Convergence of discrete conformal maps on surfaces and the determinant of the discrete Laplacian on a simplex

ICERM

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Convergence of circle packing maps to conformal maps

Circle Packing Convergence Theorem

The theorem as stated in Rodin-Sullivan-1987 (Based on Thurston's lecture from 1985) is the following:

Theorem (Rodin-Sullivan 1987). *The isomorphism* $C_\epsilon \to \bar{C}_\epsilon$ *of circle packings determines an approximate mapping which, as* $\epsilon \to 0$*, converges to a conformal homeomorphism of* R *with the unit disk.*

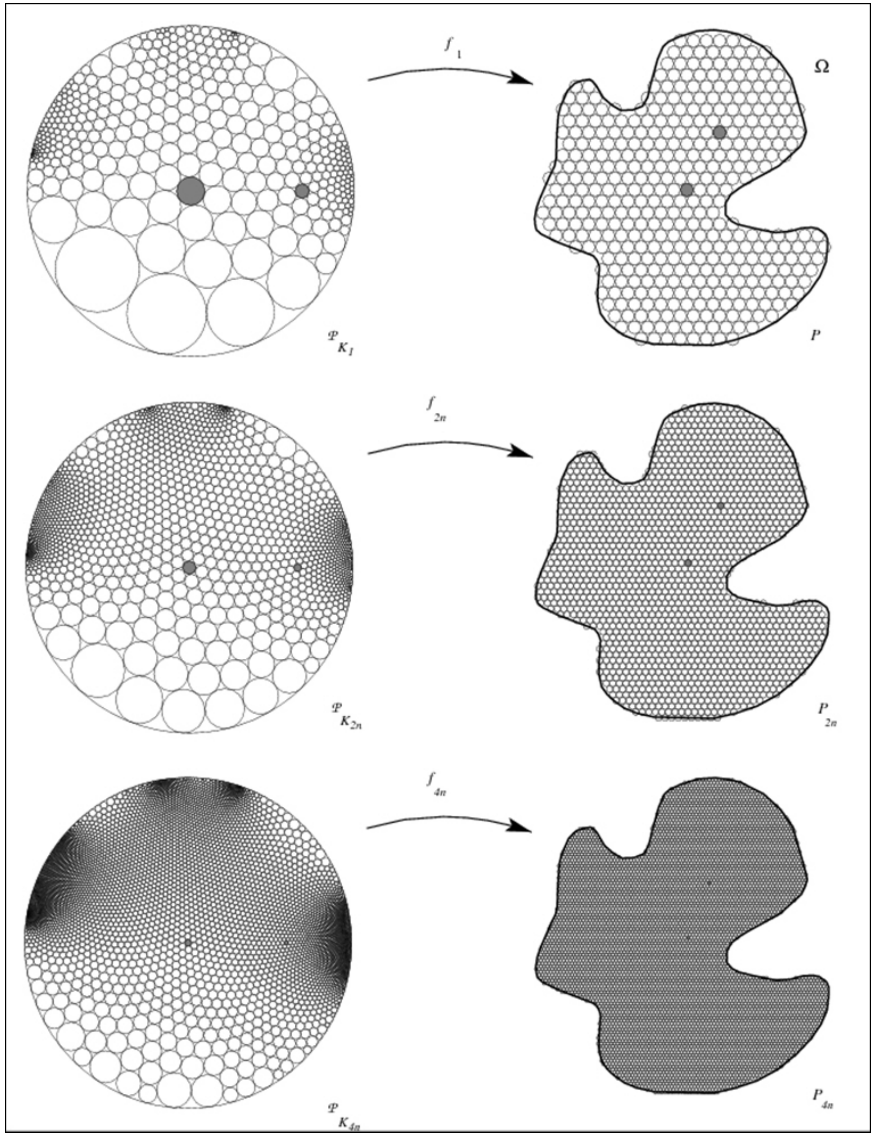
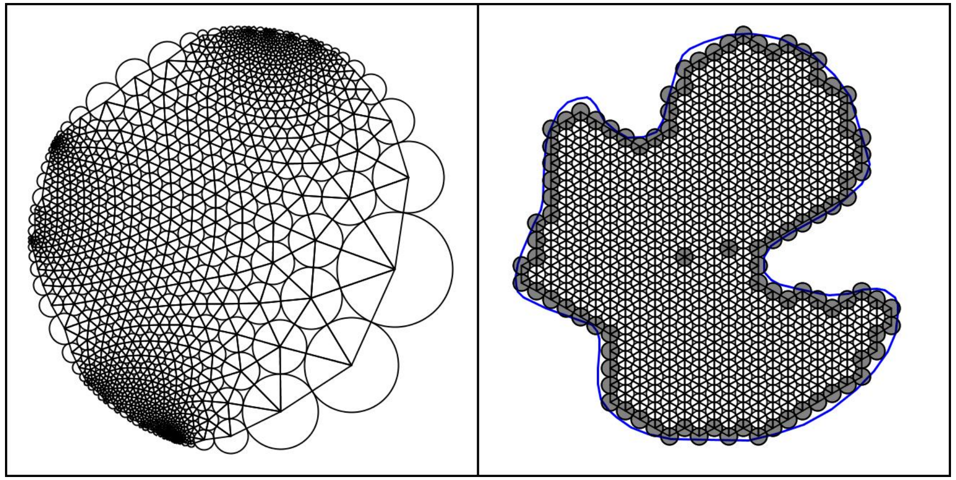
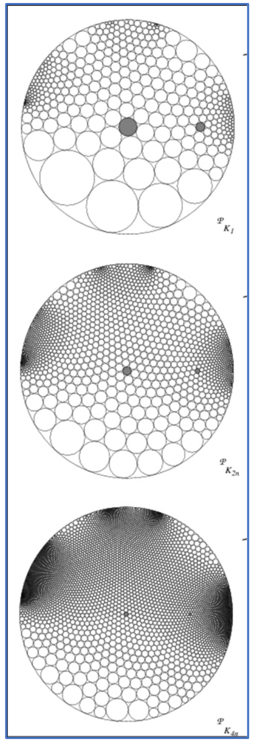


Figure credit: Stephenson

Elements of the proof of the Rodin-Sullivan Theorem

1. Maps of circle packings correspond to PL maps.
2. Need to fix something (to eliminate conformal invariance of disk) (set where 2 points go)
3. Need to show the ranges converge. (from the definitions of the packings.)
4. Need to show the domains converge. (Length-Area Lemma)
5. Need to show the maps do not degenerate. (Ring Lemma)
6. Need to show maps converge to conformal. (Hexagonal packing rigidity)
7. (For us): Need to show derivatives converge. (Discrete Schwarz lemma + estimate of Hexagonal Packing constants – Rodin/He)





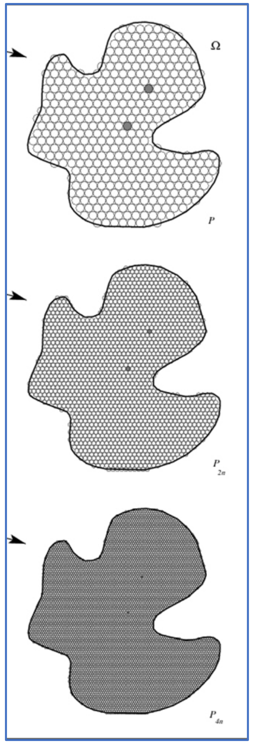


Figure credit: Stephenson

Why we might want more

1. Subdivision not clearly circle packing. (Answer: Use other discrete conformal structures)
2. Not a domain in the complex plane. (Riemannian Barycentric Coordinates)
3. What is a good PL approximation to conformal? (Answer: Local Discrete Conformal Rigidity)

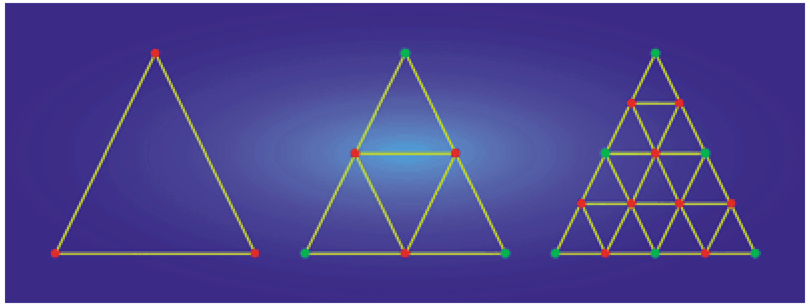


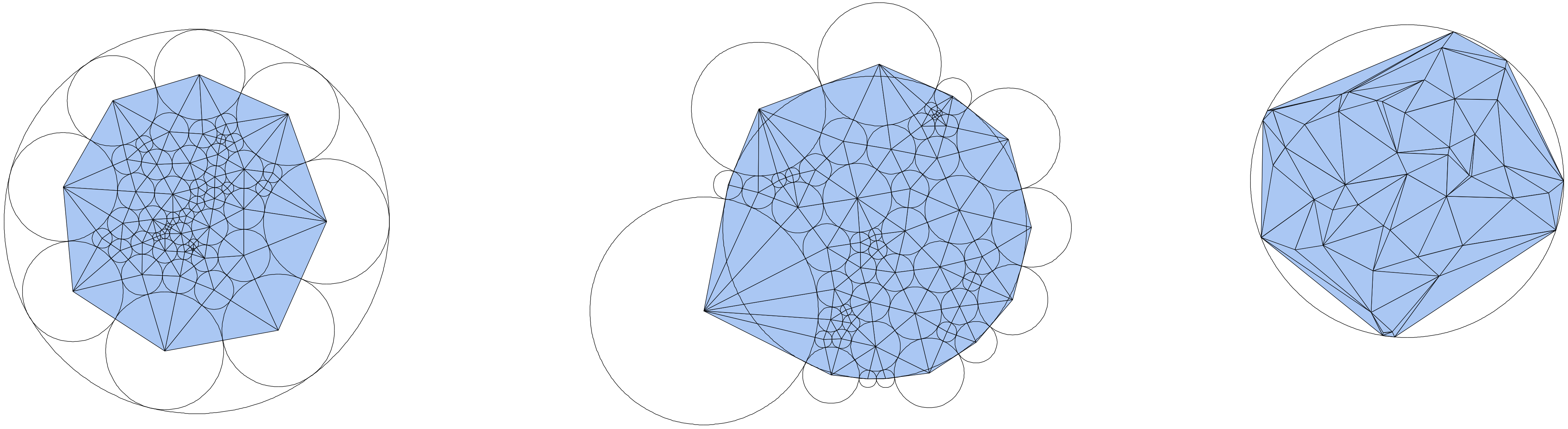
Figure credit: Math Stack Exchange

Photo by Guada Lozano: Sculpture by Ron Stansfield: Design by Roma Krebs

Brief overview of discrete conformality

(Zhang-Guo-Zeng-Luo-Yau-Guo 2014, Glickenstein-Thomas 2017) Conformal structures are parametrized by α, η and define lengths based on vertex weights f_i.

$$\ell_{ij}^2(f_i, f_j) = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}$$



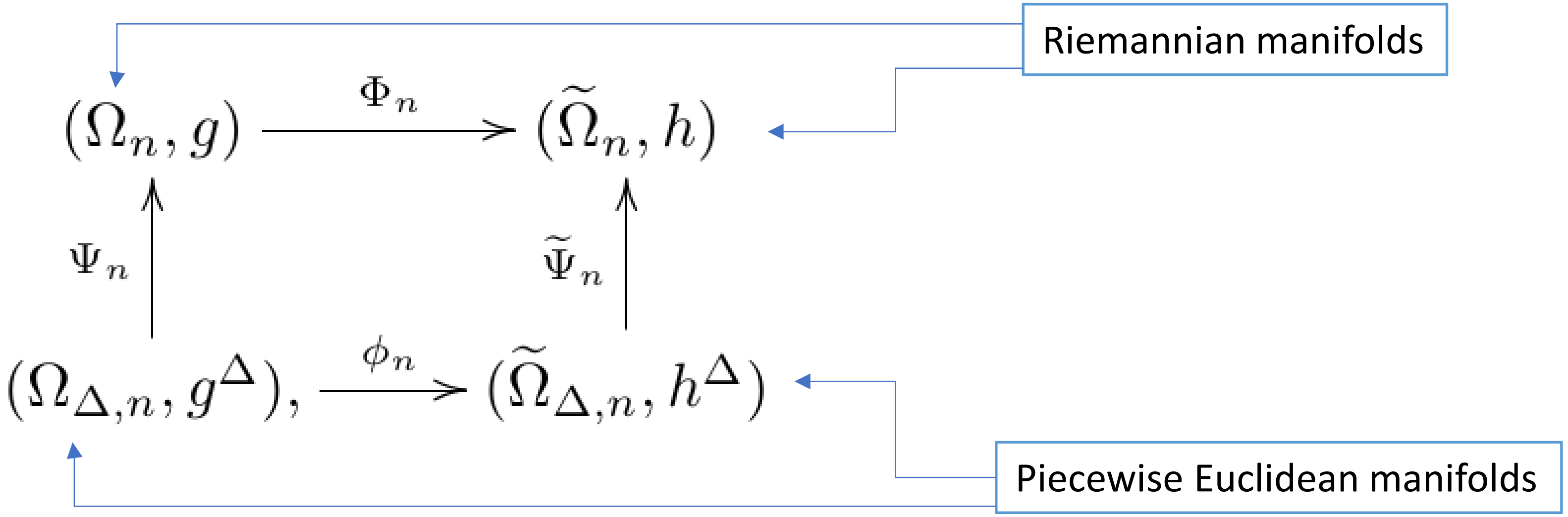
Figures: G. 2016

Setup for Theorem

Common triangulation

Vertex weights

Definition. *Let* $(\Omega, \tilde{\Omega})$ *have an admissible sequence* $\{(\Omega_n, \tilde{\Omega}_n, T_n, f_n, \tilde{f}_n)\}$ *with discrete conformal maps* $\{\phi_n : \Omega_{\Delta,n} \to \tilde{\Omega}_{\Delta,n}\}$. *For each* n *define* $\Phi_n : \Omega_n \to \tilde{\Omega}_n$ *as the composition* $\Phi_n := \tilde{\Psi}_n \circ \phi_n \circ \Psi_n^{-1}$, *where* $\Psi_n, \tilde{\Psi}_n$ *are the* ***Riemannian barycentric maps*** *on* $\Omega_{\Delta,n}, \tilde{\Omega}_{\Delta,n}$ *respectively. This map* Φ_n *is called a barycentric discrete conformal map.*



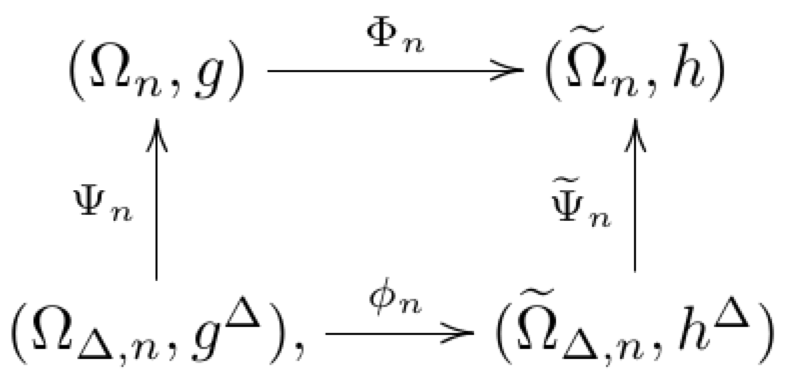
$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}$$

$$\tilde{\ell}_{ij}^2 = \alpha_i e^{2\tilde{f}_i} + \alpha_j e^{2\tilde{f}_j} + 2\eta_{ij} e^{\tilde{f}_i + \tilde{f}_j}$$

Main Theorem

Theorem (G. –Sidbury 2024). *Let* $\{(\Omega_n, \tilde{\Omega}_n, T_n, f_n, \tilde{f}_n)\}$ *be an admissible sequence for* $(\Omega, \tilde{\Omega})$ *and let* $\{\Phi_n\}$ *be the corresponding sequence of barycentric discrete conformal maps. Then the family* $\{\Phi_n\}$ *has a subsequence that converges uniformly on compact subsets of* Ω.

Furthermore, if the admissible sequence is proper, then there exists a positive continuous function e^F *such that* $\Phi_n^* h \to e^F g$ *in* L^∞ *on compact subsets of* Ω*, and hence convergence is to a conformal map.*



$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}$$

$$\tilde{\ell}_{ij}^2 = \alpha_i e^{2\tilde{f}_i} + \alpha_j e^{2\tilde{f}_j} + 2\eta_{ij} e^{\tilde{f}_i + \tilde{f}_j}$$

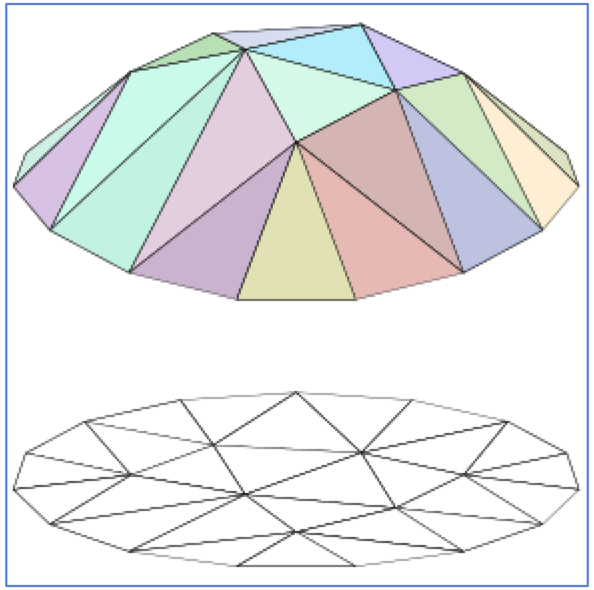
Proof of the Rodin-Sullivan Theorem to General Theorem

1. Maps of circle packings correspond to PL maps.
2. Need to fix something (to eliminate conformal invariance of disk) (set where 2 points go)
3. Need to show the ranges converge. (from the definitions of the packings.)
4. Need to show the domains converge. (Length-Area Lemma)
5. Need to show the maps do not degenerate. (Ring Lemma)
6. Need to show maps converge to conformal. (Hexagonal packing rigidity)
7. (For us): Need to show derivatives converge. (Discrete Schwarz lemma + estimate of Hexagonal Packing constants)

1. Discrete conformal maps are PL.
2. Basepoint maps to a compact set
3. *Riemannian barycentric coordinates* and exhaustions
4. *Riemannian barycentric coordinates* and exhaustions
5. Ring Lemma and fatness assumption
6. *Local discrete conformal rigidity* assumption…
7. …allows *estimate of pullback piecewise Euclidean (Riemannian) metric*

Key new elements

1. Riemannian barycentric coordinates and distortion estimates for convergence of domain and range on manifolds.



2. Local discrete conformal rigidity (instead of hexagonal rigidity) to approximate conformality for all discrete conformal structures

Riemannian center of mass for Riemannian barycentric coordiantes

Let (M, g) be a complete m-dimensional Riemannian manifold with $m > 1$ and let Δ be the n-dimensional standard simplex

$$\Delta = \left\{\lambda \in \mathbb{R}^{n+1} : \lambda^i \geq 0, \sum_i \lambda^i = 1\right\}.$$

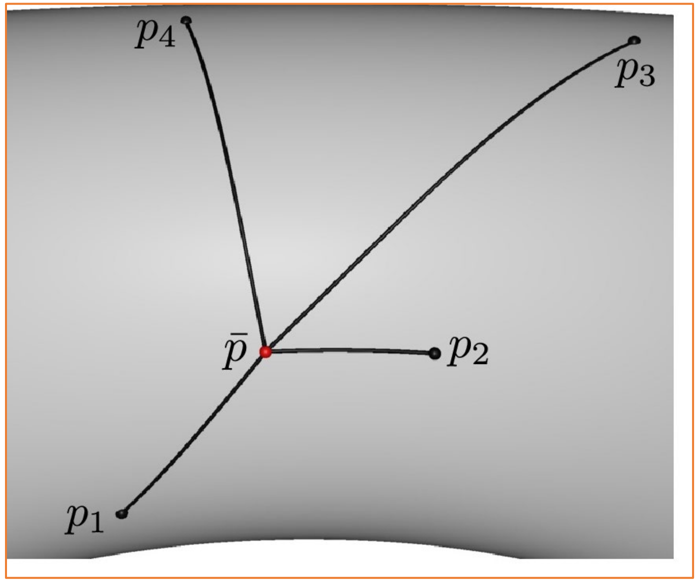


Figure credit: Mancinelli-Puppo

Let $p_0, \ldots, p_n$ be points in M and consider the function $E: M \times \Delta \to \mathbb{R}$ given by

$$E(a, \lambda) = \sum_{i=0}^{n} \lambda^i \, d_g^2(a, p_i),$$

where d_g is the Riemannian distance function on M.

Once we are in the Riemannian setting, the function $E(\cdot, \lambda)$ may not have a minimizer or it may have multiple. However,

Proposition (Karcher-77). *If the points p_i lie in a ball whose radius is less than half the convexity radius, then $E(\cdot, \lambda)$ has a unique minimizer.*

Riemannian barycentric mapping

We can now define the (Riemannian) barycentric mapping, as follows:

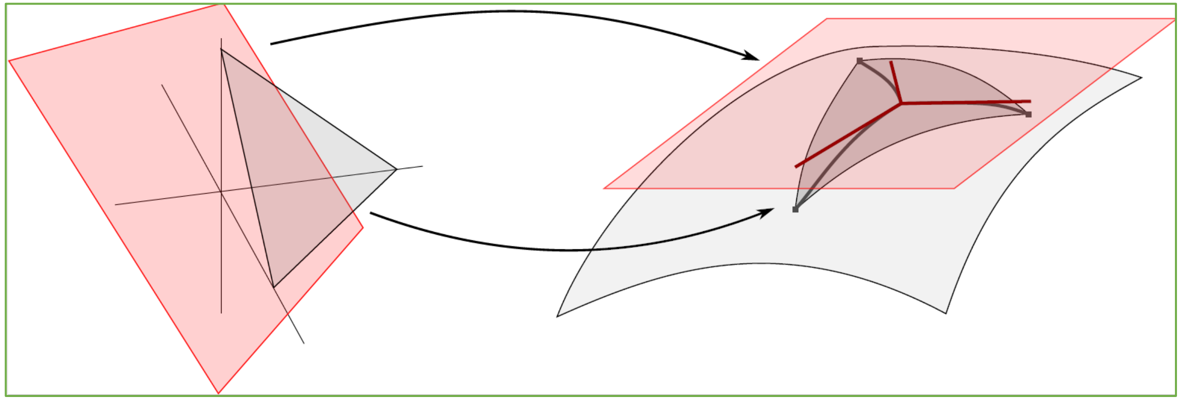
Definition. *For a given $\lambda \in \Delta$, let $\Psi(\lambda)$ be the minimizer of $E(\cdot, \lambda)$. We call Ψ the (Riemannian) barycentric mapping with respect to vertices $p_0, \dots, p_n$. Its image in M is called the corresponding Karcher simplex.*

It is shown in Karcher-77 that local minimizers of $E(\cdot, \lambda)$ for fixed λ are zeros of the section $F: M \times \Delta \to TM$ given by

$$F(a, \lambda) = \sum_{i=0}^{n} \lambda^i X_i|_a, \qquad \textit{where} \quad X_i|_x = \frac{1}{2} \mathrm{grad}_x\, d^2(\mathrm{x}, p_i) = \exp_x^{-1} p_i$$

Notice that each facet of Δ is mapped to a Karcher subsimplex and this subsimplex only depends on the vertices it contains.

Lemma. *Let Δ be the (closed) standard k-simplex. Then the Riemannian barycentric map $\Psi: \Delta \to M$ is a diffeomorphism onto its image.*



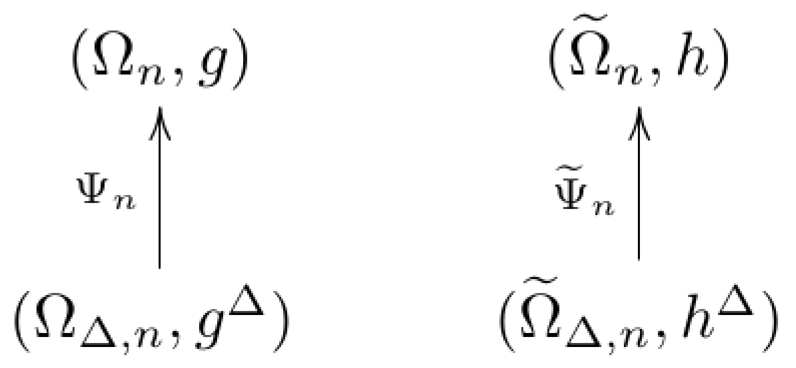
Approximation theorem for Riemannian barycentric coordinates

Theorem (von Deylan-G-Wardetzky-16). *There exist constants* $\alpha = \alpha(n, \vartheta, C_0, C_1)$, $\beta = \beta(n, \vartheta, C_0)$, *and* $\gamma = \gamma(n, \vartheta, C_0, C_1)$ *such that if* $\epsilon < \alpha$ *and* (Δ, g^Δ) *is a* (ϑ, ϵ)*-full simplex then*

$$|(\Psi^* g - g^\Delta)(v, w)| \leq \beta\epsilon^2 |v|_{g^\Delta} |w|_{g^\Delta}, \text{ and}$$

$$|\nabla^e \Psi^* g(u, v, w)| \leq \gamma\epsilon |u|_{g^\Delta} |v|_{g^\Delta} |w|_{g^\Delta}$$

for tangent vectors $u, v, w \in T_\lambda \Delta$ *at any* $\lambda \in \Delta$.



n is the dimension of the simplex, $C_0 = \| R \|_\infty$ and $C_1 = \| \nabla R \|_\infty$.

g^Δ is the metric for the Euclidean simplex with the same edge lengths as the Karcher (Riemannian) simplex

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}$$

$$\tilde{\ell}_{ij}^2 = \alpha_i e^{2\tilde{f}_i} + \alpha_j e^{2\tilde{f}_j} + 2\eta_{ij} e^{\tilde{f}_i + \tilde{f}_j}$$

(Generalized) Hexagonal packing lemma

Let P_n' be a circle packing in $\mathbb{C}$ combinatorially equivalent to P_n, which is n-generations of circles surrounding a single circle (combinatorial disk of radius n). Suppose that c_0 is surrounded by circles $c_1, c_2, \dots, c_k$ in P_n and that $c_0', c_1', \dots, c_k'$ are the corresponding circles in P_n'. Let

$$d_1(P_n, P_n') := \left\{ \frac{r(c_j')/\mathrm{r}(c_l')}{r(c_j)/r(c_l)} : 0 \leq j, l \leq k \right\},$$

and let

$$s(P_n) := \sup_{P_n'} [d_1(P_n, P_n') - 1].$$

> **Theorem** (He-91, He-Rodin-93). *Let P_n be a circle packing such that the valence of P_n is bounded by k_0, the radii of circles in P_n are bounded above by some positive r, and there is some circle c_0 of P_n such that the carrier of P_n contains a closed disc of radius $(2n+1)r$ concentric with c_0. Then there is a constant C depending only on k_0 such that $s(P_n) = s_n \leq C/n$.*

Assumption to replace Hexagonal Packing Lemma

Condition (Local Discrete Conformal Rigidity LDCR). *We say the sequence* $\{(T_n, \mathcal{C}_n, f_n, \bar{f}_n)\}_{n=1}^{\infty}$, *satisfies Local Discrete Conformal Rigidity if there is a sequence of real numbers* $\{s_m\}$ *decreasing to zero such that for any vertex* $v \in T_n$ *that is the center of a combinatorial disk centered at* v *of combinatorial radius at least* m *there exists* N_m *such that if* $n \geq N_m$ *and* w *is adjacent to* v *then*

$$\left| \frac{e^{f_n(v)}}{e^{f_n(w)}} \Big/ \frac{e^{\bar{f}_n(v)}}{e^{\bar{f}_n(w)}} - 1 \right| \leq s_m.$$

Note that we can always take a subsequence such that we can replace N_m *with* m.

We now describe the setting in which we will use this assumption. We will assume that we have $M_{\Delta,n} = (M, T_n, \mathcal{C}_n, f_n)$ and $N_{\Delta,n} = (N, T_n, \mathcal{C}_n, \bar{f}_n)$, where we note that M and N are diffeomorphic and that the triangulation and conformal structures are the same for both $M_{\Delta,n}$ and $N_{\Delta,n}$.

We can now say that the sequence of pairs $\{(M_{\Delta,n}, N_{\Delta,n})\}_{n=1}^{\infty}$ satisfies Local Discrete Conformal Rigidity if $\{(T_n, \mathcal{C}_n, f_n, \bar{f}_n)\}_{n=1}^{\infty}$ satisfies Local Discrete Conformal Rigidity.

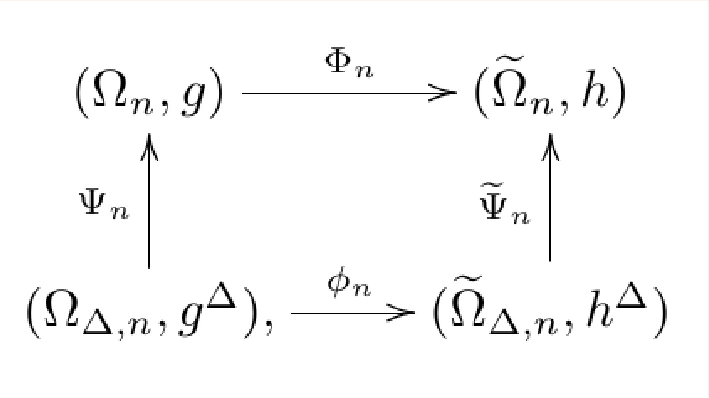
Main estimate

Proposition. *Assume M and N are Riemannian surfaces and let $\Omega \subset M$ and $\tilde{\Omega} \subset N$ be diffeomorphic embedded submanifolds with admissible sequence $\{(\Omega_n, \tilde{\Omega}_n, T_n, f_n, \tilde{f}_n)\}$.*

Then there is some constant C for which the following estimate holds:

$$(1-\beta\epsilon_n)^2(1-Cs_m)e_n^F|X|_g^2 \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon_n)^2(1+Cs_m)e_n^F|X|_g^2,$$

where β is the Riemannian barycentric constant, $\{\epsilon_n\}$ is the sequence of maximum edge lengths for the generalized triangulated exhaustions $\{(\Omega_n, T_n)\}$ of Ω and $\{(\tilde{\Omega}_n, T_n)\}$ of $\tilde{\Omega}$, and $\{s_m\}$ is the sequence of LDCR constants.



The functions e_n^F are the piecewise linear interpolations of the exponential of the ratios of vertex weights.

Assume $Q \in \mathbb{N}$ large enough so that the point p lies within Ω_n for all $n > Q$ and assume that for each such n, the closest vertex v to p is the center of a realized closed combinatorial disc of radius m in T_n.

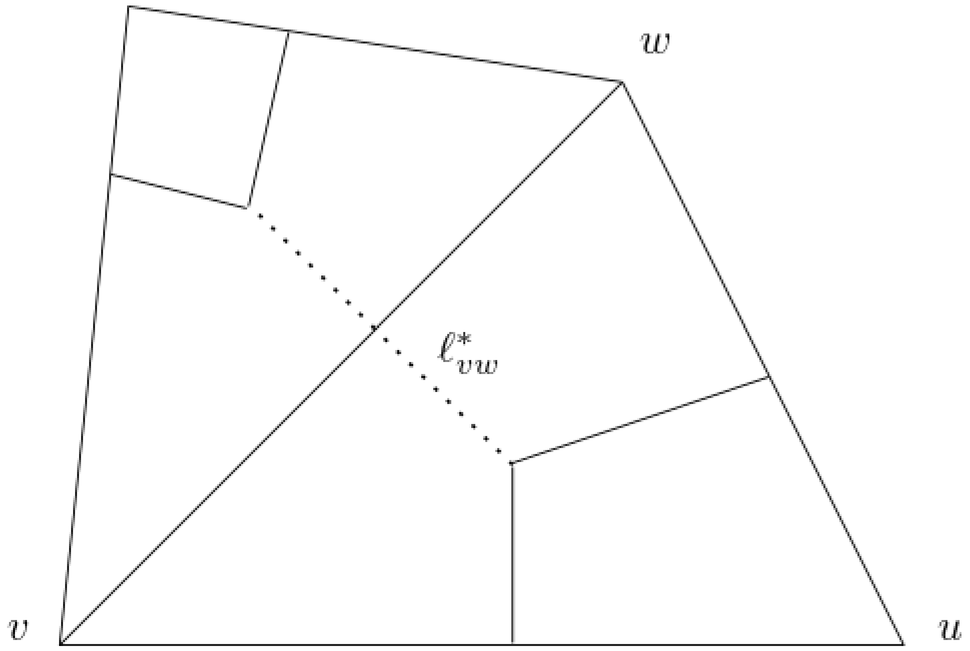
Finite Volume Laplacians

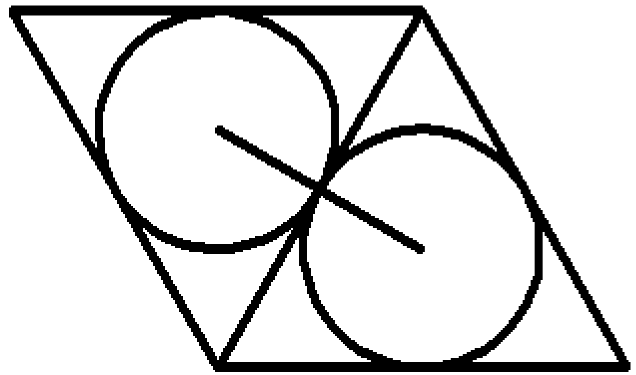
- Recall the weighted graph Laplacian is:

$$Lf_i = \sum_{ij\in E} w_{ij}\left(f_j - f_i\right).$$

- For finite volumes: $w_{ij} = \frac{\ell_{ij}^*}{\ell_{ij}}$
- These give the cotan formula corresponding to circumcentric duals and give sums of inscribed radii for circle packing.
- These Laplacians come up naturally in discrete conformal angle variations: if the change in the conformal factor at vertex v is u_v, then the change in curvature is

$$\frac{d}{dt}K_v = -Lu_v = -\sum_{w\in V(T)} \frac{\ell_{vw}^*}{\ell_{vw}}\left(u_w - u_v\right).$$



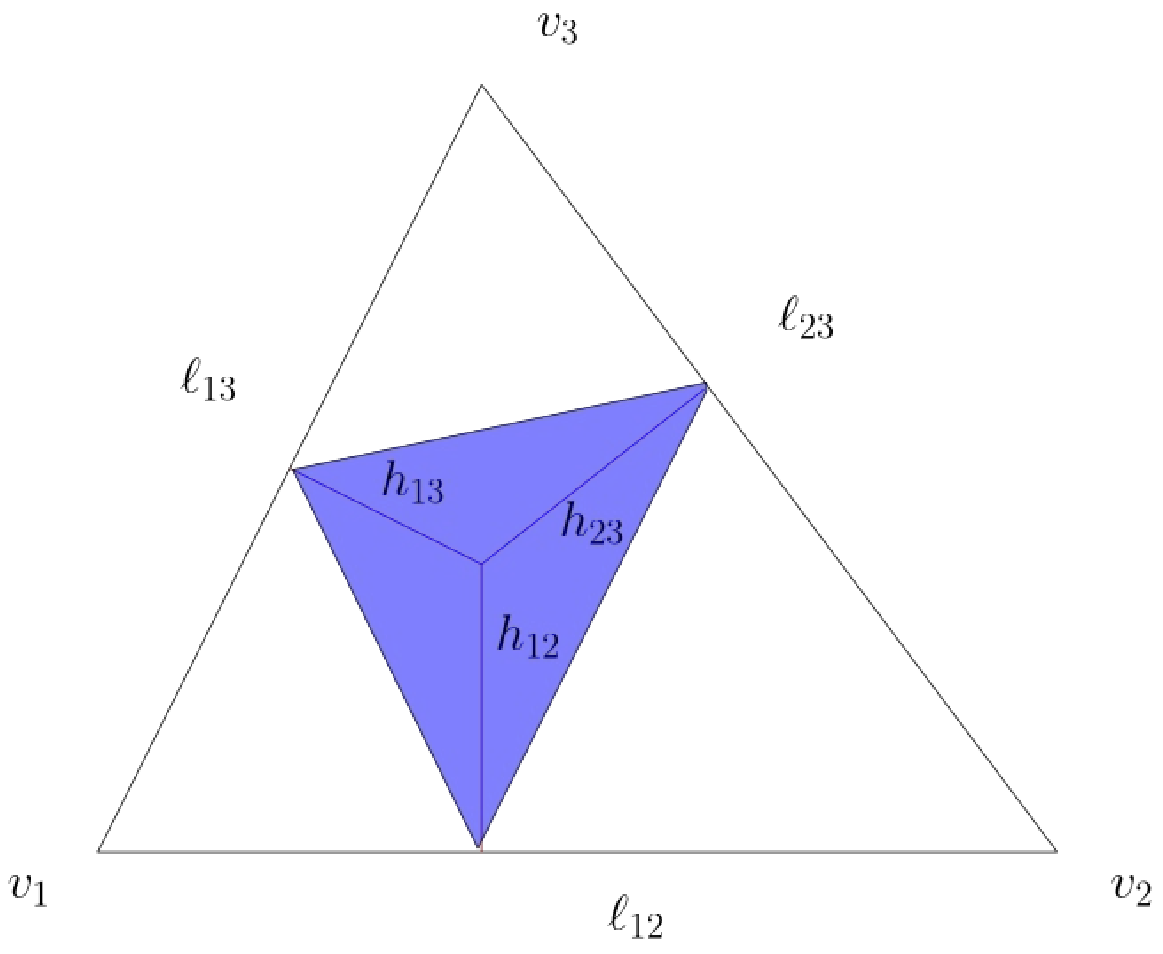


Laplacian determinant in 2D

Definition. *The Laplacian determinant is* $K^*(G) = \det(-\hat{L}_{00})$, *where* $\hat{L}_{00}$ *denotes the Laplacian matrix with the first row and first column removed.*

Theorem. (G. 2007) *For triangles, we have*

$$K^*(T) = \frac{\text{Area}(A^*)}{\text{Area}(A)}.$$



Proof:

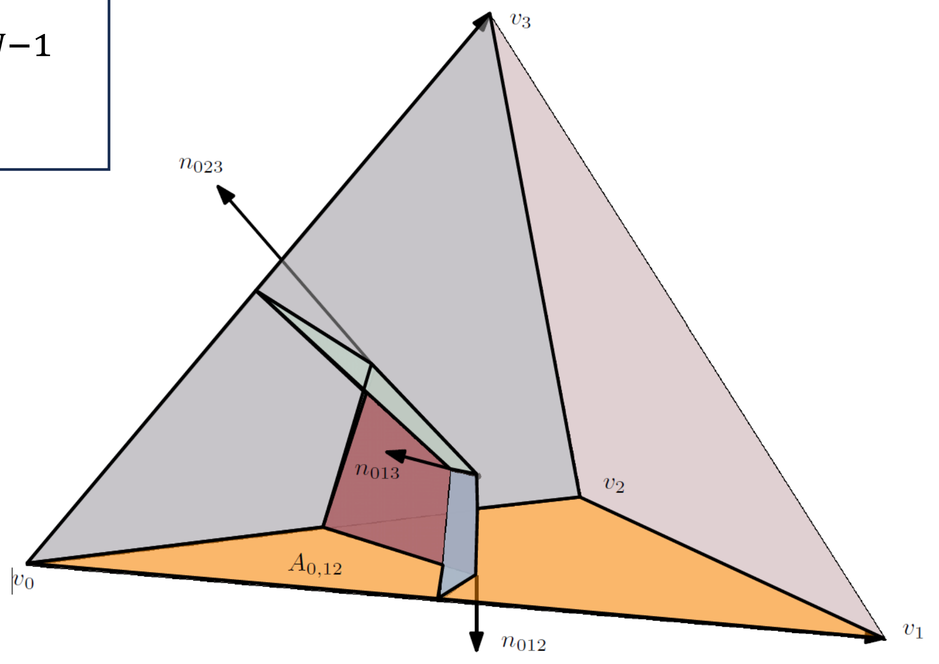
$$L = \begin{bmatrix} -\frac{h_{12}}{\ell_{12}} - \frac{h_{13}}{\ell_{13}} & \frac{h_{12}}{\ell_{12}} & \frac{h_{13}}{\ell_{13}} \\ \frac{h_{12}}{\ell_{12}} & -\frac{h_{12}}{\ell_{12}} - \frac{h_{23}}{\ell_{23}} & \frac{h_{23}}{\ell_{23}} \\ \frac{h_{13}}{\ell_{13}} & \frac{h_{23}}{\ell_{23}} & -\frac{h_{13}}{\ell_{13}} - \frac{h_{23}}{\ell_{23}} \end{bmatrix}$$

$$-\det \widehat{L}_{00} = \frac{h_{12}}{\ell_{12}}\frac{h_{13}}{\ell_{13}} + \frac{h_{12}}{\ell_{12}}\frac{h_{23}}{\ell_{23}} + \frac{h_{13}}{\ell_{13}}\frac{h_{23}}{\ell_{23}} = \frac{A^*_{123}}{A_{123}}$$

Normal vectors to a simplex

- *Let $n_{\hat{0}}, n_{\hat{1}}, \dots, n_{\hat{N}}$ denote the outward pointing normals to an N-simplex σ with vertices $v_0, \dots, v_n$ in $\mathbb{R}^N$, where $n_{\hat{j}}$ is orthogonal to the plane containing all vertices except v_j, and such that the length of $n_{\hat{j}}$ is equal to the area of the face containing all vertices except v_j.*
- *Then for any $j = 0, \dots, N$,*

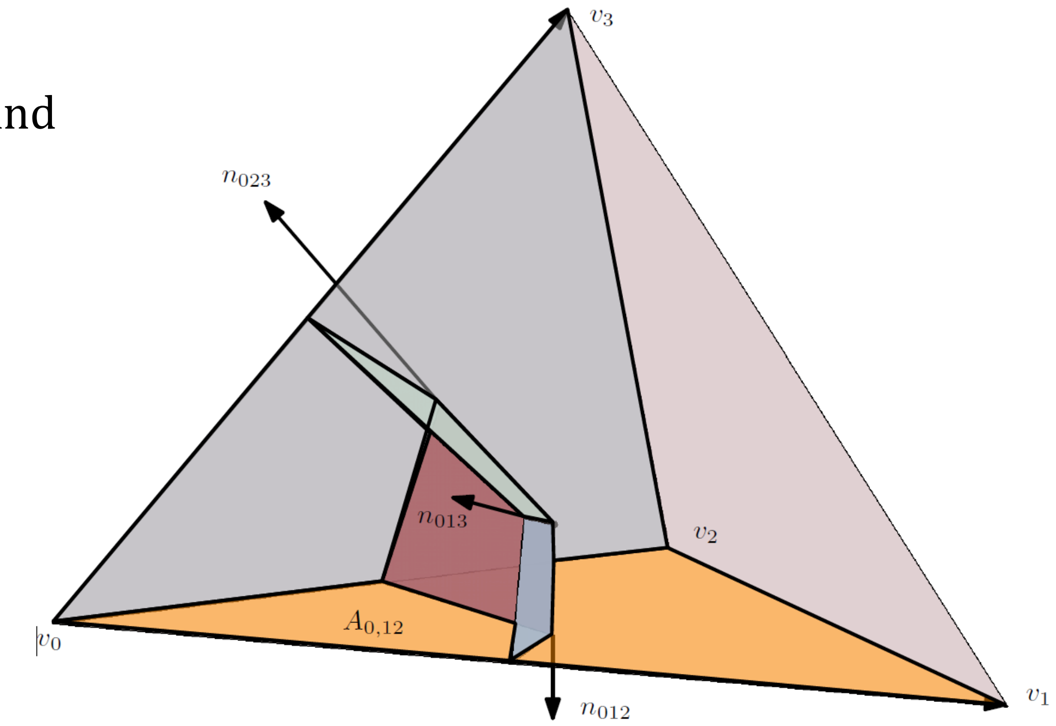
$$\det\left[n_{\widehat{0}} \quad n_{\widehat{1}} \quad \cdots \quad \widehat{n_{\hat{j}}} \quad \cdots \quad n_{\widehat{N}}\right] = \frac{(-1)^{N+j} N^N}{N!} \mathrm{Vol}(\sigma)^{N-1}$$



How to relate normal vectors to vertex vectors

- $$\begin{bmatrix} n_{\hat{0}}^T & N\mathrm{Vol}(\bar{\sigma}_{\hat{0}}) \\ n_{\hat{1}}^T & N\mathrm{Vol}(\bar{\sigma}_{\hat{1}}) \\ \vdots & \vdots \\ n_{\hat{N}}^T & N\mathrm{Vol}(\bar{\sigma}_{\hat{N}}) \end{bmatrix} \begin{bmatrix} v_0 & v_1 & \cdots & v_N \\ 1 & 1 & \cdots & 1 \end{bmatrix} = N\mathrm{Vol}(\sigma) I_{N+1},$$

where $\bar{\sigma}_{\hat{j}}$ denotes the N −simplex with vertices the origin and the $(N-1)$ −simplex excluding vertex j



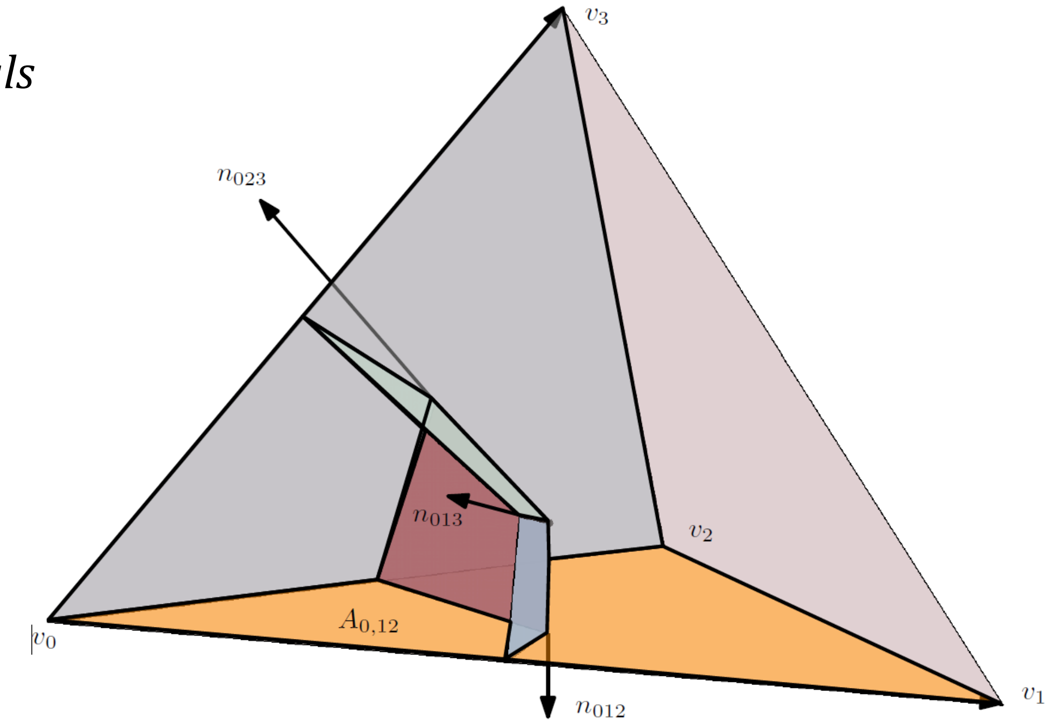
Laplacian determinant

Definition. *The Laplacian determinant is* $K^*(T) = \det(-\hat{L}_{00})$, *where* $\hat{L}_{00}$ *denotes the Laplacian matrix with the first row and first column removed.*

Theorem. (G.-Doehrman 2022) *Let* $T = [0,1,\dots,N]$ *be a* N*-simplex realized with vertices* $v_0,\dots,v_N \in \mathbb{R}^N$ *with duality structure determined by* $C_T \in \mathbb{R}^N$.

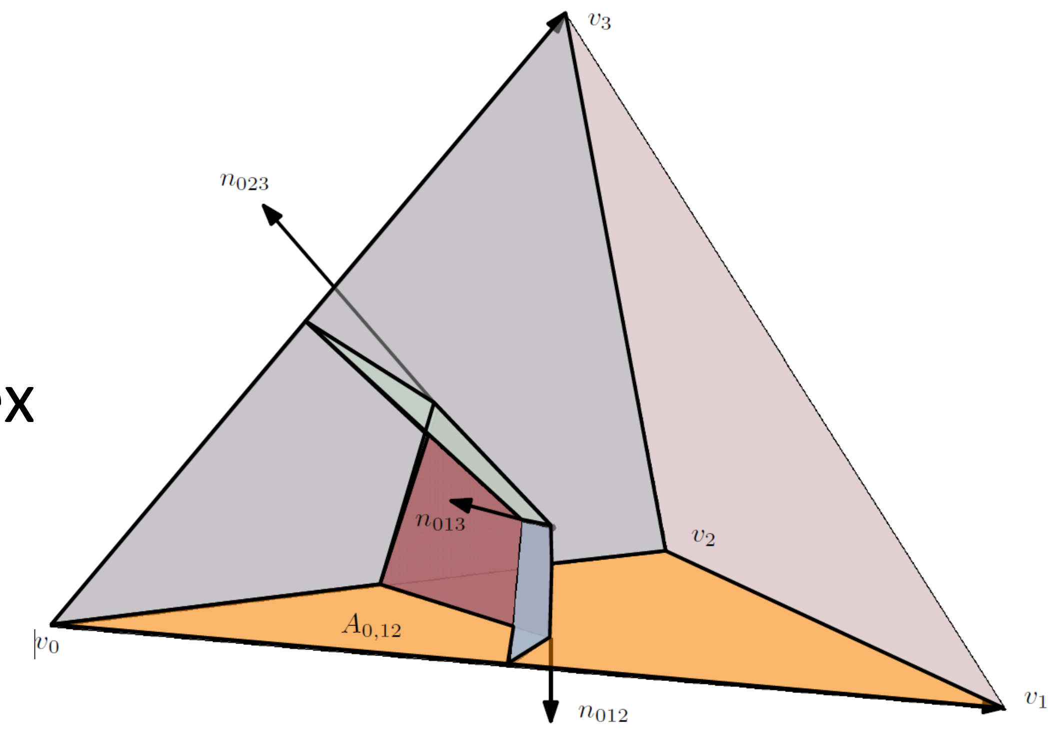
Let the dual simplex $T^\#$ *be the simplex determined by the normals*
$n_i = -\sum_j n_{ij}$ *where* $n_{ij} = \frac{\ell_{ij}^*}{\ell_{ij}}(v_j - v_i)$,
Then, we have

$$K^*(T) = \frac{N^N \mathrm{Vol}(T^\#)^{N-1}}{(N!)^2 \mathrm{Vol}(T)}.$$



Key idea for the proof

- $n_i = -\sum \frac{\ell_{ij}^*}{\ell_{ij}}(v_j - v_i)$
- These n_i determine the faces of a simplex by Minkowski's Theorem
- Hence we have $-Lv^T = n^T$
- Now take the determinant of both sides
- Question: What is the simplex $T^\#$



Connection to the space of spheres

- The coordinate of the sphere is (v, a, d) corresponds to the sphere

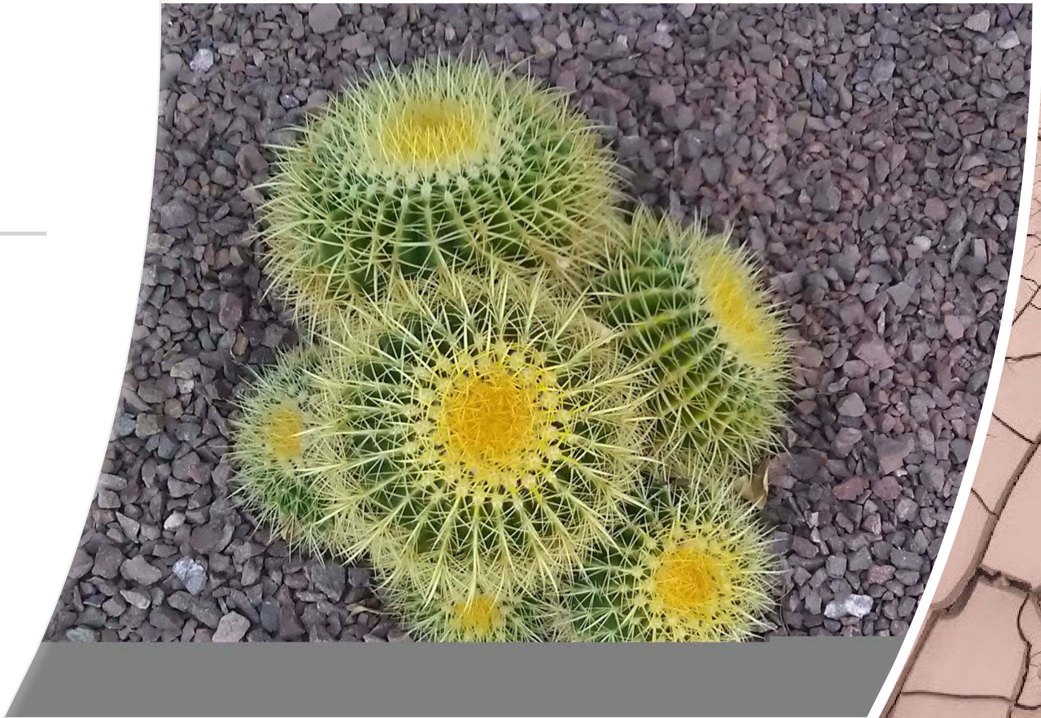
$$a|x|^2 - 2x \cdot v - d = 0$$

- $$\begin{bmatrix} n_{\hat{0}}^T & 0 & N\mathrm{Vol}(\bar{\sigma}_{\hat{0}}) \\ n_{\hat{1}}^T & 0 & N\mathrm{Vol}(\bar{\sigma}_{\hat{1}}) \\ \vdots & 0 & \vdots \\ n_{\hat{N}}^T & 0 & N\mathrm{Vol}(\bar{\sigma}_{\hat{N}}) \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} v_0 & v_1 & \cdots & v_N \\ 1 & 1 & \cdots & 1 \\ -|v_0|^2 & -|v_1|^2 & \cdots & -|v_N|^2 \end{bmatrix} = N\mathrm{Vol}(\sigma) I_{N+2}$$
- We can now expand the Laplacian to a new matrix:

$$\begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} v_0^T & 1 & -|v_0|^2 \\ v_1^T & 1 & -|v_1|^2 \\ \vdots & \vdots & \vdots \\ v_N^T & 1 & -|v_N|^2 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix} = \text{Simplex!}$$

$$\det \begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} = \pm N \det \hat{L}_{00}$$

Thank you!



Full assumptions of the theorem

We begin by recalling the definition of a generalized triangulated exhaustion.

Definition 10. *Let (M, g) be a complete Riemannian manifold and let $\Omega \subset M$ be an embedded submanifold, without boundary. Assume that there exists a sequence of compact submanifolds $\Omega_n \subset M$ such that the following properties are satisfied for each compact $K \subset \Omega$:*

1. *There exists $N \in \mathbb{N}$ such that $K \subset \Omega_n$ for every $n > N$.*
2. *For every open subset U containing Ω, there exists $Q \in \mathbb{N}$ such that $\Omega_n \subset U$ for every $n > Q$.*
3. *Each Ω_n is equipped with a triangulation T_n with geodesic edges and Riemannian barycentric map Ψ_n making each Euclidean simplex (Δ, g^{Δ}) that has at least one vertex in K a (ϑ, ϵ_n)-full simplex where $\{\epsilon_n\}$ is a sequence of real numbers decreasing to zero.*

In this case, we say that the sequence of pairs $\{(\Omega_n, T_n)\}$ is a generalized triangulated exhaustion of Ω.

Admissible Sequences

Let $\{f_n\}$ *and* $\{\tilde{f}_n\}$ *be sequences of discrete conformal factors for* $\Omega_{\Delta,n}$ *and* $\tilde{\Omega}_{\Delta,n}$ *respectively such that for each* n *the piecewise flat manifolds* $(\Omega_n, T_n, \ell(f_n))$ *and* $(\tilde{\Omega}_n, T_n, \ell(\tilde{f}_n))$ *are discrete conformal under the discrete conformal structure* $\mathcal{C}_n$ *with discrete conformal map* ϕ_n*.*

Assume that the following conditions hold:

1. *The sequence* $\{T_n, \mathcal{C}_n, f_n, \tilde{f}_n\}$ *satisfies the* ***LDCR condition****.*
2. *The piecewise flat surfaces* $(\Omega_n, T_n, \ell(f))$ *and* $(\tilde{\Omega}_n, T_n, \ell(\tilde{f}))$ *satisfy the* ***Ring Lemma condition****.*
3. *For each compact* $K \subset \Omega$*, the ratio of discrete conformal factors* $e^{\tilde{f}_n(v)}/e^{f_n(v)}$ *has a* ***uniform upper bound*** $H = H(K)$ *depending only on the compact set* K*.*
4. *There is some point* $x \in \Omega$ *such that the image set* $\{\Phi_n(x)\}$ *is contained in some compact subset* $V \subset N$*.*

In this case, we say that $\{(\Omega_n, \tilde{\Omega}_n, T_n, f_n, \tilde{f}_n)\}$ *is an* ***admissible sequence*** *for* $(\Omega, \tilde{\Omega})$*.*

If in addition the LDCR constants s_m *are such that there is some positive constant* α *such that* $s_m \leq \alpha/m$*, then we say that* $\{(\Omega_n, \tilde{\Omega}_n, T_n, f_n, \tilde{f}_n)\}$ *is a* ***proper*** *admissible sequence for* $(\Omega, \tilde{\Omega})$*.*