Discrete Uniformization Problem for non-compact polyhedral surfaces

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Joint work with Yanwen Luo

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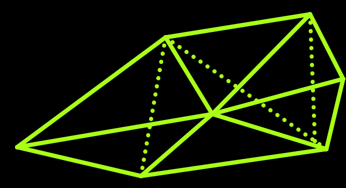
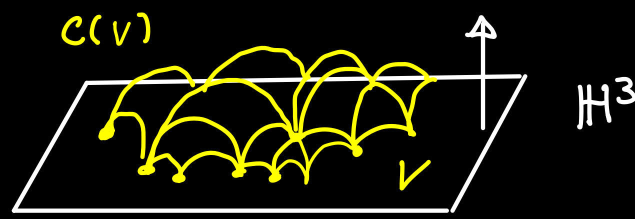
§ 1. Cauchy Rigidity



Thm. If V, W finite sets in $\mathbb{R}^3$ s.t., $\partial C_E(V) \underset{iso}{\cong} \partial C_E(W) \Rightarrow C_E(V) \underset{iso}{\cong} C_E(W)$.

$V \subset \mathbb{C}$, $C(V)$ convex hull in $\mathbb{H}^3$

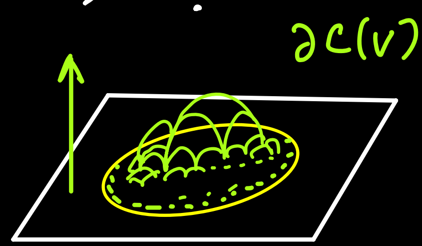


Problem 1. If V, W discrete in $\mathbb{C}$ and $\partial C(V) \underset{iso}{\cong} \partial C(W)$

$$\Rightarrow C(V) \underset{iso}{\cong} C(W) \quad ?$$

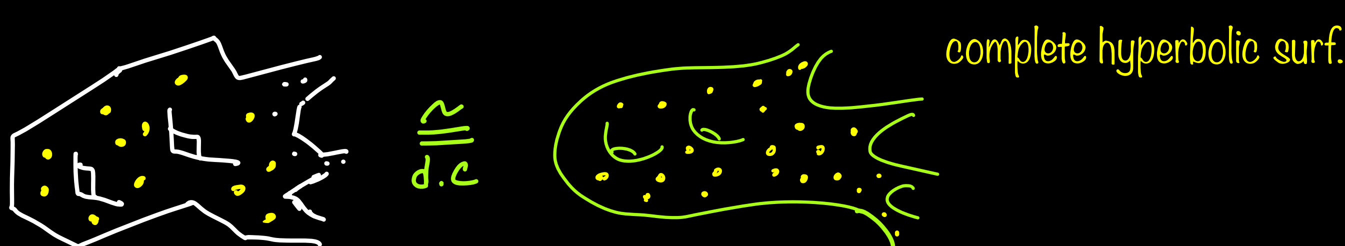
Problem 2. If V, W discrete in $\mathbb{D}$ with $\partial V = \partial W = S^1$ and

$$\partial C(V) \underset{iso}{\cong} \partial C(W) \Rightarrow C(V) \underset{iso}{\cong} C(W) \quad ?$$



Main Thm (Luo-L). Problem 2 is true.

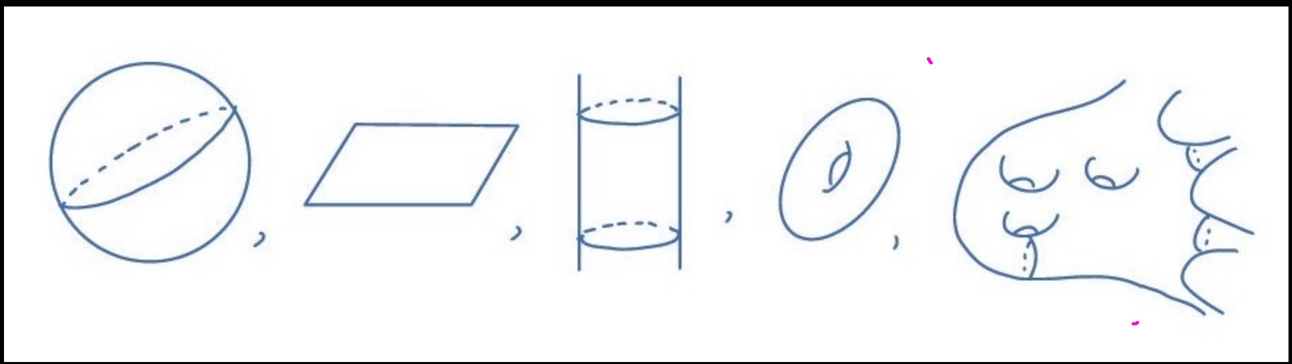
Corollary. $\forall$ polyhedral surface of non-abelian fundamental group is discrete conformal to a unique standard model.



S = connected orientable topological surface $\qquad S = \mathbb{C}$ - Cantor Set

§ 2 The uniformization thm of Poincaré-Koebe

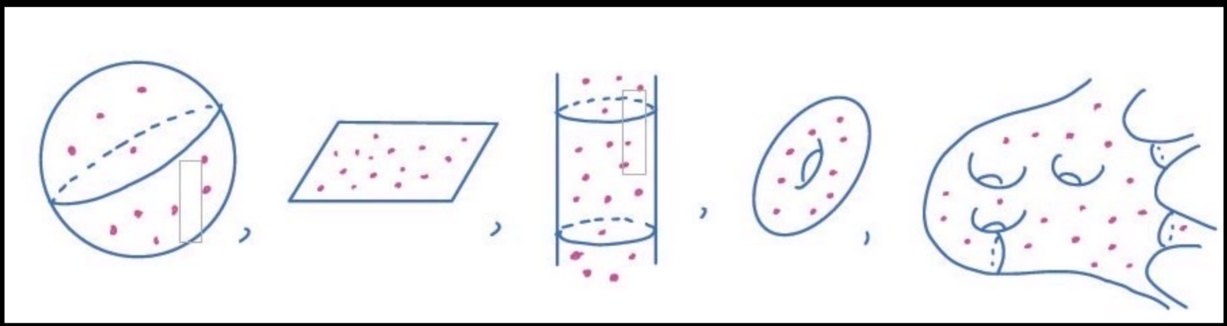
Thm. $\forall\ (S, g),\ \exists\ \lambda : S \longrightarrow \mathbb{R}_{>0}$ s.t., $(S, \lambda g)$ is a complete Riemannian metric of constant curvature $k = 0, 1, -1$.



<u>Discrete Uniformization Problem (DUP)</u>

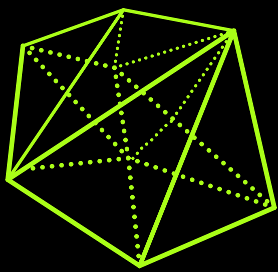
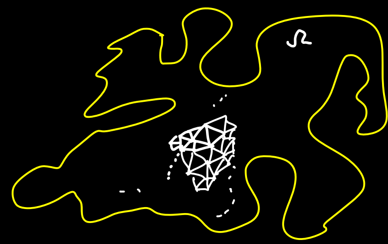
$\forall$ polyhedral surface (S, V, d), $\exists$ a complete constant curvature $0, 1, -1$ metric d' on S s.t.,

$$(S, V, d) \underset{d.c}{\cong} (S, V, d'),$$ which is a polyhedral.



§ 3. Polyhedral surfaces and discrete conformality

V discrete subset of S, d = cone metric on S with cone points in V

Polyhedral

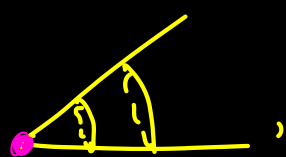
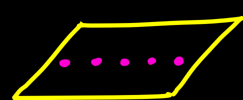
Def. A polyhedral surface is (S, V, d) s.t.,

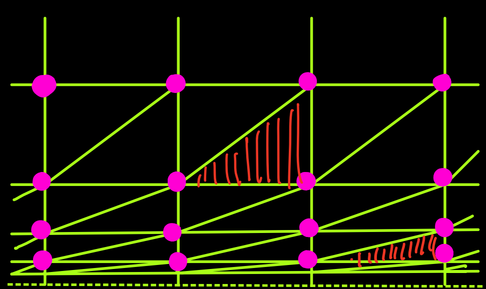
(1). (S, V, d) admits a geodesic triangulation with vertex set V,

(2). the circumdisk of each triangle is cpt and is contained in S,

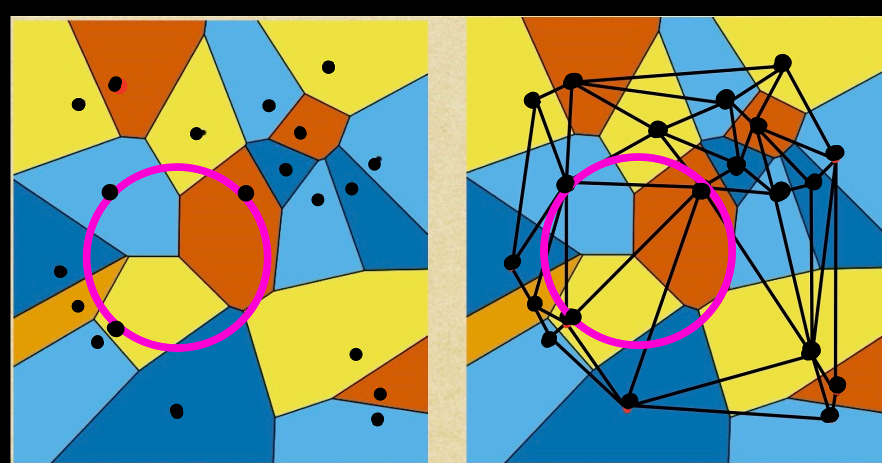
(3). the interior of each circumdisk contains no points in V.

Fact: If S is closed, every cone metric surface (S, V, d) is polyhedral.

Not polyhedral :  , $(\mathbb{C}, \mathbb{Z}, d_E) =$ 



triangulated upper half plane

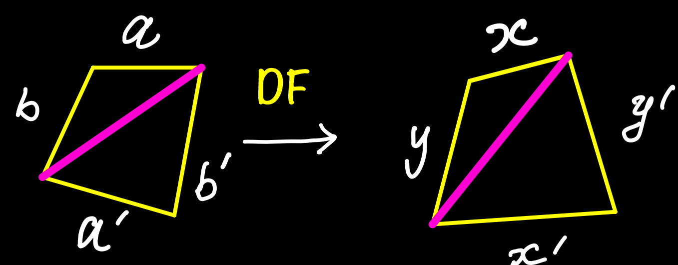


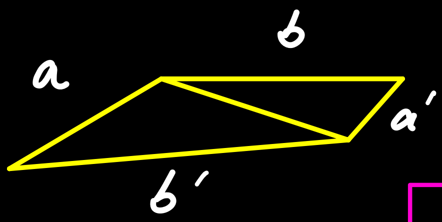
Discrete conformal

F conformal $\Leftrightarrow$ DF: $R^2 \to R^2$ circle preserving.

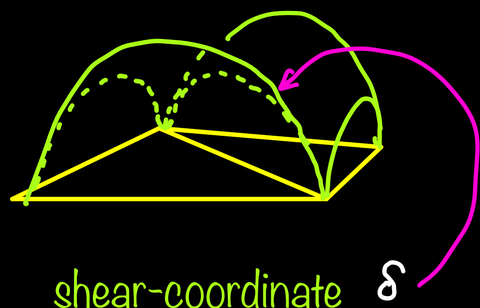
$\Leftrightarrow$ DF preserves length cross ratios:

$$\frac{aa'}{bb'} = \frac{xx'}{yy'}$$





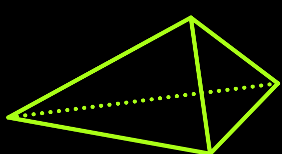
$$\boxed{\frac{aa'}{bb'} = e^{\delta}}$$



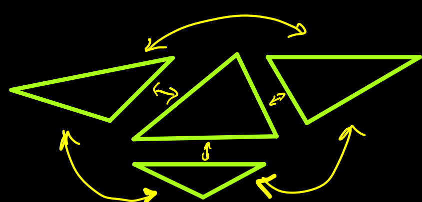
Bobenko-Pinkall-Springborn

Given polyhedral (S, V, d), produce a Delaunay $\mathcal{T}$ with faces $\Delta(\mathcal{T})$.

Then $(S, V, d) \underset{\text{iso}}{\cong} \bigsqcup_{\tau \in \Delta(\mathcal{T})} \tau \,/\, \text{iso}$

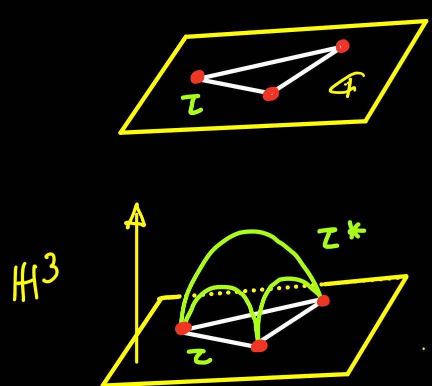


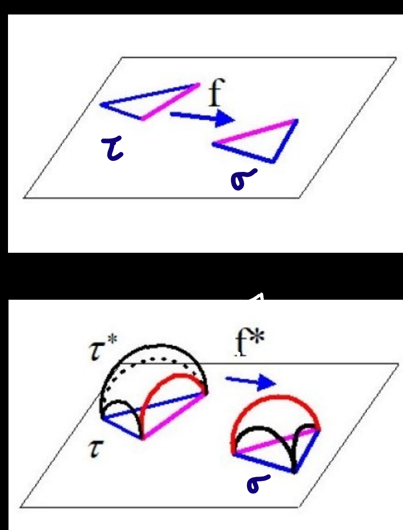
$\underset{\text{iso}}{\cong}$

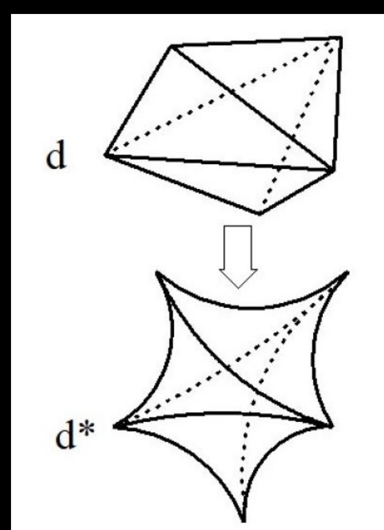


isometric gluing

$\forall \quad \tau \in \Delta(\mathcal{T}) \rightsquigarrow$ ideal hyperbolic τ^*, produce hyperbolic d^*

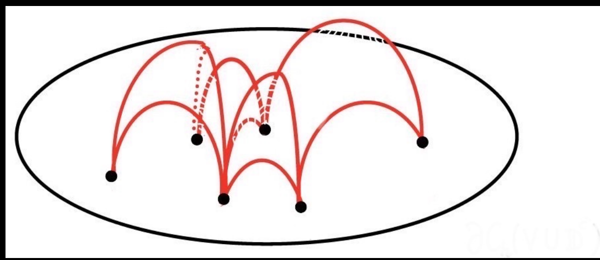


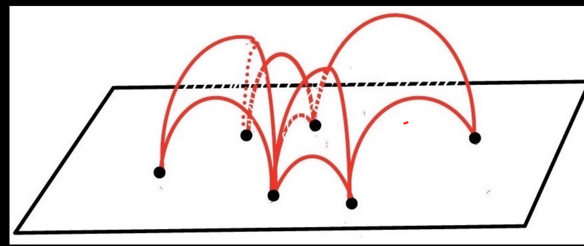




same works for H^2 or S^2

Eg. d^* for the standard models : $(\mathbb{D}, V, d_H)$ + $(\mathbb{C}, V, d_E)$





$d^* \cong \partial C(V)$, boundary of convex hull of V in $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$

Def. Two polyhedral surfaces (S, V, d_1) and (S, V, d_2) are discrete conformal (d. c.) if $\exists$ isometry

$$F: (S-V, d_1^*) \to (S-V, d_2^*) \quad \text{s.t.,} \quad F \simeq id.$$

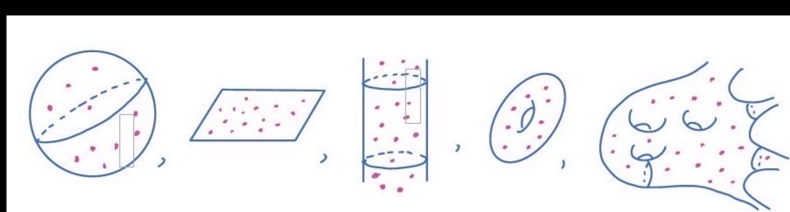
Prop. If (S, V, d) is a Euclidean or hyperbolic polyhedral surface $\Rightarrow$

(1). d^* is independent of the choice of Delaunay triangulations,

(2). d^* is complete hyperbolic.

RM. If d is complete hyperbolic on S, V subset of S, then d^* is defined $S-V$.

DUP. $\forall$ polyhedral surface (S, V, d), $\exists$ a complete constant curvature $k=0,1,-1$ metric d' on S s.t., $(S, V, d) \underset{d.c.}{\cong} (S, V, d')$. Furthermore (S,V,d') is polyhedral.



RM. DUP holds for closed surfaces (Rivin, Fillastre, Gu-L-Sun-Wu, Gu-Guo-L-Sun-Wu)

§4. Proof of classical unif thm. Go to the universal cover.

Assume S is simply connected and non-cpt

- $\exists$ (S, d) conformal to $\mathbb{H}^2$ or $\mathbb{C}$ — Schwarz, Liouville
- $!$ Conformal $\varphi : \mathbb{H}^2 \to \mathbb{H}^2 \Rightarrow \varphi \in \mathrm{Iso}(\mathbb{H}^2)$.

  (conformal $\varphi : \mathbb{C} \to \mathbb{C} \Leftrightarrow \varphi(z) = az+b$).

Discrete case hyperbolic or Euclidean polyhedral surf (S,V,d), simply connected

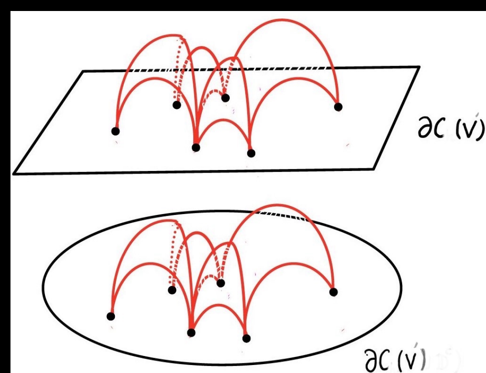
Step 1 $\cdot\,\exists$ $\forall\ (S, V, d) \overset{}{\underset{d.c.}{=\!=}} (\mathbb{D}, V', d_H)$ or $(\mathbb{C}, V', d_E)$

$\Longleftrightarrow$ hyperbolic $(S-V, d^*) \underset{iso}{\cong} \partial C(V')$

d^*

$\underset{iso}{\cong}$

Weyl Problem



Step 2 $\cdot\,!$ If $\varphi : (\mathbb{D}, V, d_H) \to (\mathbb{D}, W, d_H)$ dis. conf. $\Rightarrow \varphi \in \mathrm{Iso}(\mathbb{H})$.

(If $\varphi : (\mathbb{C}, V, d_E) \to (\mathbb{C}, W, d_E)$ dis. conf. $\Rightarrow \varphi(z) = az+b$.)

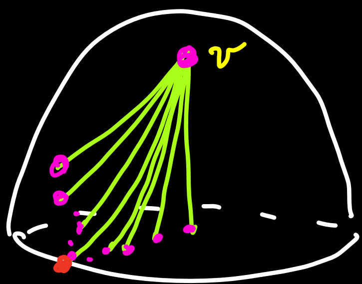
These are problems 1 & 2.

§5. Our progress

Step 1. Existence

Thm (L-Wu) $\forall$ complete hyperbolic metric d^* on a genus zero surface with at most one non-cusp end is iso $\partial C(V)$, $V \subset \mathbb{D}/\mathbb{C}$.

But $(\mathbb{D}, V, d_H)$ or $(\mathbb{C}, V, d_E)$ may not be polyhedral.



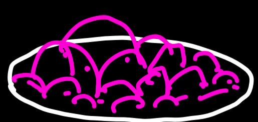
$\exists$ v in V w/ infinitely many edges from v in $\partial C(V)$.

We don't know how to prove that if d^* is associated to a polyhedral surface $(S, V', d) \Rightarrow \partial C(V)$ is polyhedral.

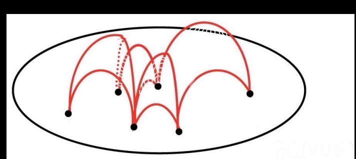
Step 2. Uniqueness (discrete Schwarz lemma, related to work of Beardon-Stephenson)

Thm(Luo-L). If V_1, V_2 are discrete in $\mathbb{D}$ with limit points with $\partial V_i = \partial \mathbb{D}$

$$\partial C(V_1) \underset{iso}{\cong} \partial C(V_2) \Rightarrow C(V_1) \underset{iso}{\cong} C(V_2).$$

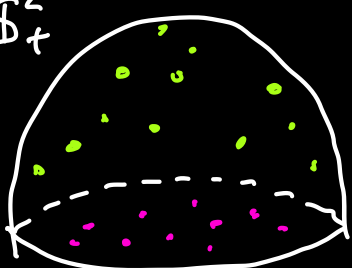


$\mathbb{D}^2$



(Cauchy-Alexandrov w/ infinite vertices).

Shifting from $\mathbb{D}$ to $\mathbb{S}^2_+$

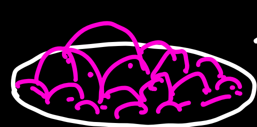


Stereographic proj

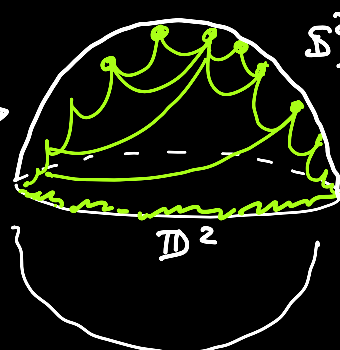
From the south pole

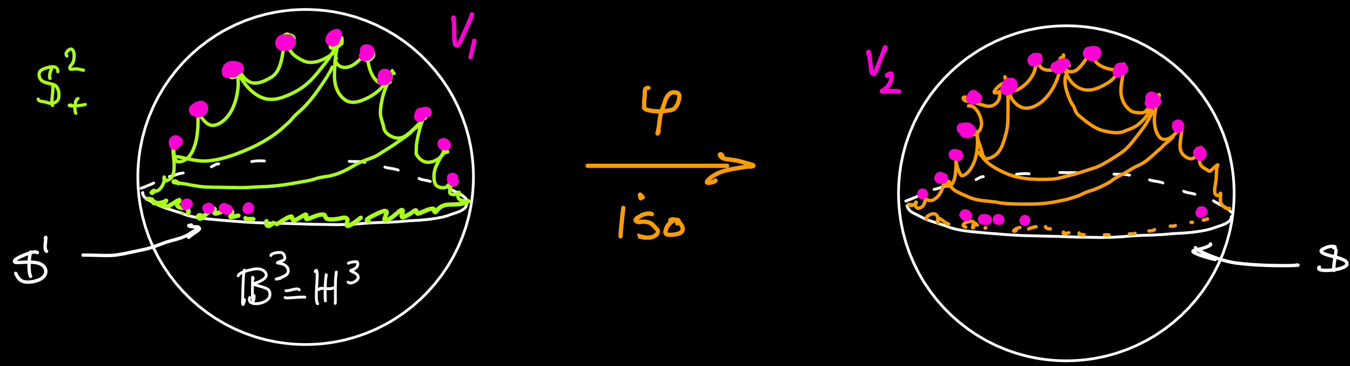
$\mathbb{D}^2 \longrightarrow \mathbb{S}^2_+$

$\partial C(V)$



$\mathbb{D}^2$

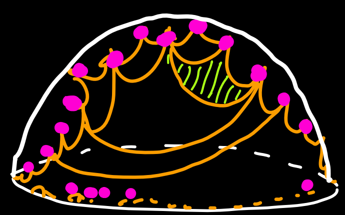




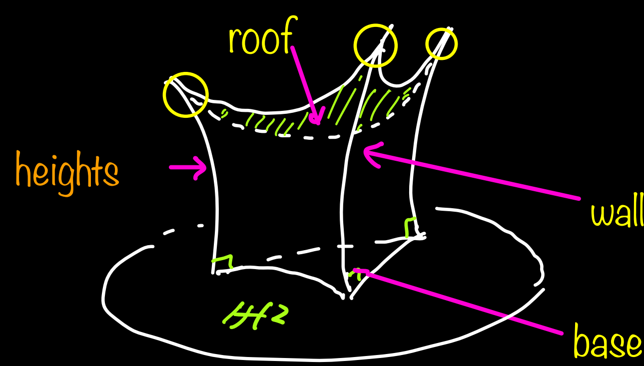
Assume now that $C(V)$ is a polyhedral convex cap

related to the work of Alexandrov, Volkov, Izmestiev, ...

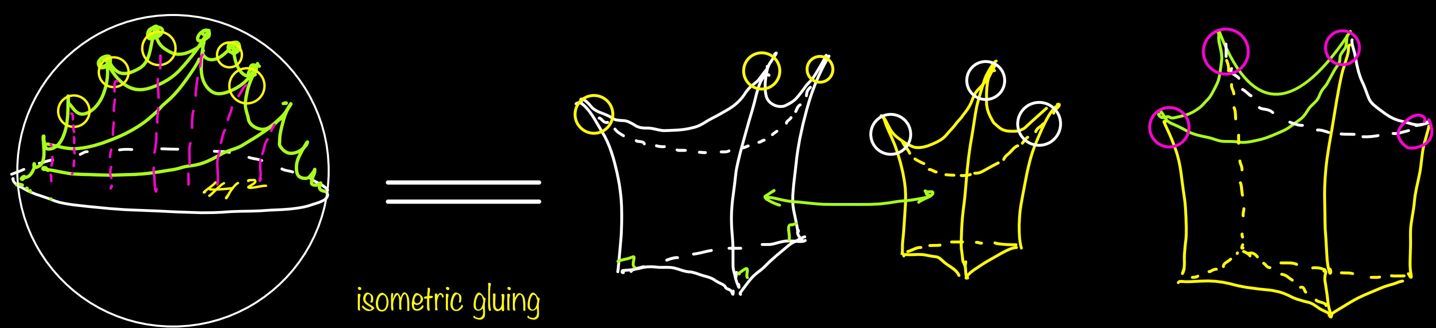
$C(V)$



union of prisms

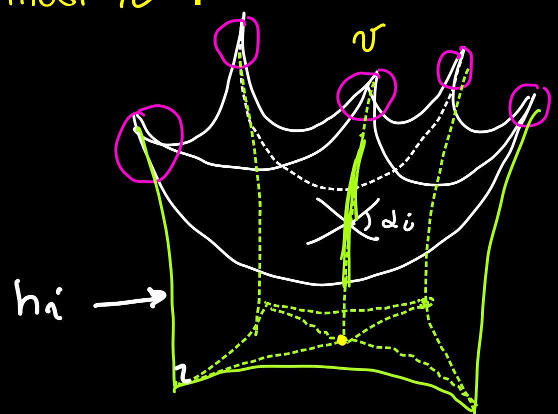


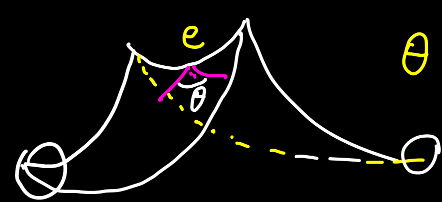
Each vertex v is decorated by a horoball for computation



Def (Convex Cap = CC)

Isometric gluing of of prisms along walls s.t. dihedral angles at all interior roof edges are at most π .





$\theta \leq \pi$

The curvature $K(v) = 2\pi - \sum_i \alpha_i$

Notation:

(D, V, d) = roof surface

(D, V, d, h) = a CC with roof (D, V, d) and height $h: V \to \mathbb{R}$.

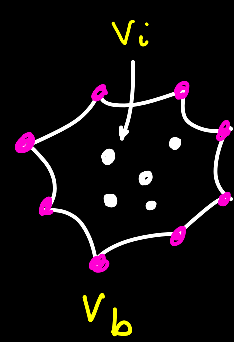
Schwarz - Pick- Ahlfors lemma

$$(\mathbb{D}, d_H) \xrightarrow{f} (\Omega, g), \quad K_g \leq -1,$$

$$f \text{ conformal} \Rightarrow d_g(f(x), f(y)) \leq d_H(x, y).$$

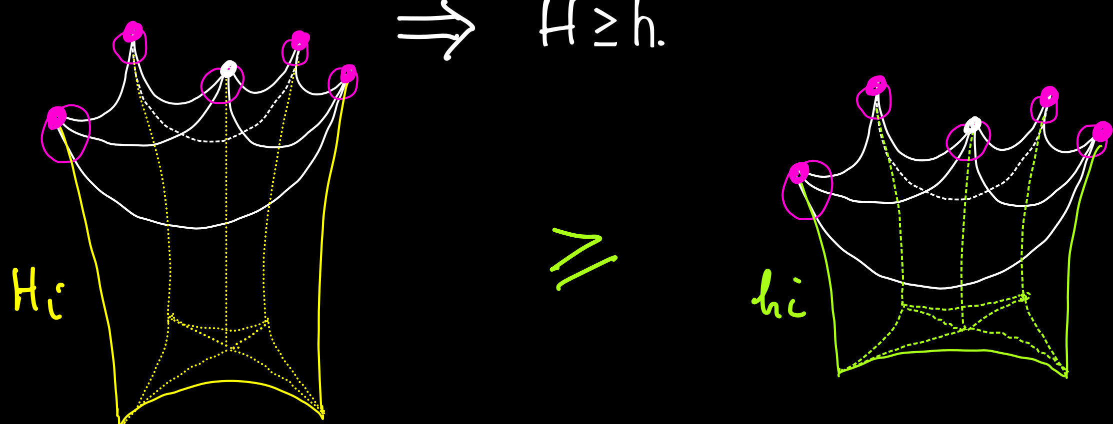
Discrete Schwarz lemma (DSL) for finite convex caps

Let (D, V, d, H) and (D, V, d, h) be two finite CC with isometric roof, $V = V_b \sqcup V_i$.



If $H|_{V_b} \geq h|_{V_b}$ and $K_H|_{V_i} \leq K_h|_{V_i}$

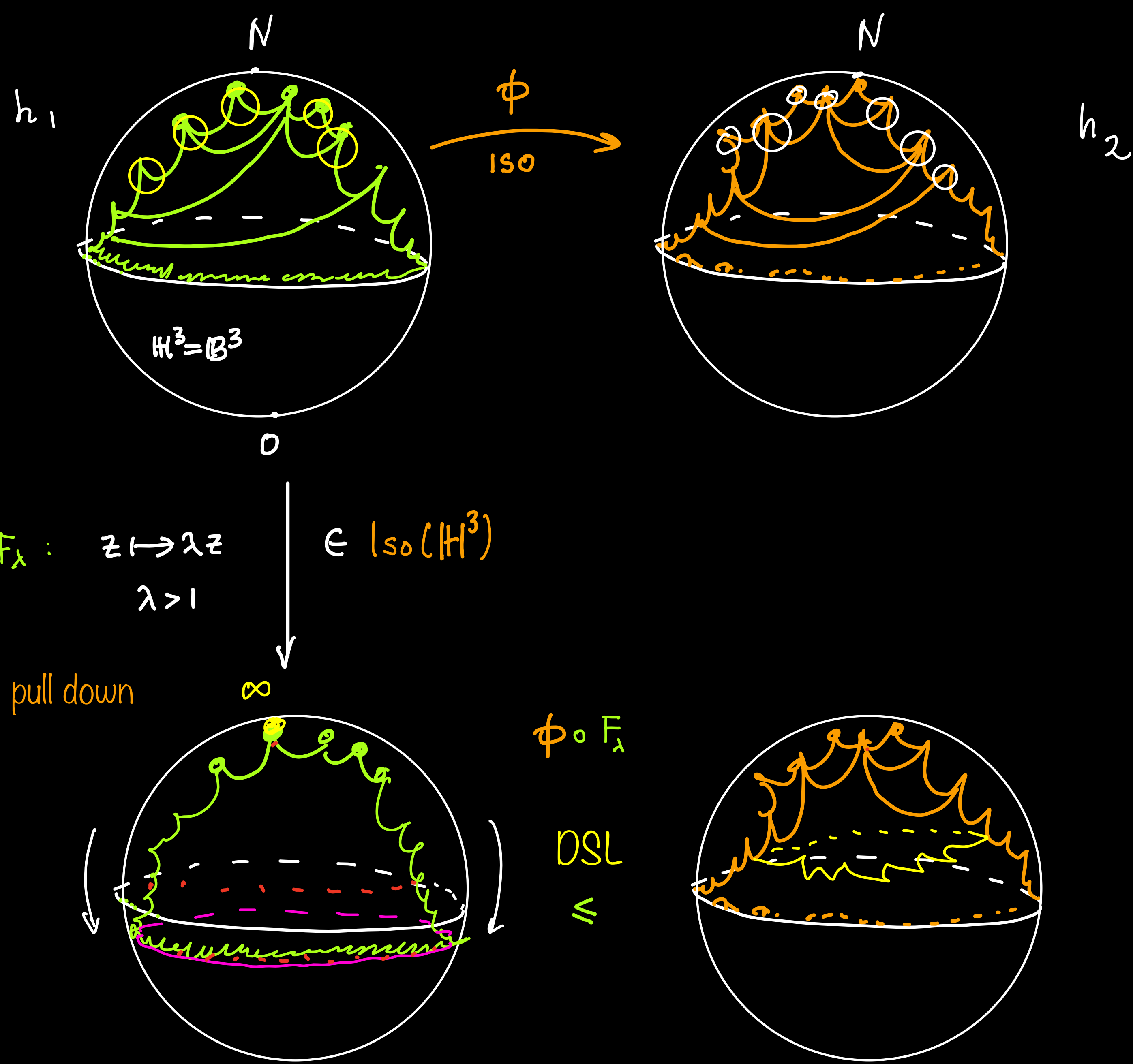
$\Rightarrow H \geq h.$



DSL $\Rightarrow$ rigidity thm using Z. He's scaling method (polyhedral case)

Two CC's with isometric roof and heights h_1, h_2.

Goal: $h_1 = h_2$. $\qquad \phi(N) = N$, if $h_1(N) > h_2(N)$



Thank you!