CRITICAL GAUGES FOR RANDOM TILINGS

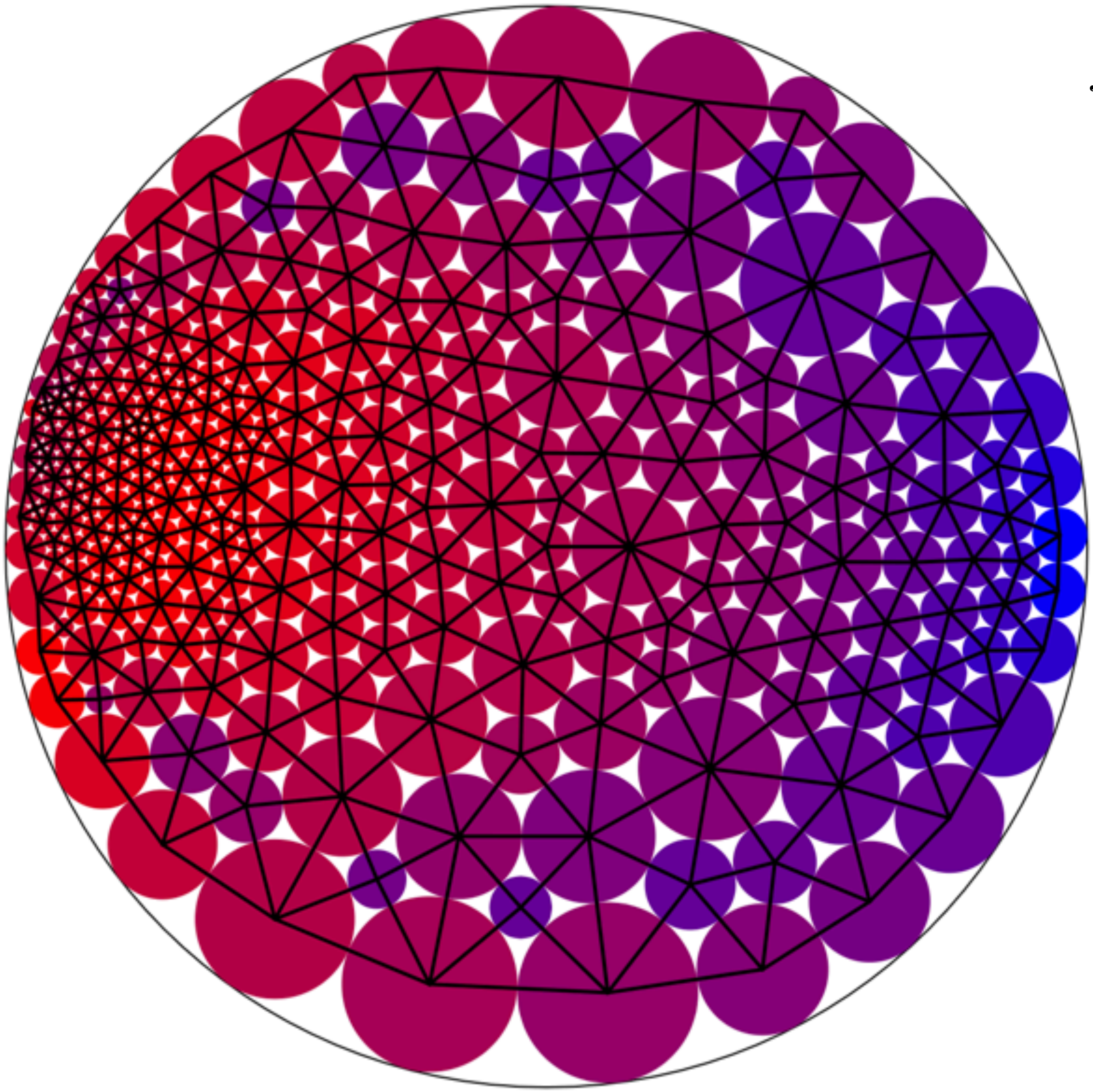
Richard Kenyon (Yale)

Catherine Wolfram (MIT)

based on earlier work with Cosmin Pohoata (Emory)

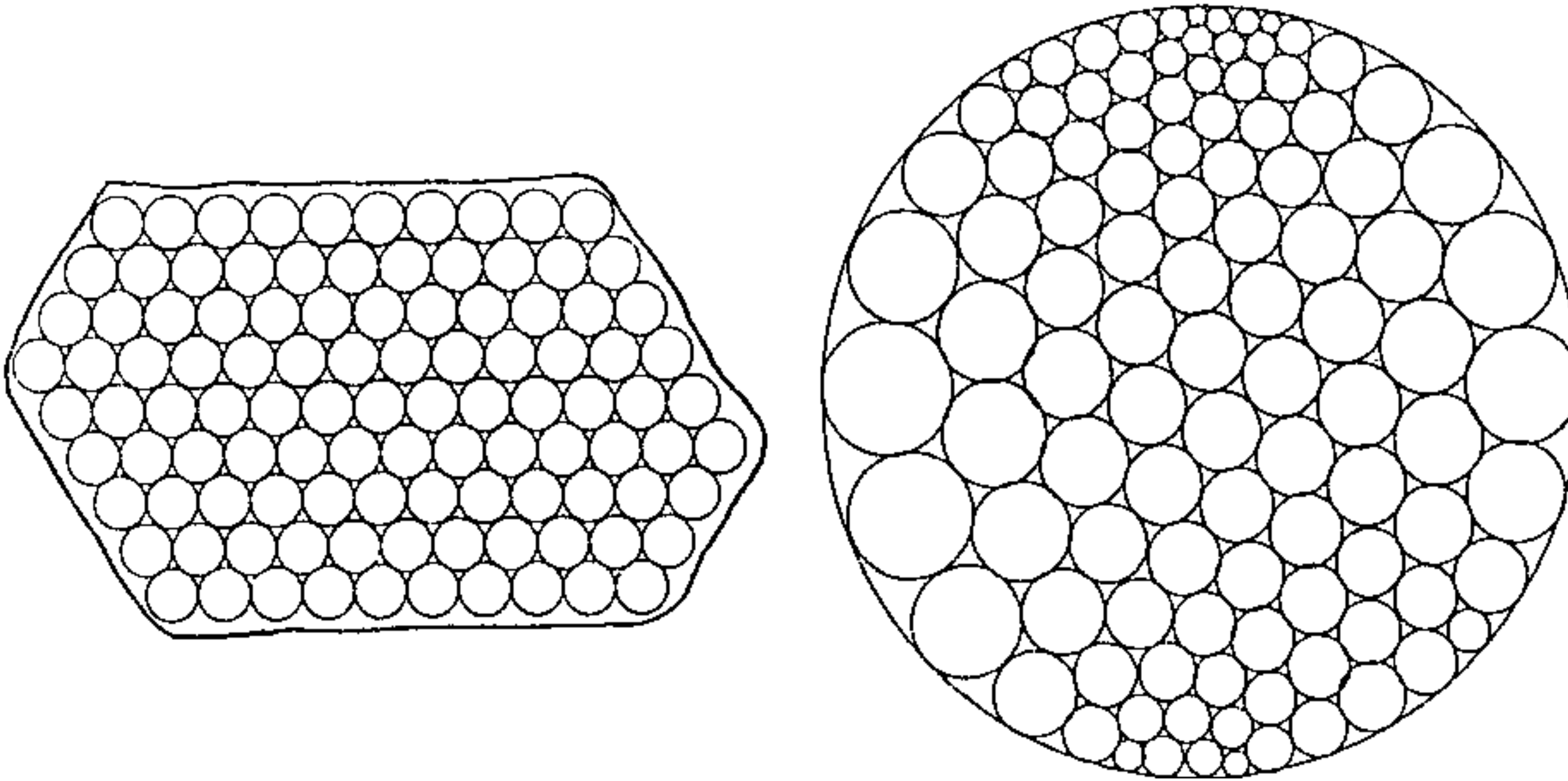
Circle packing: given a planar triangulation, find a function $f : V \to \mathbb{R}_+$

f="critical" radius



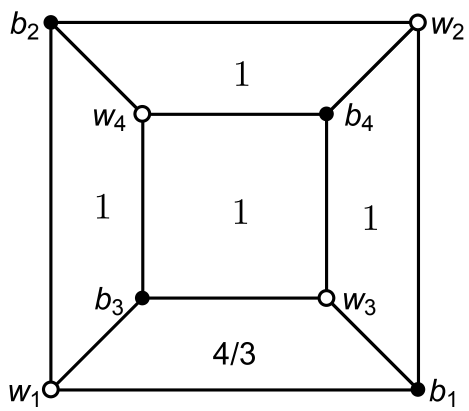
Colin-de-Verdière, Brägger, Springborn (...)

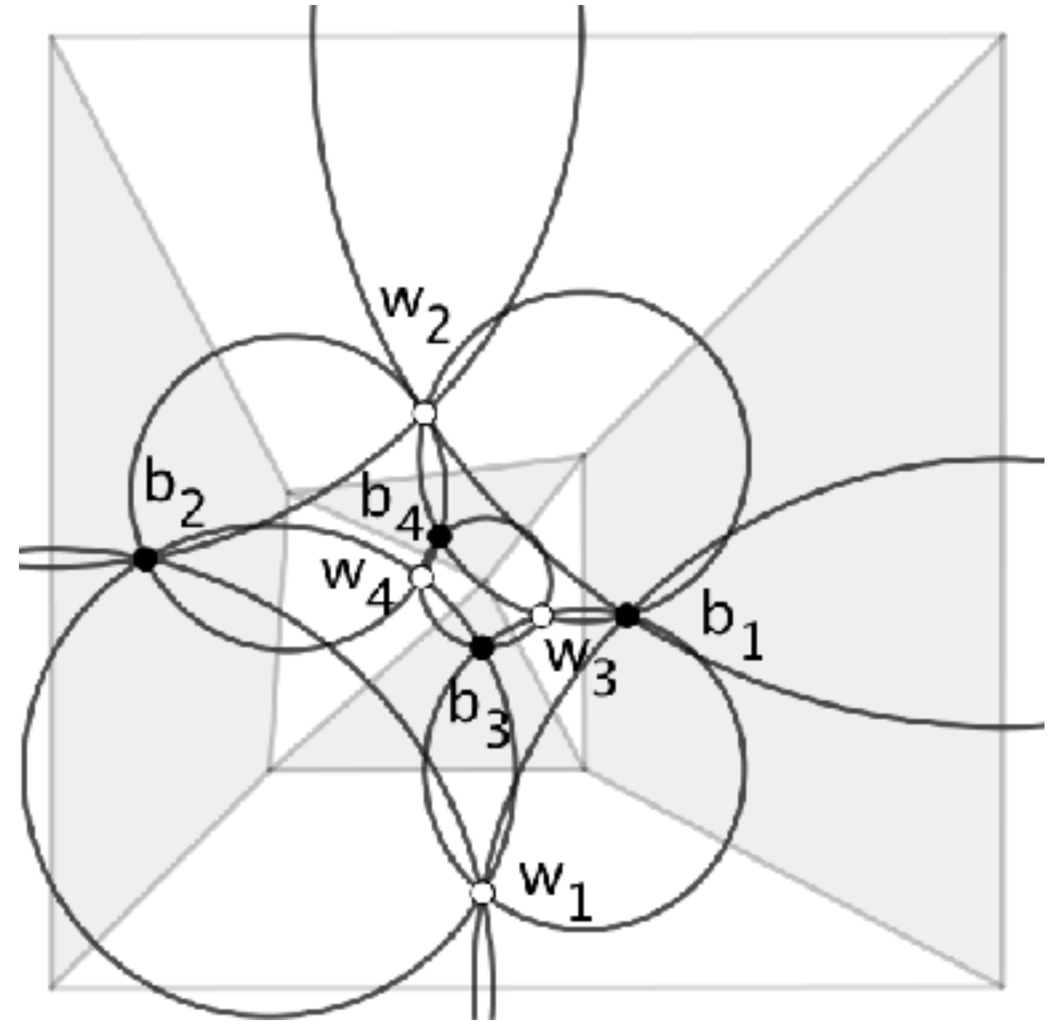
f is determined as the critical point of a convex functional $F(r_1, \ldots, r_V)$.



Riemann mapping via convergent circle packings

Thurston, Rodin-Sullivan





Circle pattern: Given a bipartite quadrangulation with positive face weights, embed it so that faces are cyclic and circle center distances have given biratios.

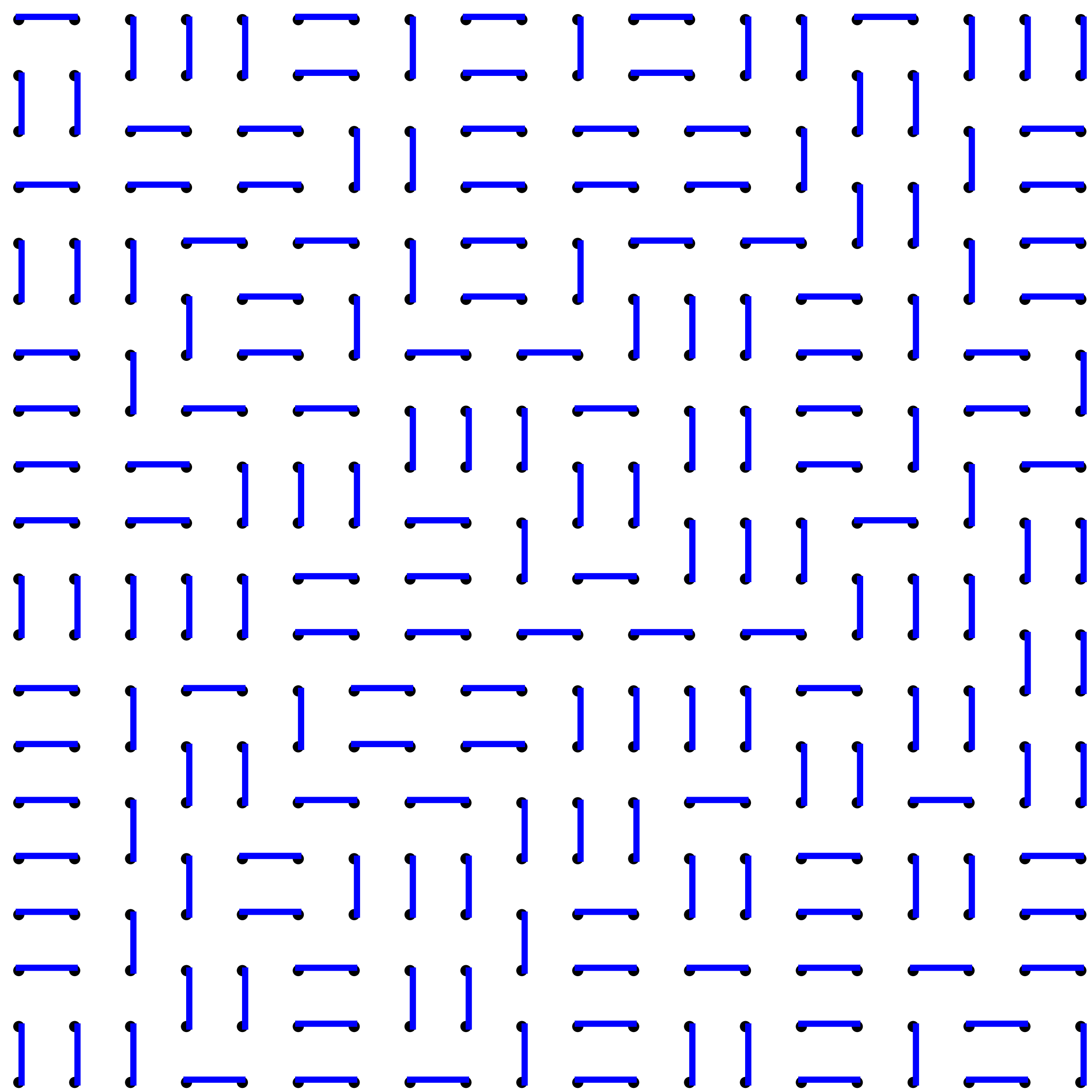
[KLRR '22] To do this find a "Coulomb gauge": Find $F : V \to \mathbb{C}$ so that

$$\sum_b K_{wb} F(b) = 0$$

$$\sum_w F(w) K_{wb} = 0$$

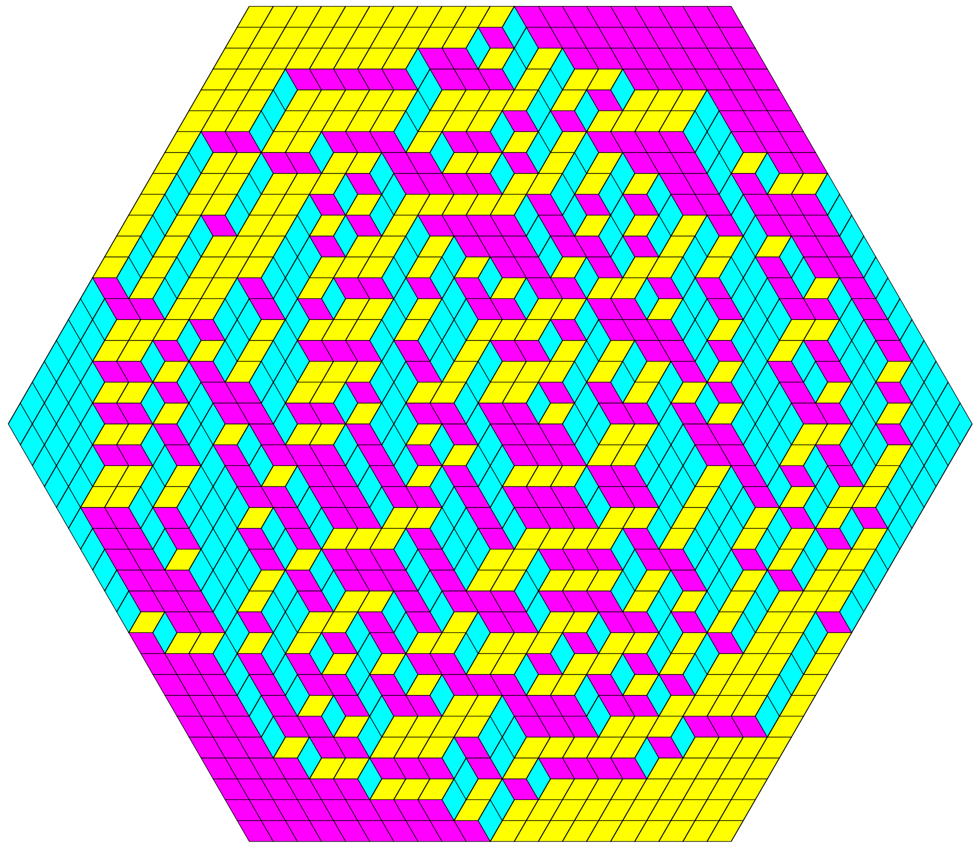
(and some boundary equations).

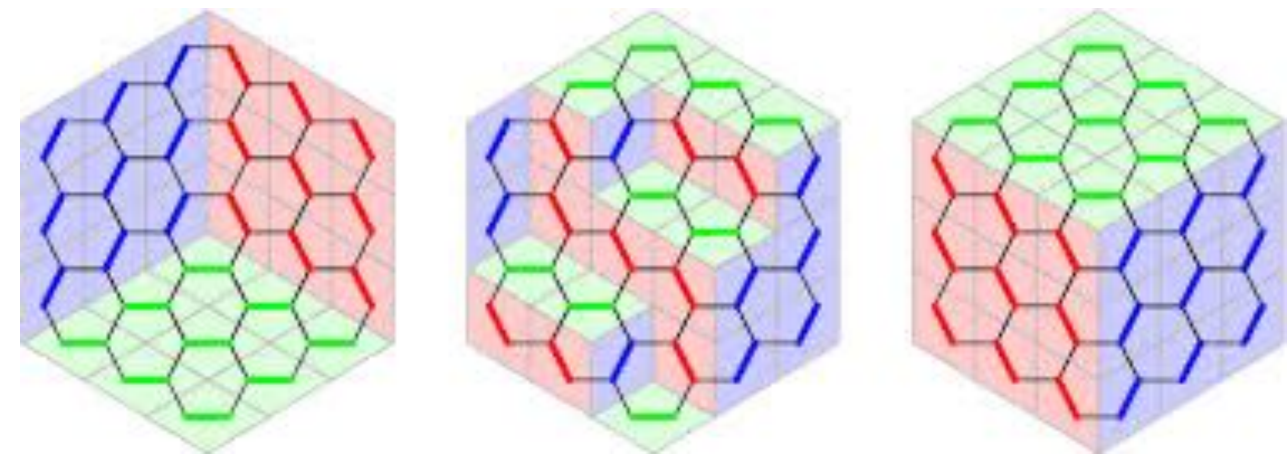
Dimer cover (or, perfect matching)



number of dimer covers of an $n \times n$ square [Kasteleyn, Temperley/Fisher 1961]

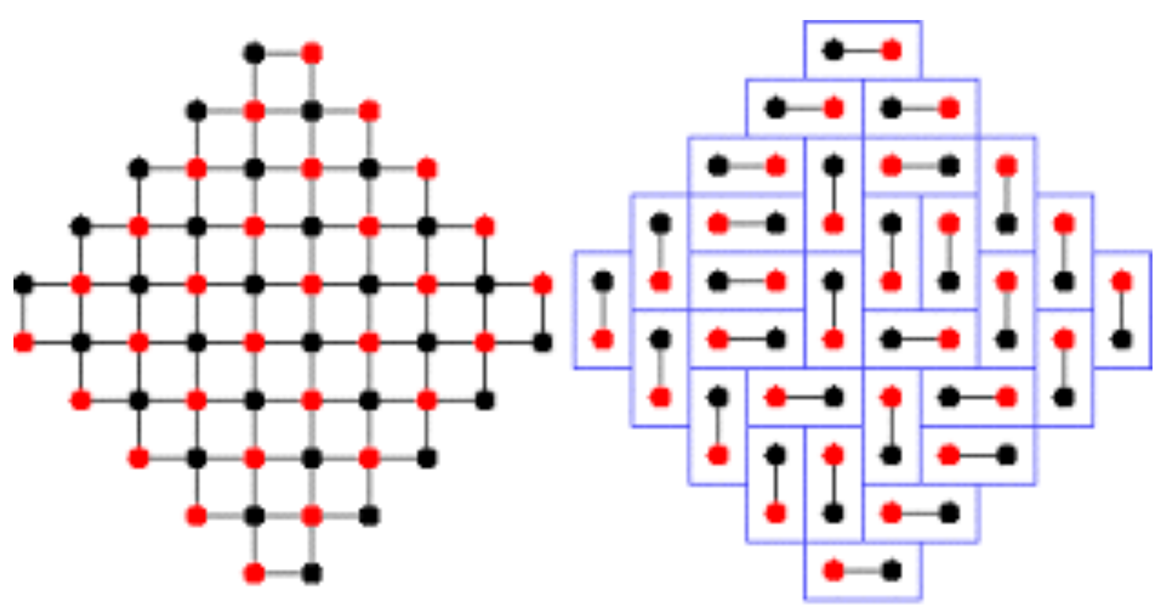
$$Z_{n\times n} = \left[\prod_{j=1}^{n}\prod_{k=1}^{n}\left(2\cos\frac{\pi j}{n+1} + 2i\cos\frac{\pi k}{n+1}\right)\right]^{1/2}$$





Number of "boxed plane partitions" $= \displaystyle\prod_{i,j,k=1}^{n} \frac{i+j+k-1}{i+j+k-2}$

Macmahon (1896)

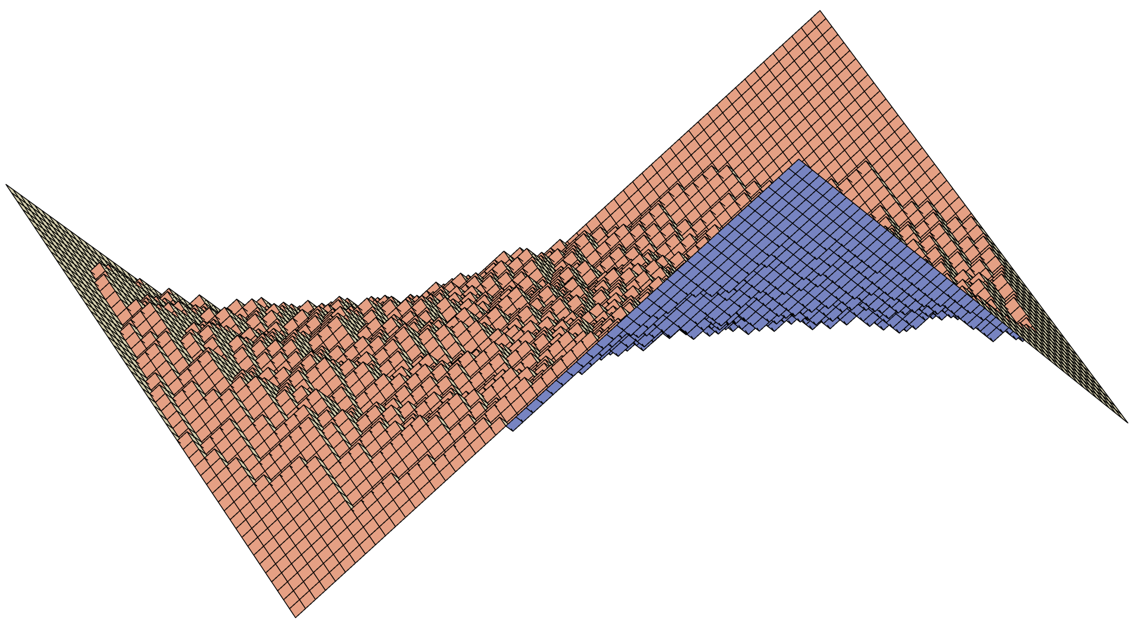


Number of dimer covers of the "Aztec diamond" $= 2^{\binom{n}{2}}$.

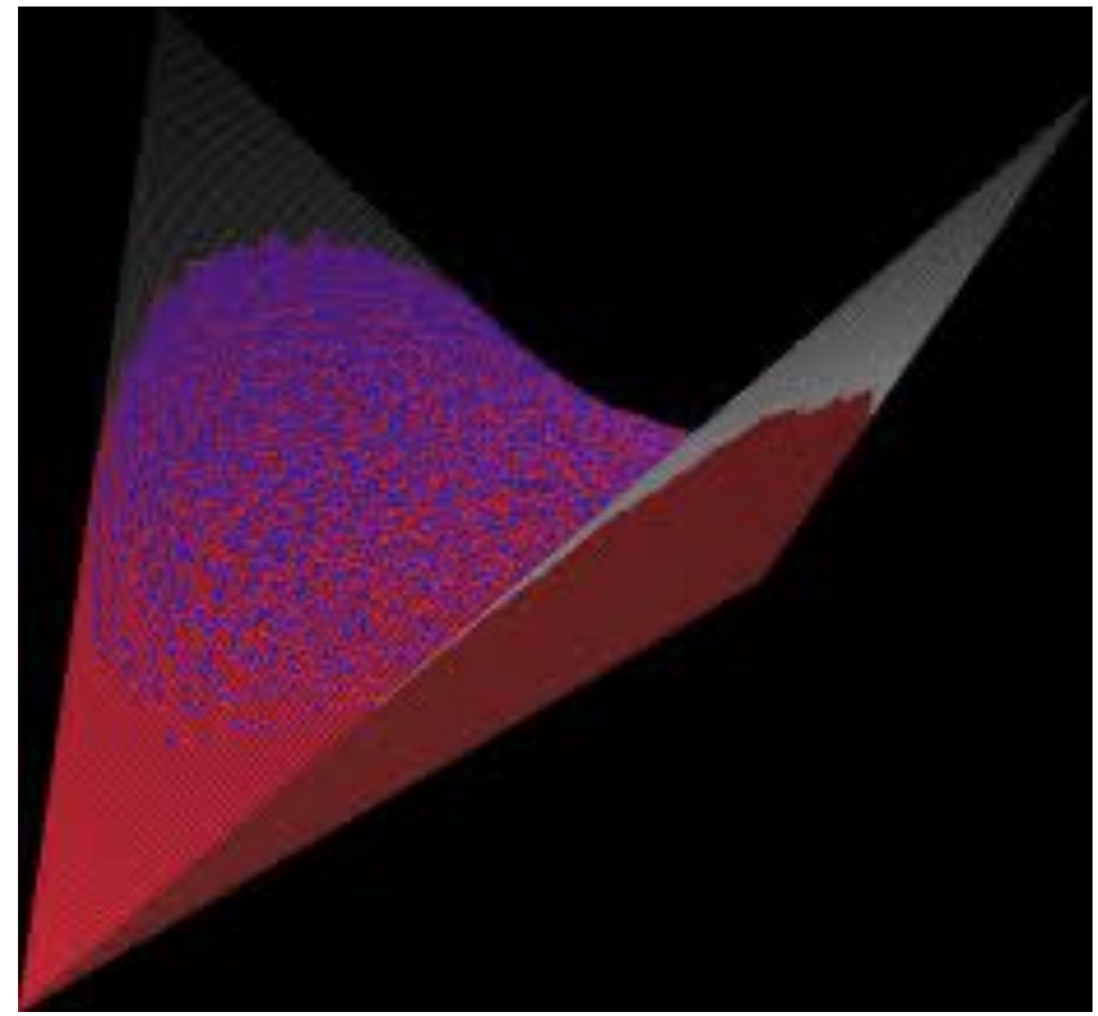
Elkies Kuperberg Larsen Propp (1992)

Limit shapes

(energy-minimizing surfaces)



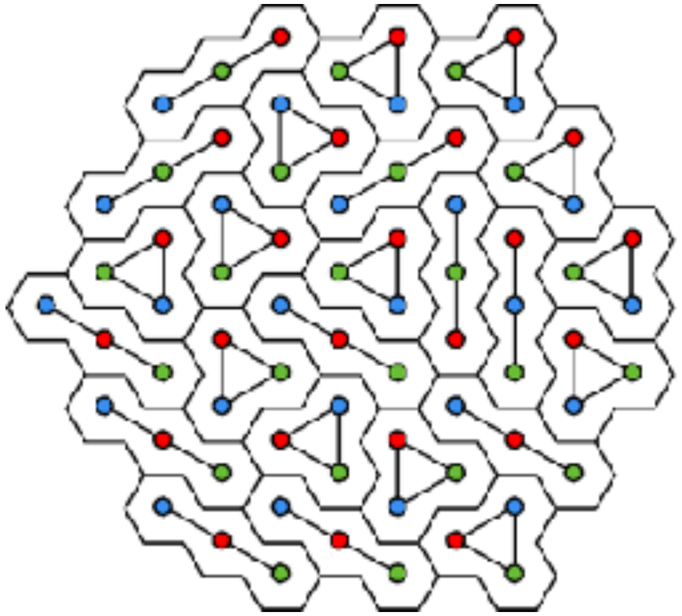
Cohn Larsen Propp '98



[picture by A. Borodin]

Cohn Elkies Propp '96

Counting tilings of regions with fixed tile shapes is NP-hard.



We define a variant of the tiling problem with easier (asymptotic) counting:

The multinomial tiling problem [K-Pohoata 2021].

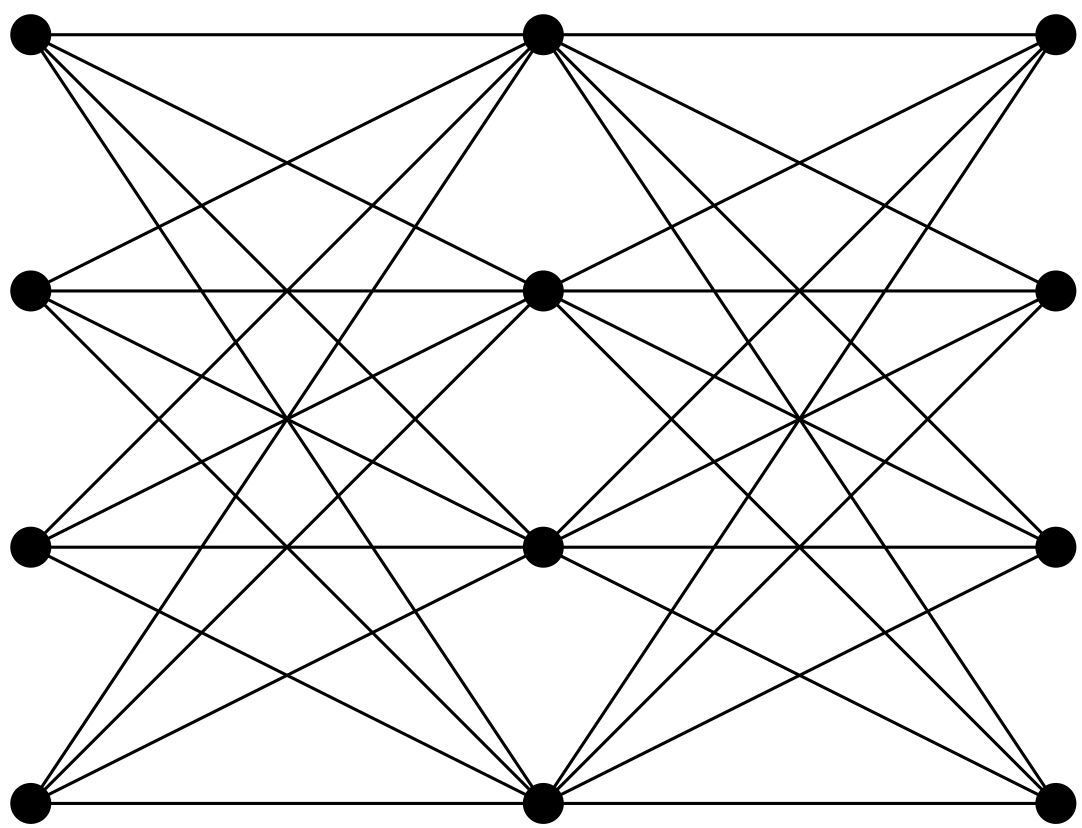
$G = (V, E)$ is a finite graph. Let $T = \{t_1, \ldots, t_k\}$ be the tiles; $t_i \subset V$.

Let G_n be the n-fold *blow-up graph* of G:

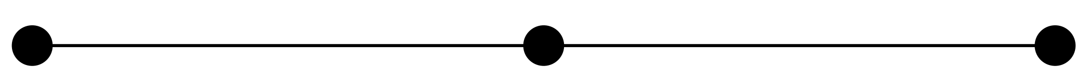
G_n has vertices $V \times \{1, 2, \ldots, n\}$

G_n has edges $(u, i) \sim (v, j)$ whenever $u \sim v$.

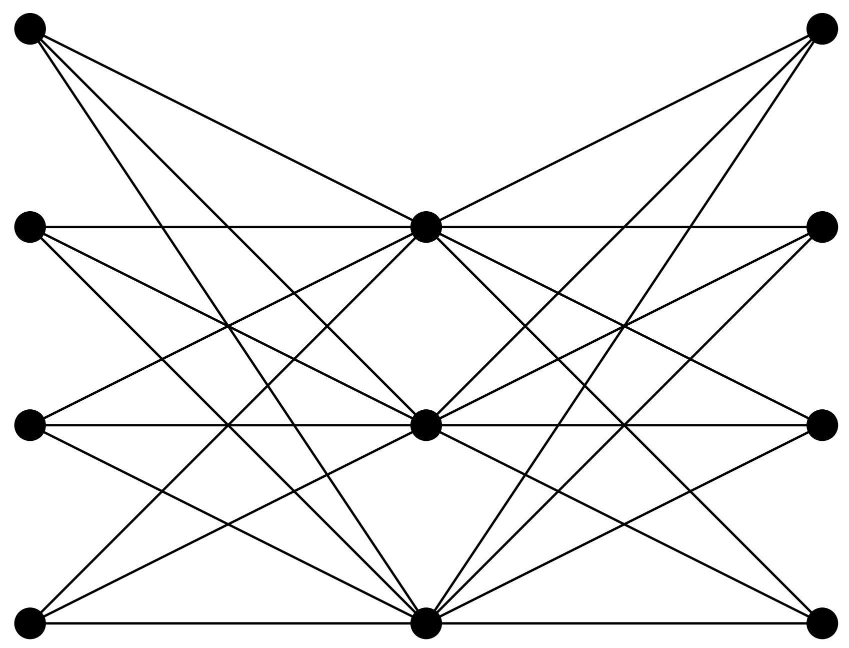
G_4



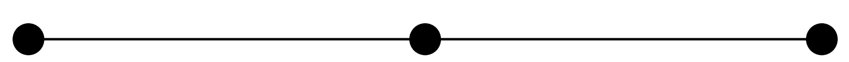
G



We can also let n vary from vertex to vertex: $\mathbf{n} = (n_1, \ldots, n_V)$.



$G_{\mathbf{n}}$



G

Let $Z(\mathbf{n})$ be the number of tilings of G_n by lifts of $\{t_1, \ldots, t_k\}$.

Let x_v a variable for each vertex v of G.

Let $P(\mathbf{x}) = \sum_t \prod_{v \in t} x_v$ be the "tiling polynomial".

Thm [K'-Pohoata 2021]:

$$Z := \sum_{\mathbf{n} \geq 0} Z(\mathbf{n}) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = e^P.$$

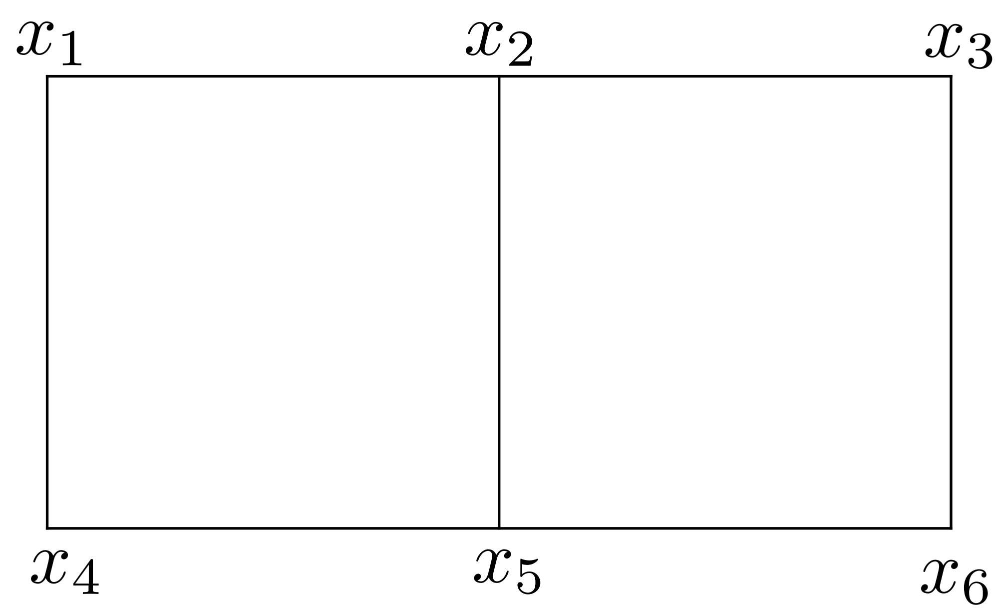
$$\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} := \prod_v \frac{x_v^{n_v}}{n_v!}$$

Note: if use K tiles: $\frac{P^k}{K!}$.

We'll now assume that $T = \{\text{edges of } G\}$ (i.e. the dimer model).

Probabilistic interpretation

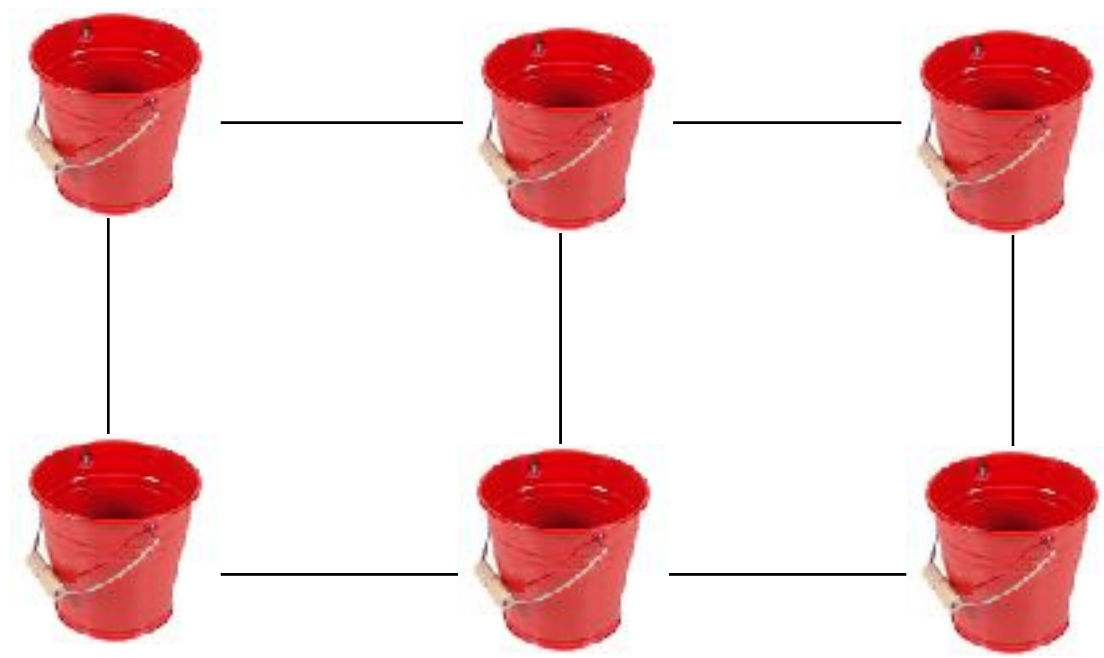
Think of P as a (scaled) probability generating function.



$$P = x_1x_2 + x_1x_4 + x_2x_3 + x_2x_5 + x_3x_6 + x_4x_5 + x_5x_6$$

Probabilistic interpretation

Think of P as a (scaled) probability generating function.

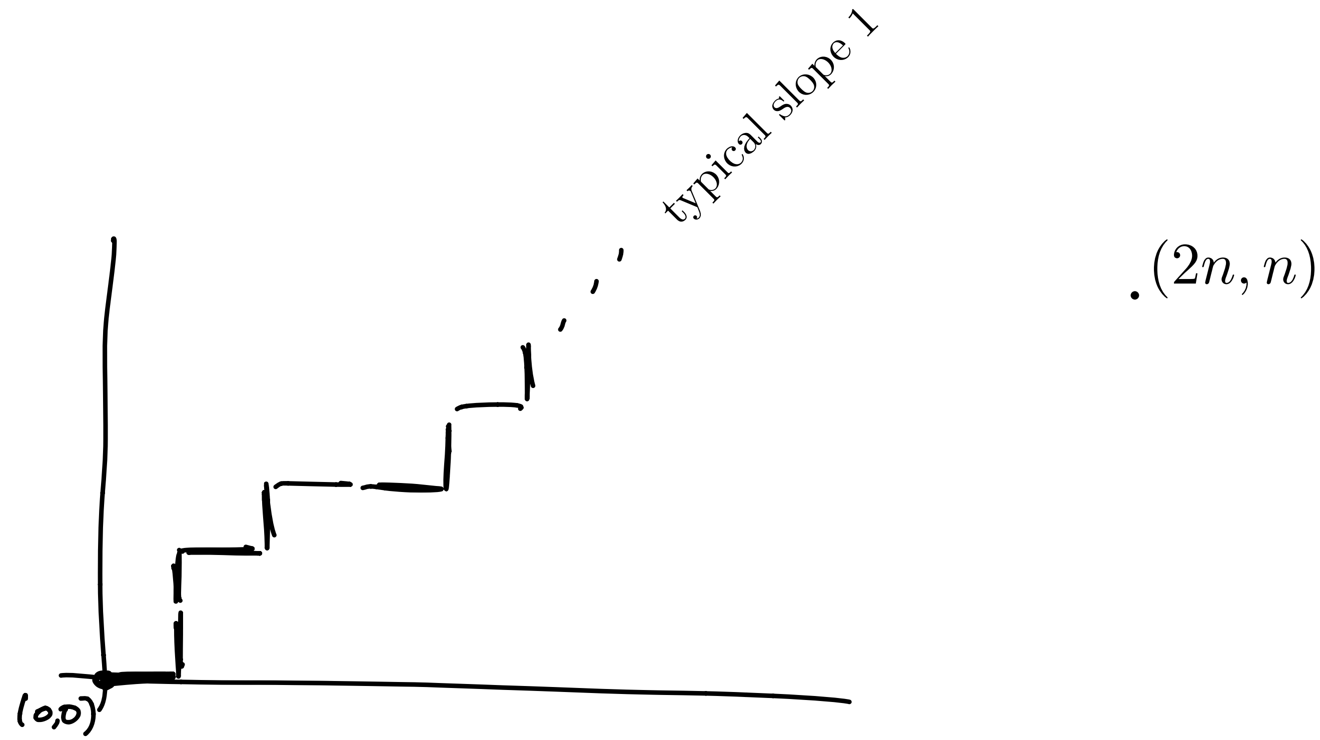


$$P = x_1x_2 + x_1x_4 + x_2x_3 + x_2x_5 + x_3x_6 + x_4x_5 + x_5x_6$$

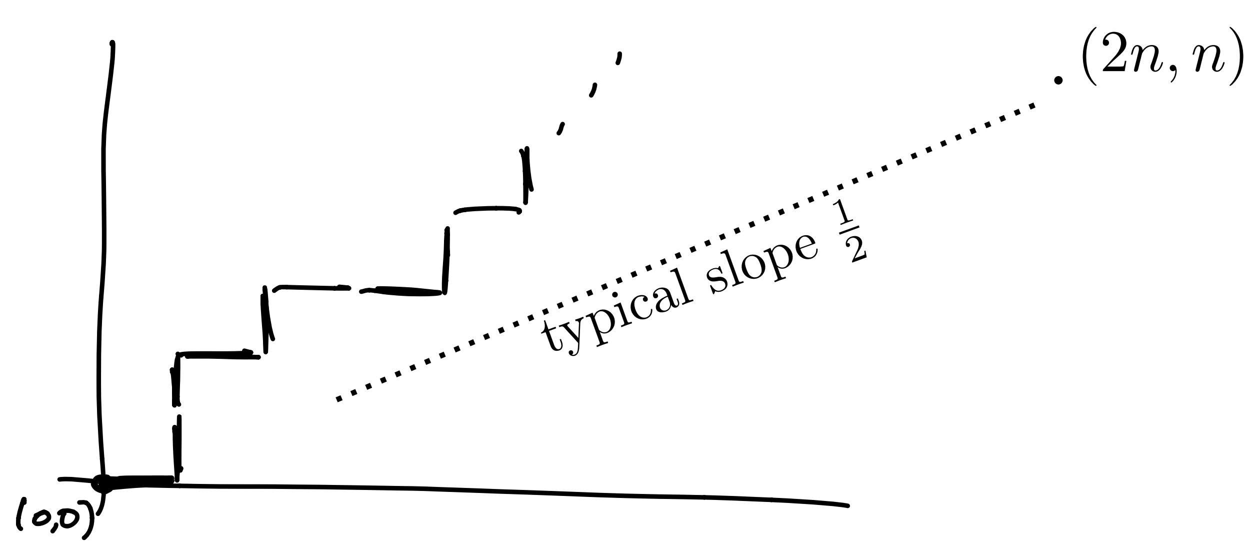
Select an edge at random; drop a ball into its two buckets. Repeat K times.
Condition on the event that all buckets filled after $K = 3n$ steps.

Problem: central buckets fill up faster...

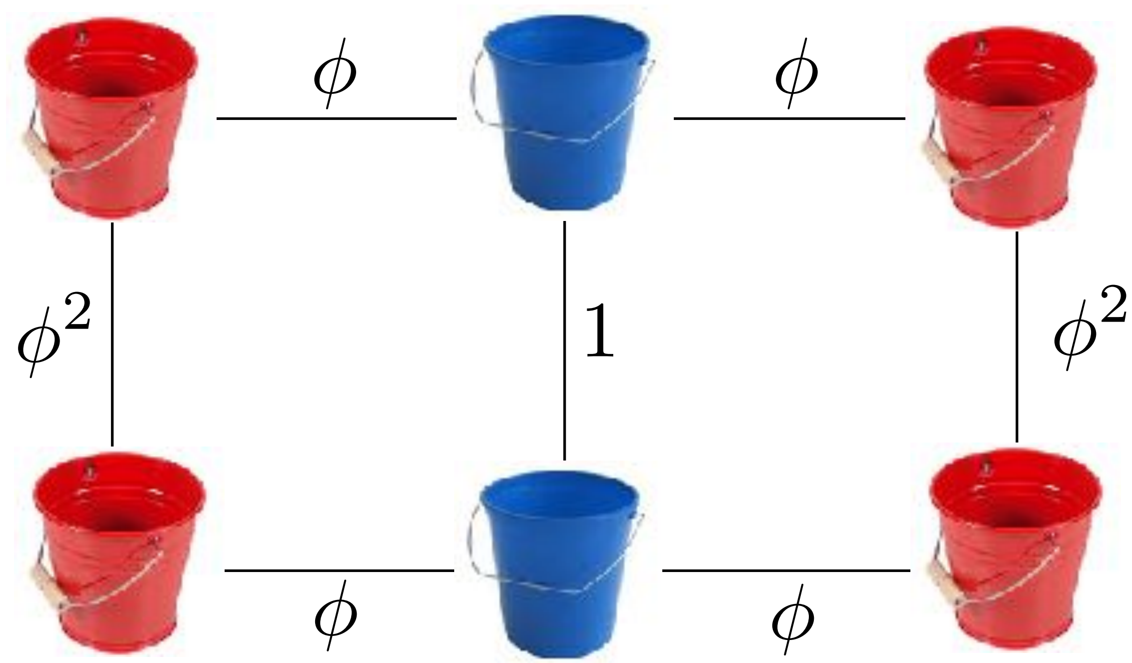
Analogous problem: find coefficient of $x^{2n}y^n$ in $(x+y)^{3n}$.



Bias by changing x to $\frac{2}{3}x$ and y to $\frac{1}{3}y$.



Gauge change: change edge probabilities to $pr(uv) = f(u)f(v)$, so that all buckets fill *at the same rate*, but the conditional distribution is unchanged.



$= \phi$

$$\phi^2 + \phi = 2\phi + 1$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

Asymptotics

Let K = number of dimers = $\frac{1}{2}\sum n_v$.

Suppose $\mathbf{n} \to \infty$ with $\frac{n_v}{K} \to \alpha_v$.

(So α_v is the fraction of dimers covering v.)

Thm[KP]: We have $Z(\mathbf{n}) = K! e^{cK+o(K)}$ where

$$c = \log P(\mathbf{x}) - \sum_v \alpha_v \log(x_v/\alpha_v)$$

and where the x_v are the (essentially) unique positive solution to

$$x_v \frac{\partial}{\partial x_v} \log P = \alpha_v.$$

Here c is a strictly convex function of the $\{\log x_v\}$.

we call $\{x_v\}$ the *critical gauge.*

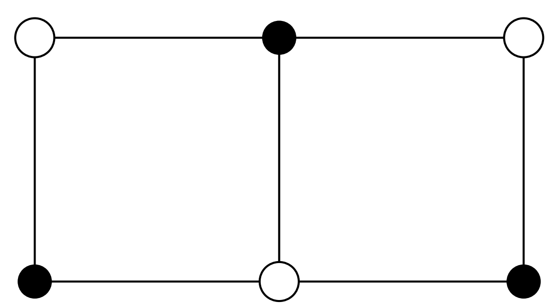
If $\mathbf{n} \equiv n$ the critical gauge (up to scale) satisfies

$$\boxed{\sum_{u \sim v} x_u x_v = 1.}$$

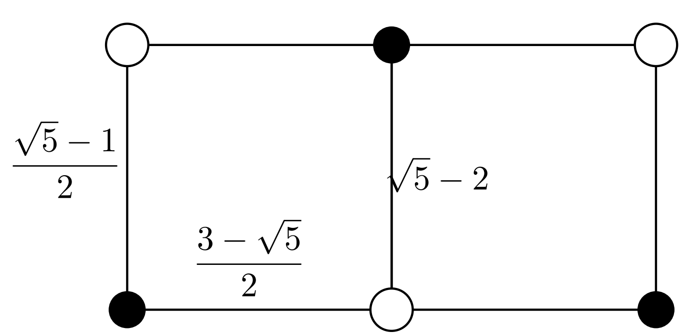
i.e. it is the gauge in which the sum of edge weights around a vertex is 1.

Then "dimer probabilities" (edge fractions) are $\boxed{x_u x_v.}$

Example.

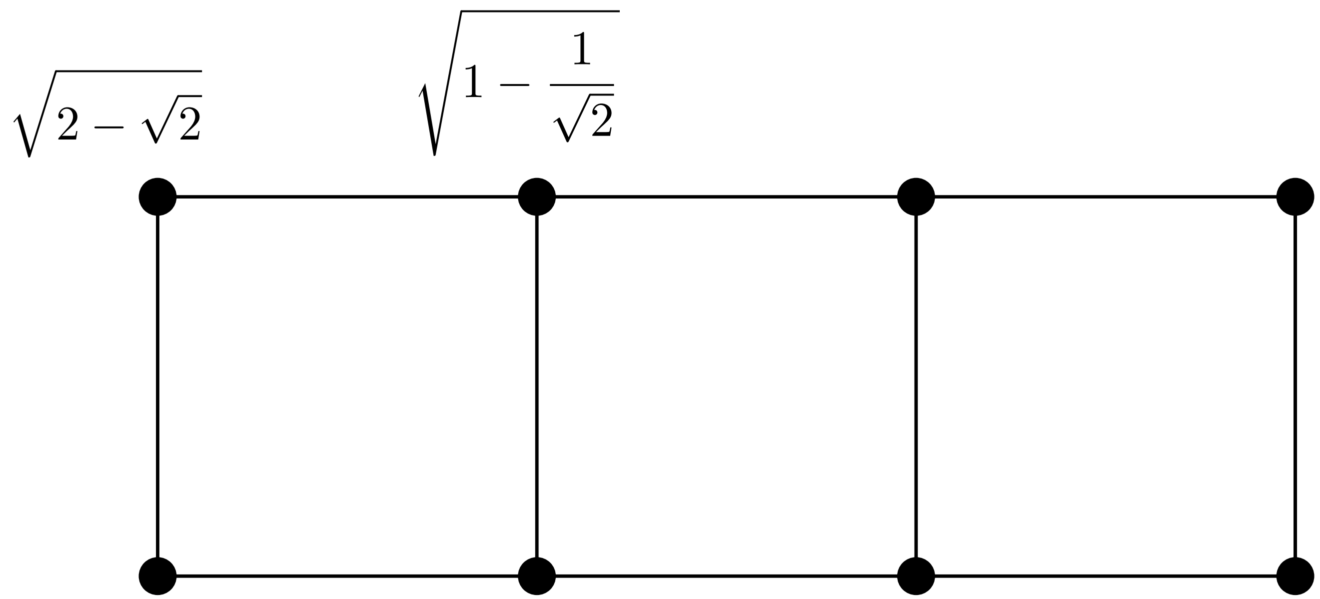


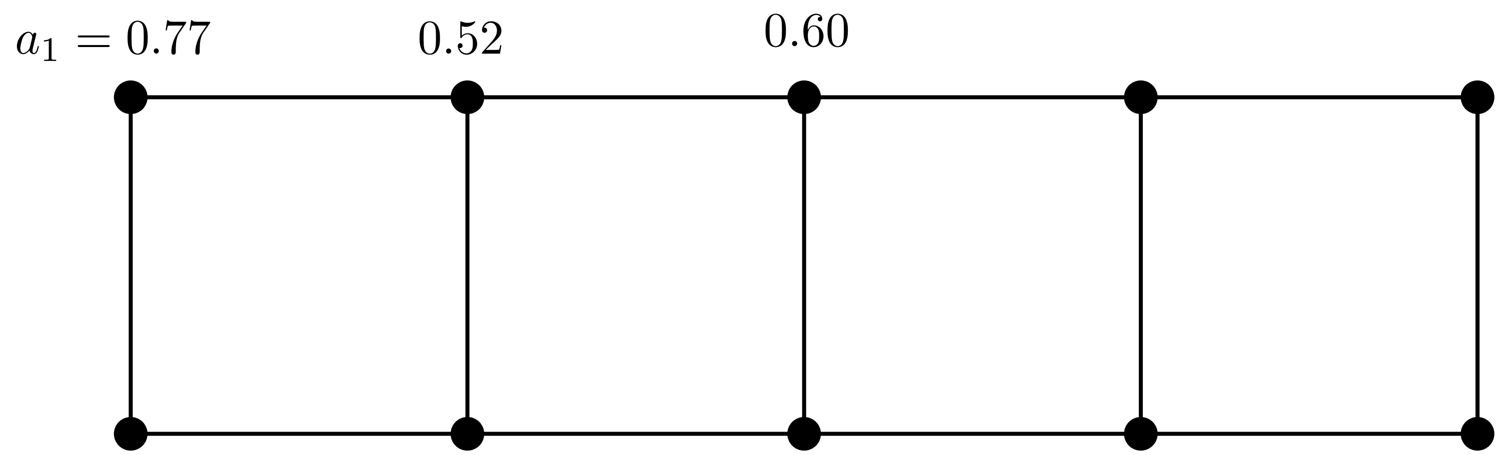
$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_5 & 0 \\ 0 & 0 & x_6 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & \frac{3-\sqrt{5}}{2} & 0 \\ \frac{3-\sqrt{5}}{2} & \sqrt{5}-2 & \frac{3-\sqrt{5}}{2} \\ 0 & \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}$$



"make the adjacency matrix bistochastic"

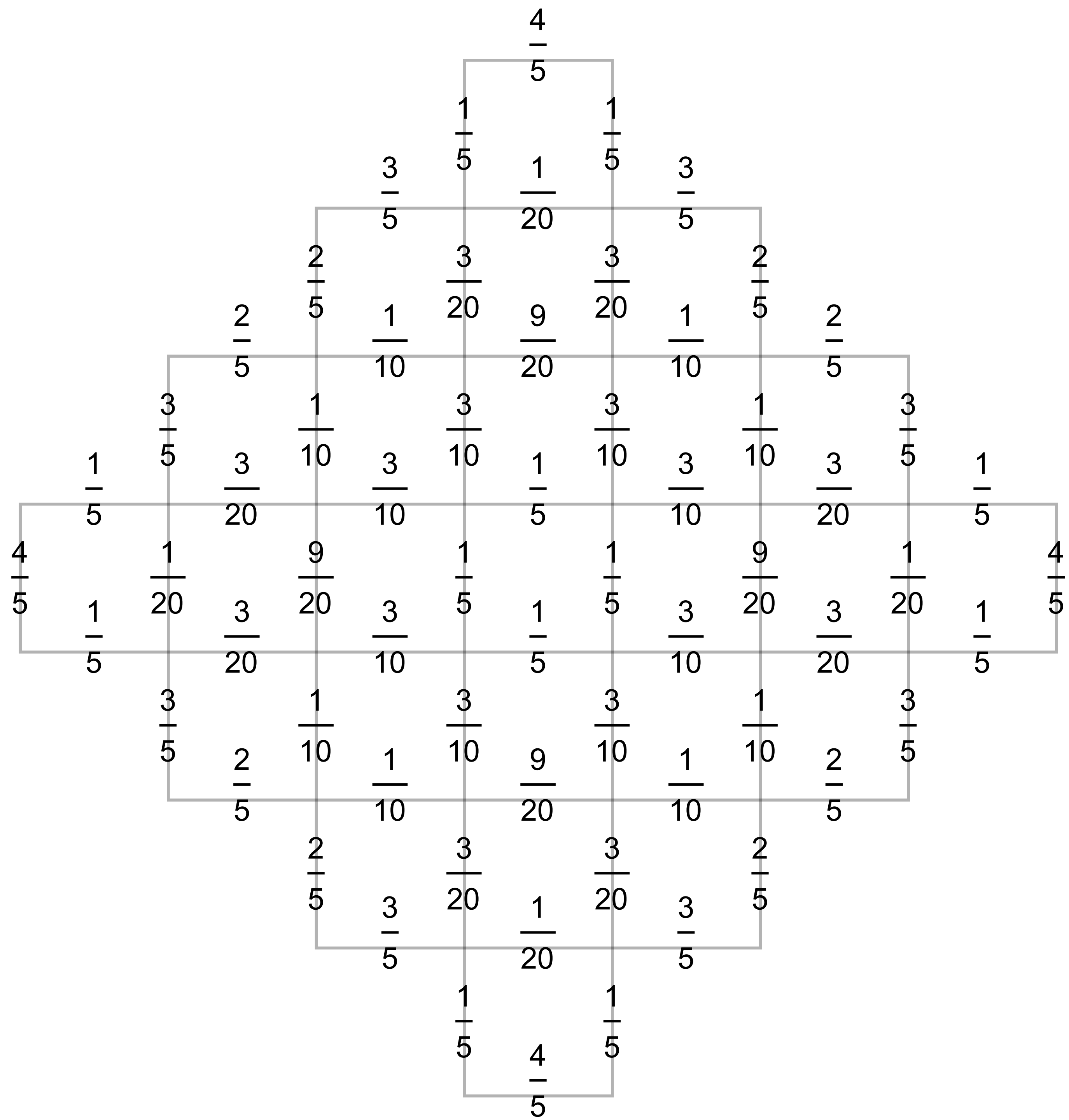
critical gauges for some simple graphs



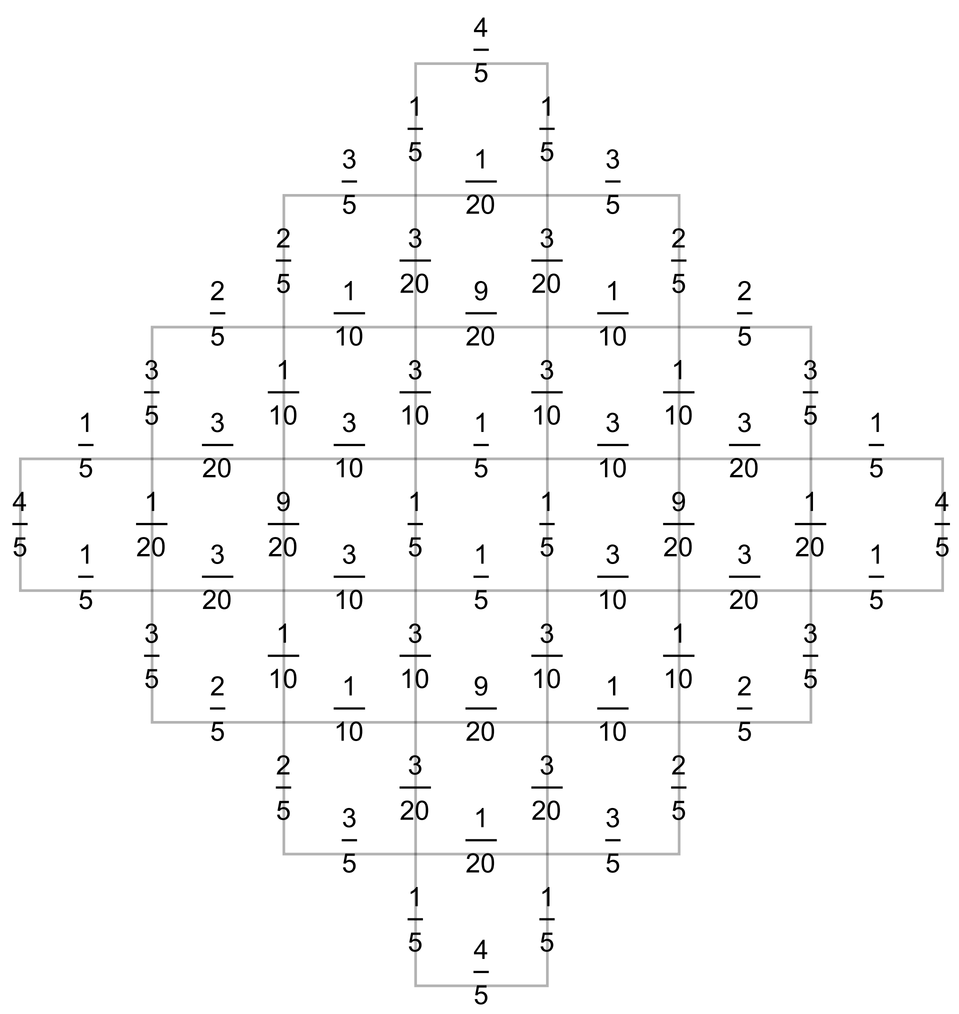


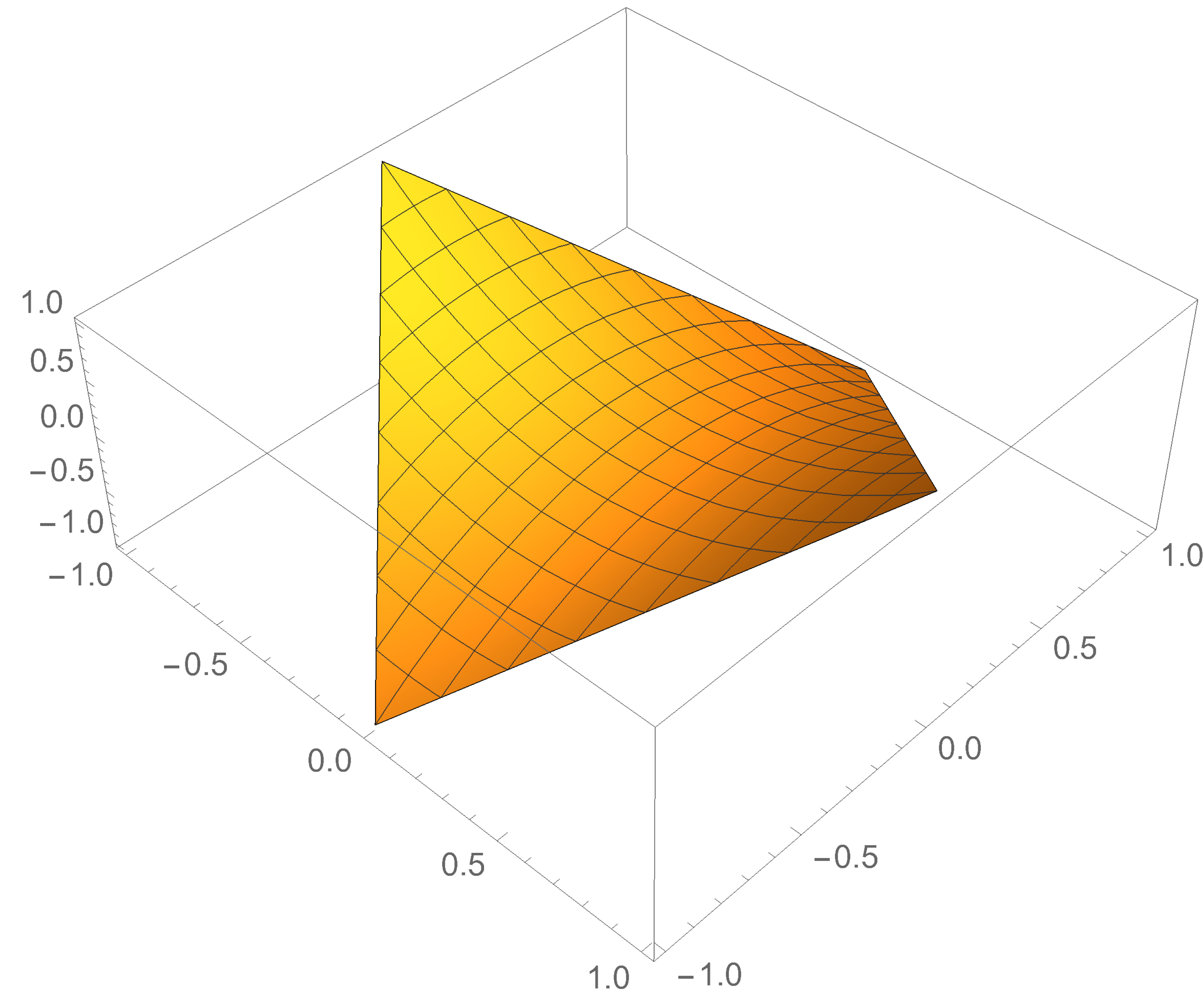
$$3a_1^6 - 4a_1^4 + 3a_1^2 - 1 = 0$$

critical gauge for Aztec diamond



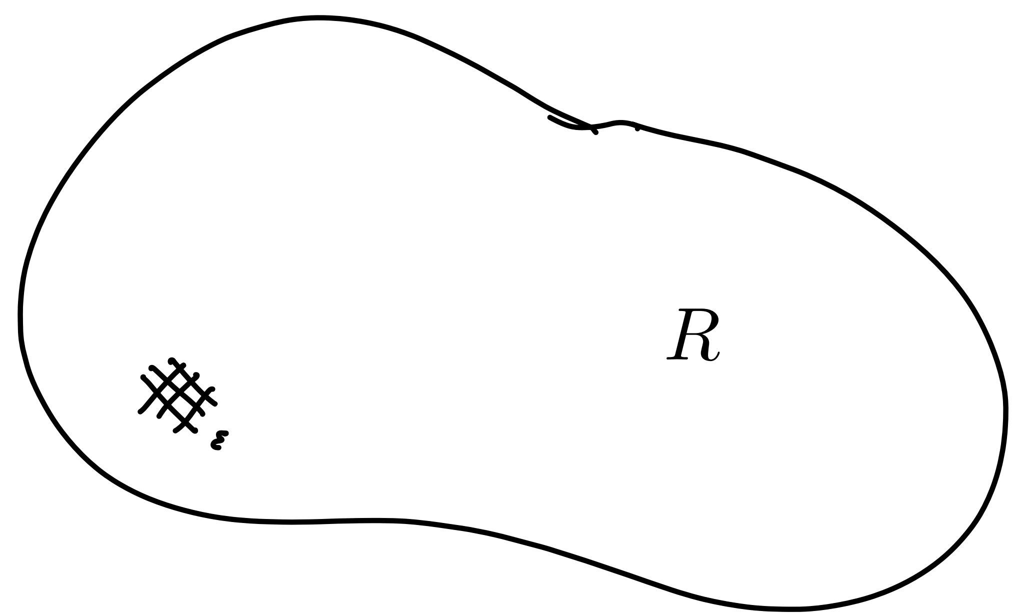
For the critical gauge as above, the tile fractions (edge probabilities) are $x_u x_v$.





The scaling limit height function for the aztec diamond is $h(x, y) = x^2 - y^2$.

Variational principle:



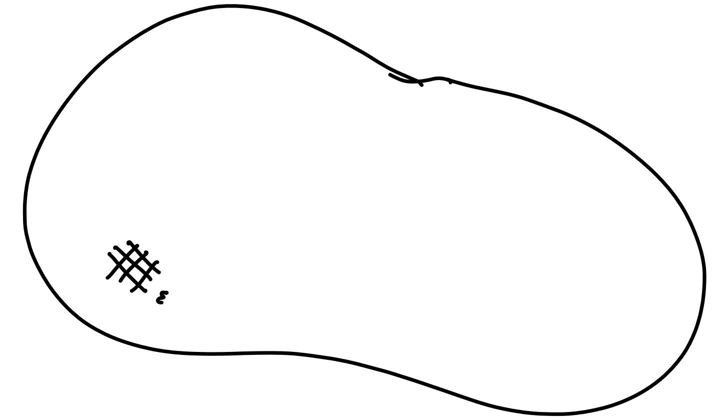
A region R in $\mathbb{R}^2$ and approximating graph $R_\epsilon \subset \mathbb{Z}^2$.

In limit $\epsilon \to 0$, what is the critical gauge?

What is the limiting height function?

(Note: boundary height is determined by choice of boundary conditions for R_ϵ)

Variational principle:



Thm [K-Wolfram]: For multinomial dimers on the scaling limit of (rotated) $\mathbb{Z}^2$, on a domain R with boundary height function $u : \partial R \to \mathbb{R}$, the limit height function h is the unique function with $h|_{\partial R} = u$ maximizing

$$\mathrm{Ent}(h) = \iint_R \sigma(\nabla h)\,dx\,dy$$

where

$$\sigma(s,t) = -\frac{1-s}{2}\log\frac{1-s}{2} - \frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-t}{2}\log\frac{1-t}{2} - \frac{1+t}{2}\log\frac{1+t}{2}.$$

and $(s,t) \in [-1,1]^2$.

The Euler-Lagrange equation for the limiting height function h is

$$\frac{h_{xx}}{1-h_x^2} + \frac{h_{yy}}{1-h_y^2} = 0.$$

General solutions can be written in terms of ${}_2F_1$'s.

The critical gauge (on black vertices) is

$$f(x,y) = e^{\frac{1}{\epsilon}(H(x,y)+o(1))}$$

where

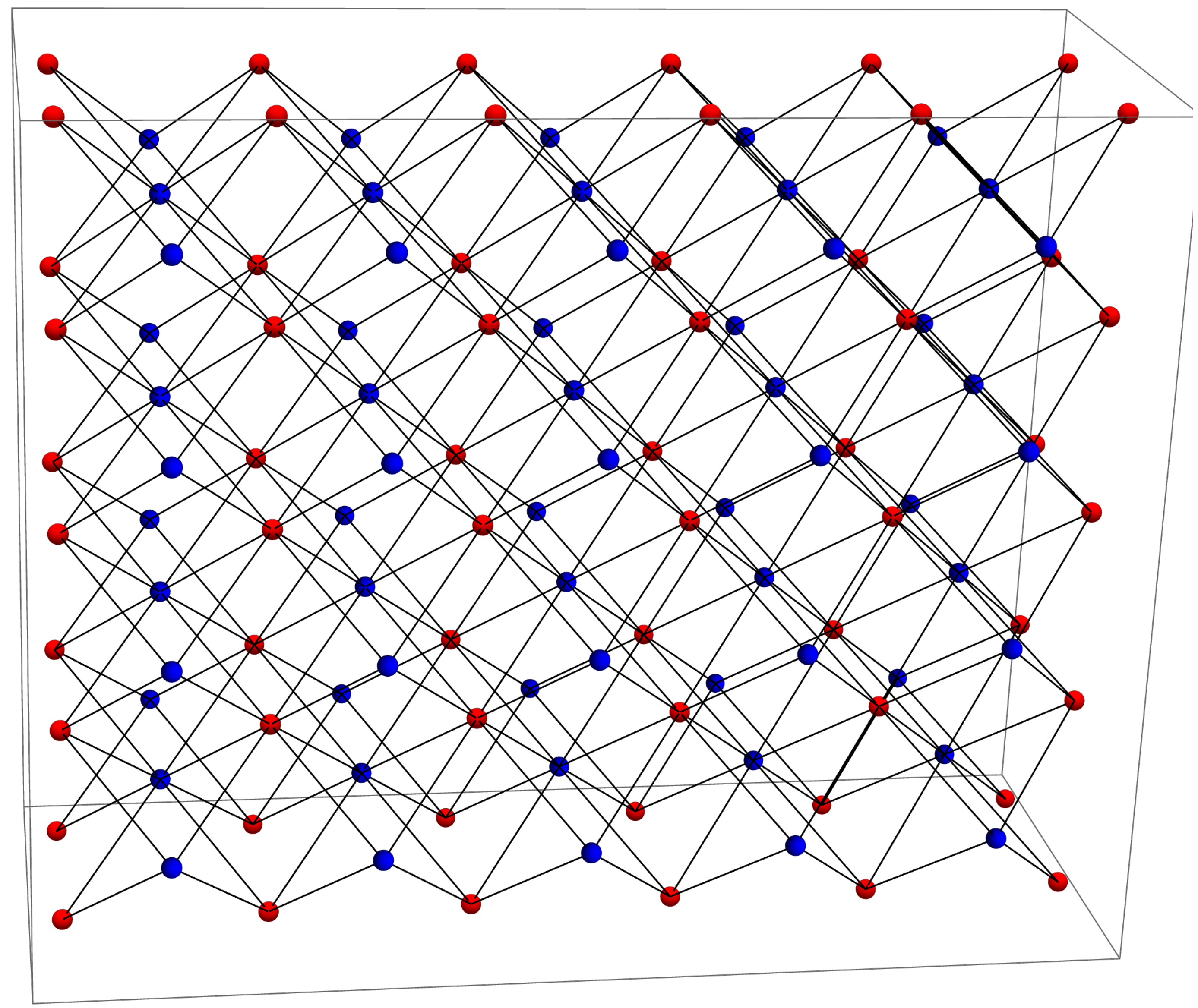
$$\frac{e^{H_x}}{(1+e^{H_x})^2}H_{xx} + \frac{e^{H_y}}{(1+e^{H_y})^2}H_{yy} = 0$$

PDEs for h and H are “dual”:

EL equation for h is equivalent to $H_{xy} = H_{yx}$.

Critical gauge equation for H is equivalent to $h_{xy} = h_{yx}$.

Multinomial dimers on the “3D Aztec diamond” (on BCC lattice in $\mathbb{Z}^3$)



Reds: $a \times b \times c$ box
Blues: $(a+1) \times (b-1) \times (c-1)$ box

$$abc = (a+1)(b-1)(c-1)$$

Dimers in 3D are not described by a height function, but a (divergence free) vector field

The critical gauge is given by

$$x(i,j,k) = \frac{\binom{a}{i}}{\binom{b}{j}\binom{c}{k}}$$

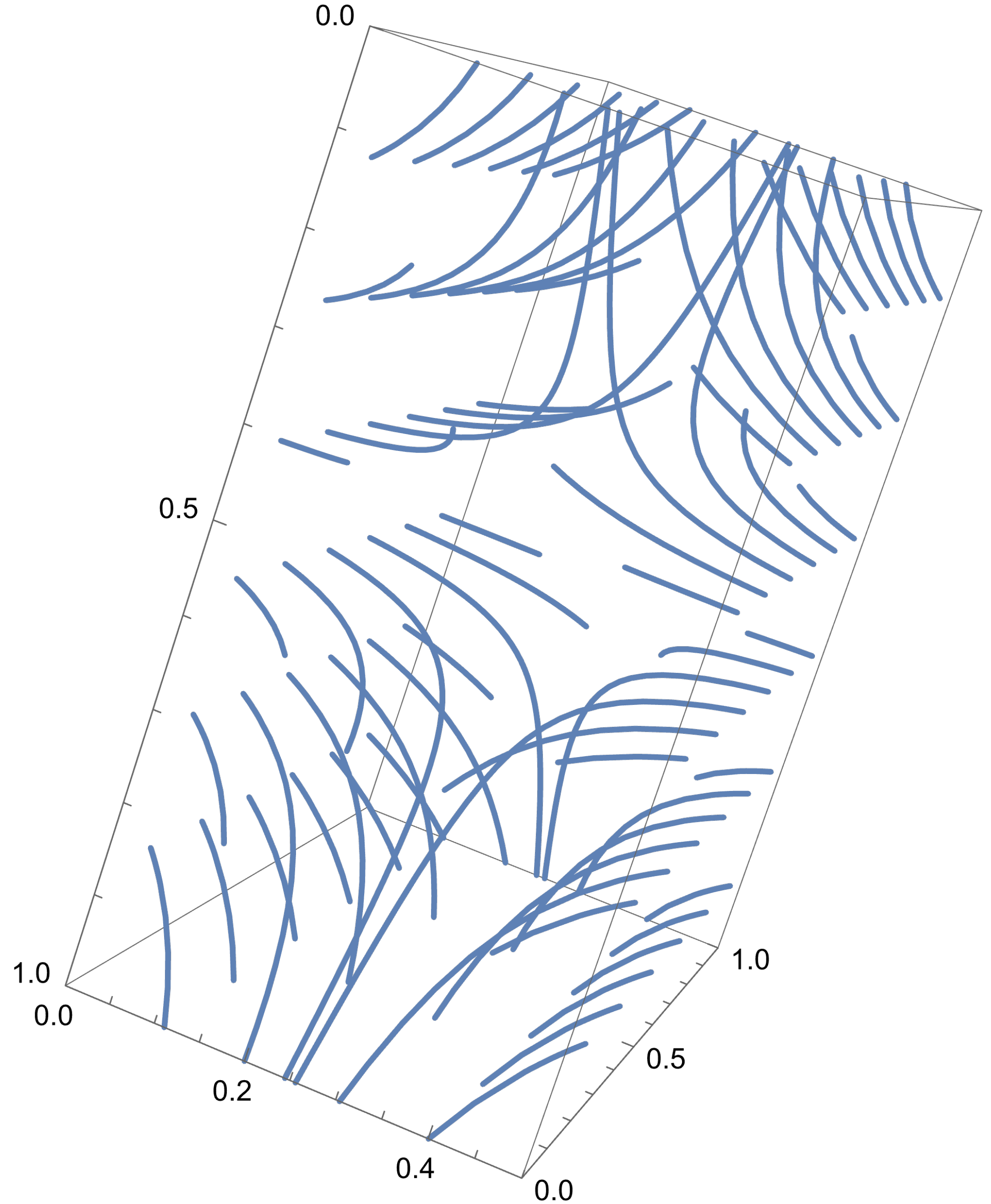
at red vertices and

$$x(i',j',k') = \frac{\binom{b-1}{j}\binom{c-1}{k}}{\binom{a+1}{i+1}} \frac{bc}{(b+1)(c+1)}$$

at blue vertices.

The limit vector field in $[0,\alpha] \times [0,\beta] \times [0,\gamma]$ is

$$\left(\frac{2x}{\alpha} - 1, 1 - \frac{2y}{\beta}, 1 - \frac{2z}{\gamma}\right)$$



integral curves of the vector field

PDE for H:

$$\frac{\partial}{\partial x}\left(\frac{1}{1+e^{H_x}}\right)+\frac{\partial}{\partial y}\left(\frac{1}{1+e^{H_y}}\right)+\frac{\partial}{\partial z}\left(\frac{1}{1+e^{H_z}}\right)=0.$$

This is equivalent to the divergence-free condition for $\vec{u}$.

$$\vec{u}=(u,v,w)=\left(\frac{1-e^{H_x}}{1+e^{H_x}},\frac{1-e^{H_y}}{1+e^{H_y}},\frac{1-e^{H_z}}{1+e^{H_z}}\right).$$

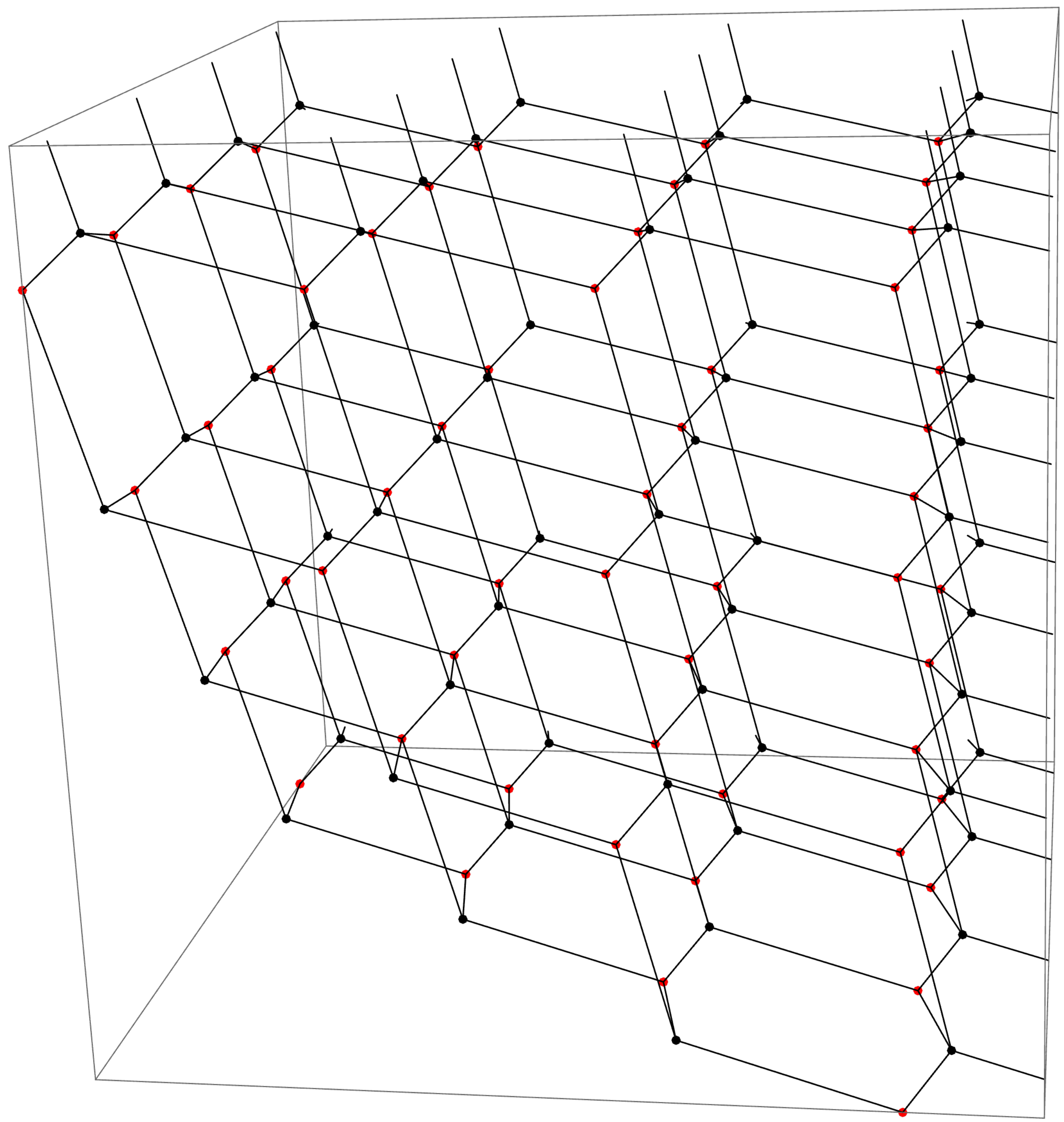
The EL equations for $\vec{u}$ are the mixed partial conditions for H:

$$\frac{u_y}{1-u^2}=\frac{v_x}{1-v^2} \qquad H_{xy}=H_{yx}$$

$$\frac{v_z}{1-v^2}=\frac{w_y}{1-w^2} \qquad H_{yz}=H_{zy}$$

$$\frac{w_x}{1-w^2}=\frac{u_z}{1-u^2}. \qquad H_{xz}=H_{zx}$$

3D "Honeycomb" model (diamond lattice dimer model)



This graph also has an explicit gauge involving trinomial coefficients.

The criticality equation for H:

$$\frac{\partial}{\partial x}\left(\frac{H_x}{1+H_x+H_y+H_z}\right)+\frac{\partial}{\partial y}\left(\frac{H_y}{1+H_x+H_y+H_z}\right)+\frac{\partial}{\partial z}\left(\frac{H_z}{1+H_x+H_y+H_z}\right)=0$$

Scaling limit vector field on "truncated orthant" $\{x+y+z>1\}$:

$$(u,v,w)=\left(\frac{x}{(x+y+z)^3},\frac{y}{(x+y+z)^3},\frac{z}{(x+y+z)^3}\right)$$

THANK YOU