \sim S$ if $\sup\{d(R(t), S(t)) : t \geq 0\} < \infty$.

How do you proceed from combinatorial/topological hypotheses to an analytic conclusion?

Weight functions, combinatorial moduli

- *shingling* (locally-finite covering by compact, connected sets) $\mathcal{T}$ on a surface S, ring (or quadrilateral) $R \subset S$
- *weight function* ρ on $\mathcal{T}$: $\rho\colon \mathcal{T} \to \mathbb{R}_{\geq 0}$
- ρ*-length* of a curve, ρ*-height* H_ρ of R, ρ-area A_ρ of R, ρ*-circumference* C_ρ of R
- *moduli* $M_\rho = H_\rho^2/A_\rho$ and $m_\rho = A_\rho/C_\rho^2$
- *moduli* $M(R) = \sup_\rho H_\rho^2/A_\rho$ and $m(R) = \inf_\rho A_\rho/C_\rho^2$
- The sup and inf exist, and are unique up to scaling. (This follows from compactness and convexity.)

Combinatorial Riemann Mapping Theorem

- Now consider a sequence of shinglings of S.
- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli
- **Axiom 2.** Existence of local rings with large moduli
- *conformal sequence* of shinglings: Axioms 1 and 2, plus mesh locally approaching 0.

Theorem (Cannon): If $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on a topological surface S and R is a ring in S, then R has a metric which makes it a right-circular annulus such that analytic moduli and asymptotic combinatorial moduli on R are uniformly comparable.

- J. W. Cannon, The combinatorial Riemann mapping theorem, *Acta Math.* **173** (1994), 155–234.

Cannon-Swenson

- G a Gromov-hyperbolic group, Γ a locally finite Cayley graph, base vertex $\mathcal{O}$
- space at infinity Γ_∞: points are equivalence classes of geodesic rays based at $\mathcal{O}$
- *half-space* $H(R,n) = \{x \in \Gamma : \ d(x, R([n,\infty)) \le d(x, R([0,n])\}$
- *disk at infinity* $\mathcal{D}(R,n) = \{[S] \in \Gamma_\infty : \ \lim_{t\to\infty} d(S(t), \Gamma \setminus H(R,n)) = \infty\}$
- cover $\mathcal{D}(n) = \{\mathcal{D}(R,n) : \ R$ is a geodesic ray based at $\mathcal{O}\}$

Cannon-Swenson

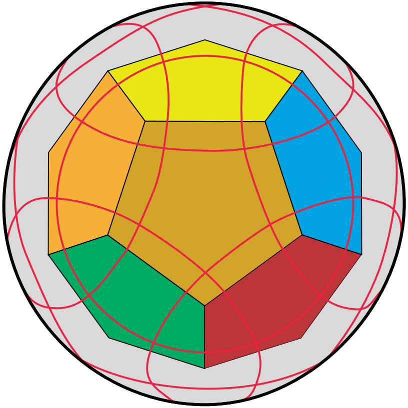
Theorem (Cannon-Swenson): In the setting of Cannon's conjecture, it suffices to prove that the sequence $\{\mathcal{D}(n)\}_{n\in\mathbb{N}}$ is conformal. Furthermore, the $\mathcal{D}(n)$'s satisfy a linear recursion.

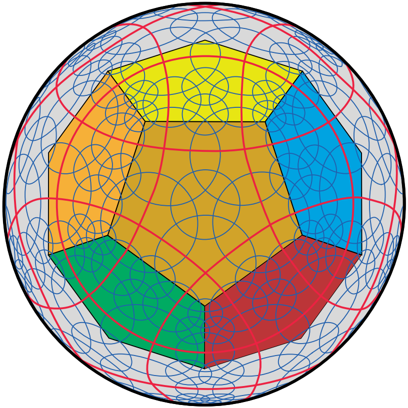
- J. W. Cannon, E. L. Swenson, Recognizing constant curvature groups in dimension 3, *Trans. Amer. Math. Soc.* **350** (1998), 809–849.
- The disks at infinity give a basis for the topology of Γ_∞.
- The CRMT implies there is a quasiconformal structure on Γ_∞. It is quasiconformally equivalent to an analytic structure. The group action is uniformly quasiconformal so by Sullivan/Tukia it is conjugate to a conformal action.
- The linear recursion follows from finite cone types.

  Theorem (Cannon): If G is a cocompact, discrete group of isometries of hyperbolic space, then G has a linear recursion.
- J. W. Cannon, The combinatorial structure of cocompact discrete hyperbolic groups, *Geom. Dedicata* **16** (1984), 123–148.

This subdivision rule on the sphere at infinity

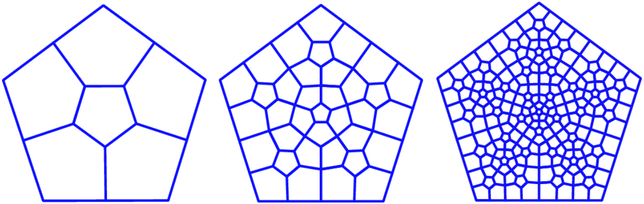
The dodecahedral subdivision rule comes from the recursion at infinity for a Kleinian group. The images are from SnapPea.



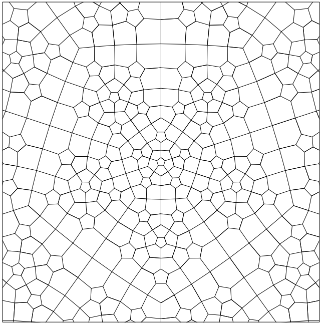


The shapes of tiles

- Given a sequence of subdivisions of a tiling, how do you understand/control the shapes of tiles?
- When can you realize the subdivisions so that the subtiles stay almost round?
- For the pentagonal expansion complex, here are the first three subdivisions of the tile type, drawn using CirclePack.



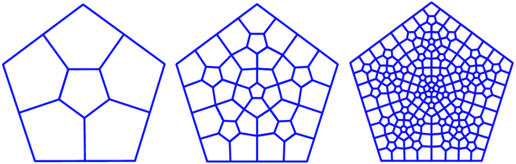
The pentagonal expansion complex



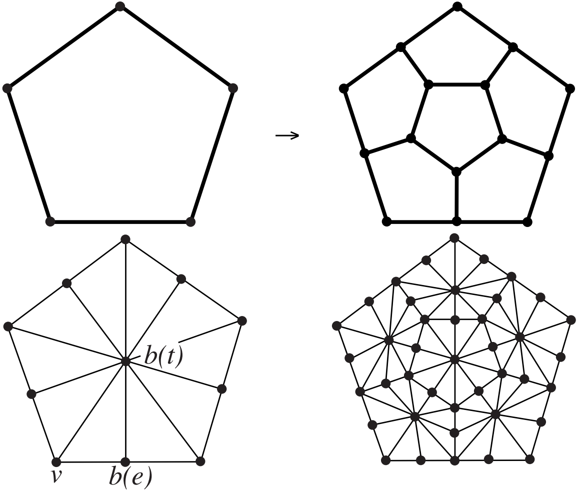
- P.L. Bowers and K. Stephenson, A “regular” pentagonal tiling of the plane, *Conform. Geom. Dyn.* **1** (1997), 58–68 (electronic.)

Expansion complexes

- An expansion $\mathcal{R}$-complex is an $\mathcal{R}$-complex X with structure map $f\colon X \to S_{\mathcal{R}}$ such that X is homeomorphic to $\mathbb{R}^2$ and there is an orientation-preserving homeomorphism $\varphi\colon X \to X$ with $\sigma_{\mathcal{R}} \circ f = f \circ \varphi$.
- Expansion complexes arise as direct limits of sequences of subdivisions.
- For the pentagonal expansion complex, Bowers and Stephenson showed that there is a conformal expanding map, $z \mapsto \lambda z$ which takes each subcomplex to its subdivision. What is λ? At the Barrett Lectures in 1998, Cannon, Kenyon, Parry and I used a rational map to find λ.

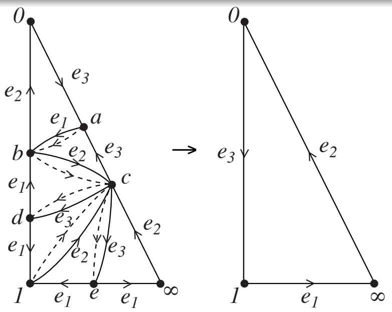


The pentagonal subdivision rule



- The pentagons, subdivided into triangles

A rational map for the triangular finite subdivision rule.



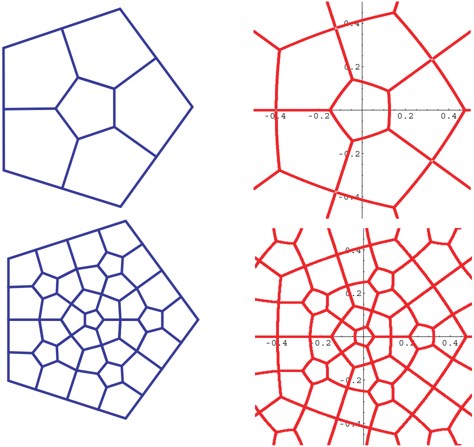
- The pentagonal subdivision rule is closely associated with a fsr (with: triangular tile types) which is realizable by the rational map

$$f(z) = \frac{2z(z + 9/16)^5}{27(z - 3/128)^3(z - 1)^2}.$$

The expansion constant for the pentagonal expansion complex is $f'(0)^{1/5} = (-324)^{1/5}$.

The pentagonal subdivision rule

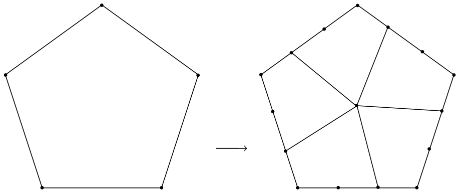
- Here are subdivisions drawn by CirclePack and by preimages under the rational map (unfolding by $z \mapsto z^5$).

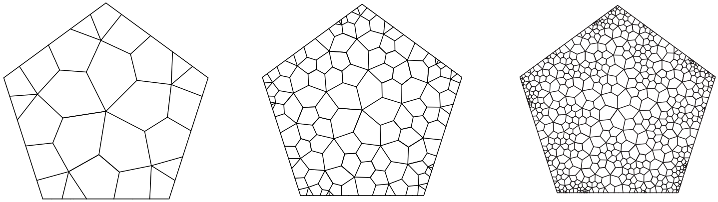


Rotational but not dihedral symmetry

- The pentagonal subdivision rule is conformal; the dihedral symmetry makes showing this much easier. For potential applications, you would like to be able to make use of rotational symmetry. The intuition is that for Cannon's Conjecture expansion complexes correspond to tangent spaces at infinity, and at fixed points of loxodromic elements you will see rotational (but not dihedral) symmetry.

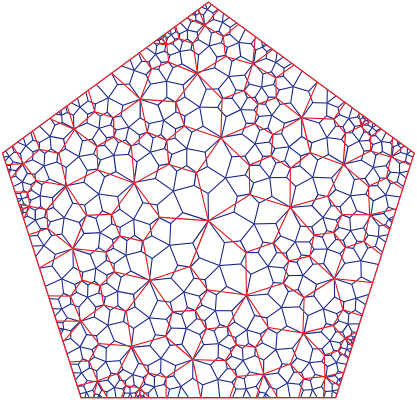
An example with rotational symmetry





Superimposed subdivisions

- Here are the third and fourth subdivisions, superimposed. Note the vertices.
- Because of the superposition of the vertices, one can prove conformality for rotationally invariant finite subdivision rules with a single tile type.



Thurston maps

- An orientation-preserving branched map $f\colon\ S^2 \to S^2$ is *postcritically finite* or a *Thurston map* if the set P_f of postcritical points is finite.
- Two such maps f and g are *equivalent* if there are homeomorphisms $h_1, h_2\colon\ S^2 \to S^2$ such that $h_i(P_f) = P_g$, $(h_1 \circ f)\big|_{P_f} = (g \circ h_2)\big|_{P_f}$, and h_1 is isotopic, rel P_f, to h_2.
- If a finite subdivision rule $\mathcal{R}$ is orientation-preserving and has model subdivision complex a 2-sphere, then the subdivision map σ is postcritically finite. (The postcritical points are vertices of $\mathcal{R}$.)
- Given a Thurston map, choose a circle going through the postcritical points. The map is *expanding* if the diameters of connected components of the preimages of the two open tiles converge to 0.

Realizing subdivision maps by rational maps

- When can the subdivision map of a finite subdivision rule be realized by a rational map?
- **Theorem (C-F-K-Pa):** Suppose $\mathcal{R}$ is an orientation-preserving finite subdivision rule which has bounded valence, mesh approaching zero, and subdivision complex a 2-sphere. If $\mathcal{R}$ is conformal (in the sense of Cannon), then it is equivalent to a rational map.
- C, F, K and Pa, Constructing rational maps from finite subdivision rules, *Conform. Geom. Dyn.* **7** (2003), 76–102 (electronic).

Realizing rational maps by fsr's

Theorem (C-F-Pa, Bonk-Meyer): If f is a critically finite rational map without periodic critical points (or, more generally, an expanding Thurston map), then every sufficiently large iterate of f is equivalent to the subdivision rule of a fsr.

- C, F, and Pa, Constructing subdivision rules from rational maps, *Conform. Geom. Dyn.* **11** (2007), 128–136 (electronic).
- M. Bonk and D. Meyer, Expanding Thurston Maps, Mathematical Surveys and Monographs, Vol. 225, 478 pp. Providence: Amer. Math. Soc. 2017
- Idea: pick a simple closed curve containing the postcritical points. For a sufficiently large iterate, that curve can be approximated by a curve in its preimage. Now use the expansion complex machinery.
- Do you need to pass to an iterate of the map?
- What about postcritically finite maps with periodic critical points? This corresponds to fsr's with unbounded valence.

Realizing rational maps by fsr's

Theorem(F-Pa-Pi): If f is a Böttcher expanding Thurston map, then every sufficiently large iterate of f is equivalent to a subdivision map.

- F. , Pa. and Pi., Expansion properties for finite subdivision rules II, *Conform. Geom. Dyn.* **24** (2020), 29–50.

Theorem(Cui-Gao-Zeng): If f is a rational Thurston map, then every sufficiently large iterate of f is a subdivision map.

- G. Cui, Y. Gao and J. Zeng, Invariant graphs in Julia sets and decompositions of rational maps, preprint (2024)

Application: Meyer's theorem

Theorem (Meyer, 2011): If f is an expanding Thurston map, then a sufficiently large iterate of f is a mating of two polynomials.

- D. Meyer, Invariant Peano curves of expanding Thurston maps, *Acta Math.* **210**, (2013), 95–171.