Optimal disc and sphere packings

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Circle Packings, Minimal Surfaces, and Discrete Differential Geometry
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1. Introduction
2. Disc packings
3. Sphere packings
4. Conclusion

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Optimal coin packings

Given infinite number of identical coins (),

how to place them on an infinite plane without overlap to maximize the covered surface?

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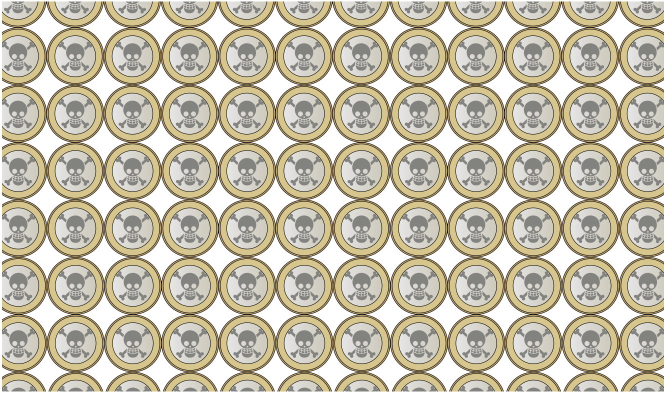
coin packing: 

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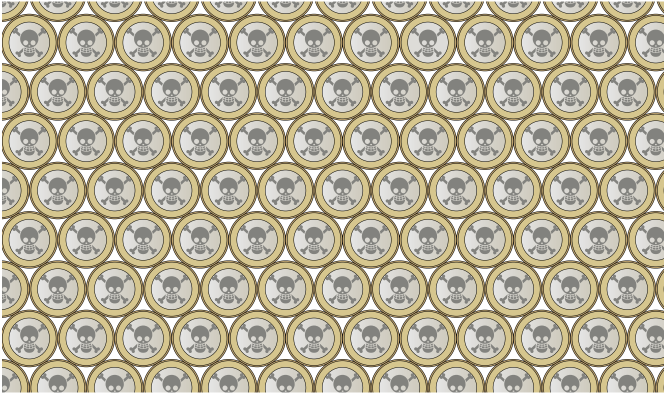
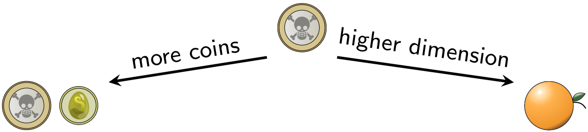
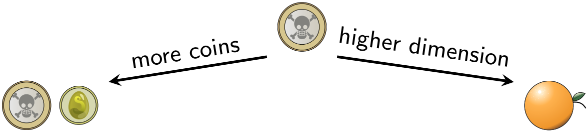
hexagonal coin packing: 

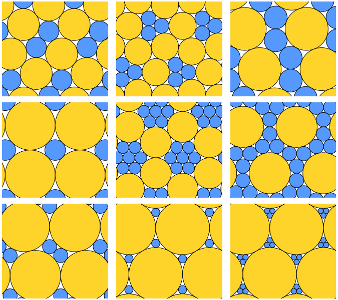
1910–1940

The hexagonal coin packing is optimal.


more coins
higher dimension

Introduction





(proved optimal in 2000-2022)

Kepler conjecture, 1611

The “cannonball” packing is optimal:

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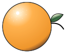

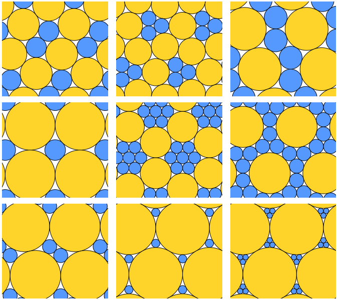

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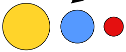

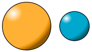

$\mathbb{R}^8, \mathbb{R}^{24}$

(Viazovska, Fields Medal 2022)

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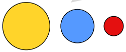

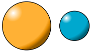

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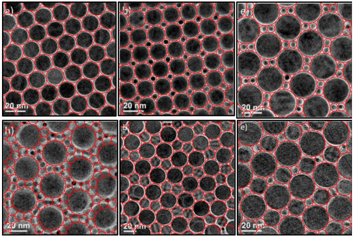
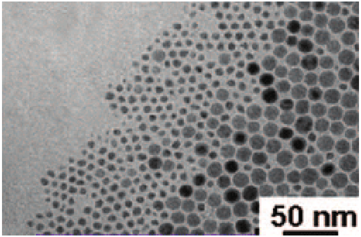
Nanomaterials and packings

combine different types of nanoparticles
self-assembly

new material → phase separation

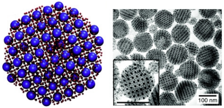


Paik et al 2015



Cheon et al 2006

Also in 3D:



Wu, Fan, Yin 2022

Chemists' question : **which sizes and concentrations allow for new materials?**

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Definitions

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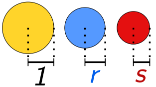
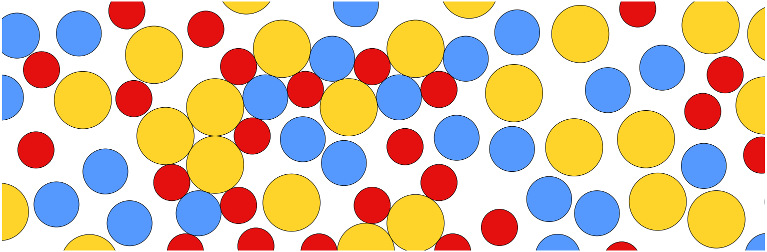


Packing P:
(in $\mathbb{R}^2$)



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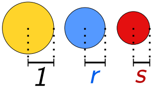

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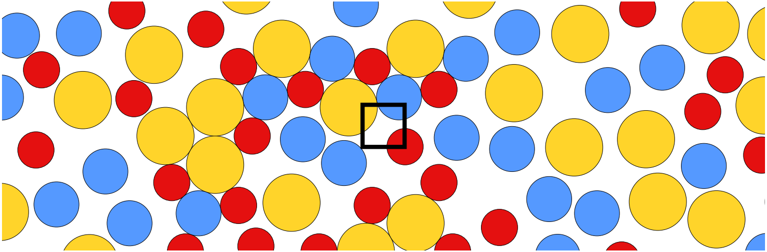

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$$\delta(P) := \limsup_{n\to\infty} \frac{area([-n,n]^2 \cap P)}{area([-n,n]^2)}$$

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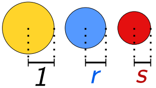

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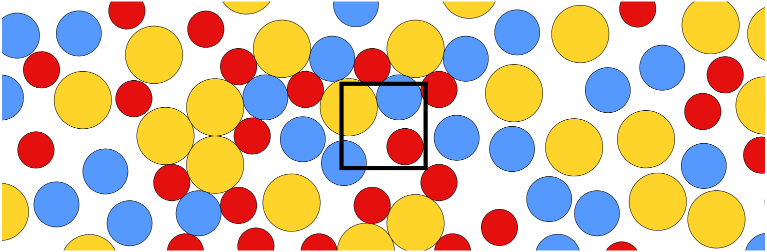

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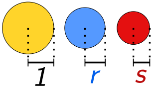

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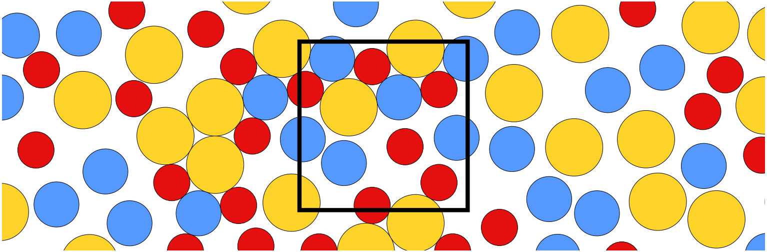

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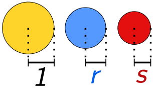

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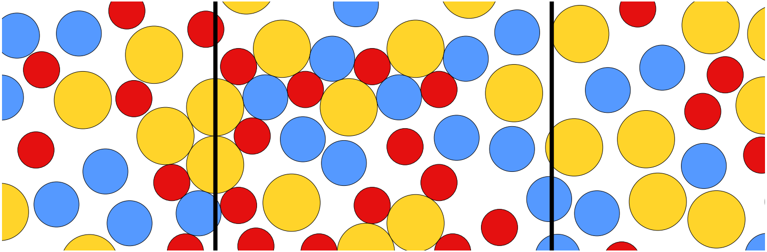

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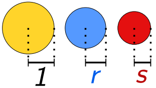

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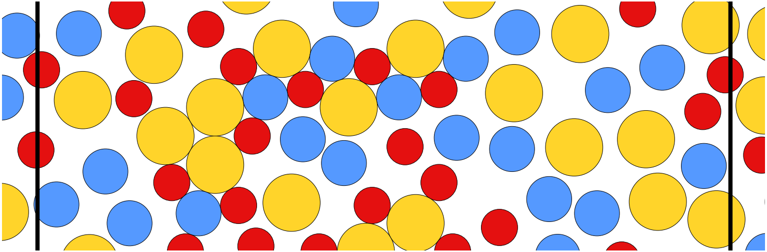

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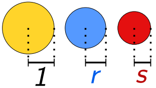

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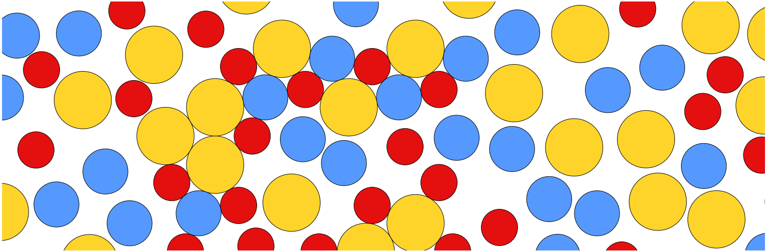

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Disc packings

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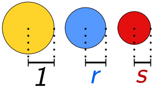

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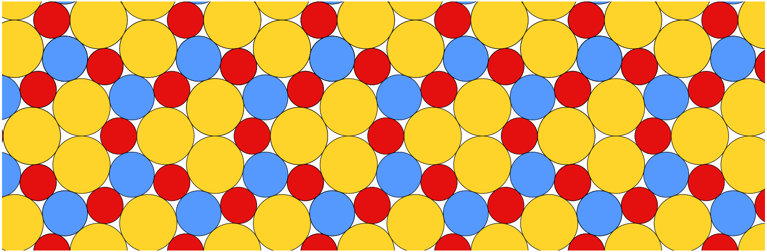

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Main Question

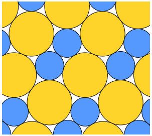
Given a finite set of discs (e.g., ),
what is the maximal density δ^* of a packing?

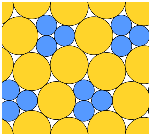
$$\delta^* := \sup_P \delta(P)$$

Optimal 2-disc packings

Theorem (Heppes 2000, 2003, Kennedy 2005, Bedaride and Fernique 2022)

Each of the following packings is optimal (densest) for discs of radii 1 and r:


$r \approx 0.63$ $\delta^* \approx 91.1\%$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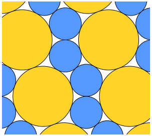
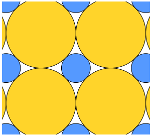
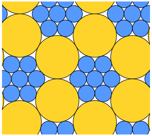

$r \approx 0.53$ $\delta^* \approx 91.4\%$


$r \approx 0.41$ $\delta^* \approx 92\%$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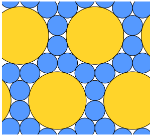
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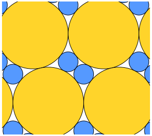
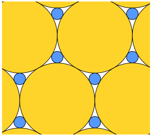

$r \approx 0.28$ $\delta^* \approx 93.2\%$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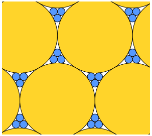
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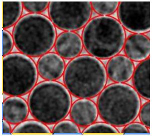

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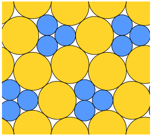

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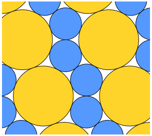

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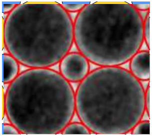

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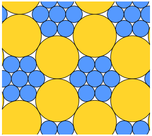

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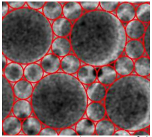

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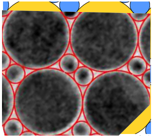

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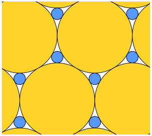

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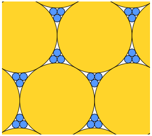

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Triangulated packings

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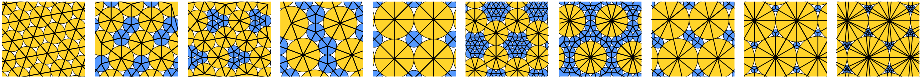

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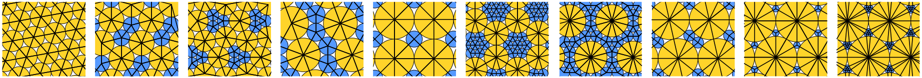
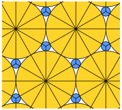


Conjecture (Connelly 2018)

If a finite set of discs allows **saturated** triangulated packings then one of them is optimal.



triangulated
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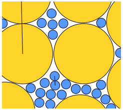
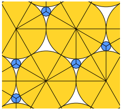


non triangulated
saturated



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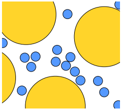

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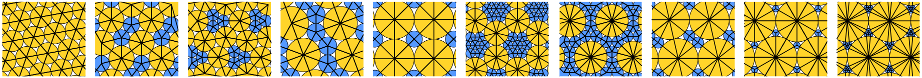

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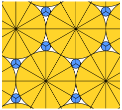

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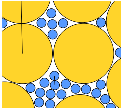

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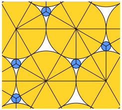

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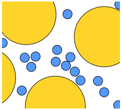

non triangulated
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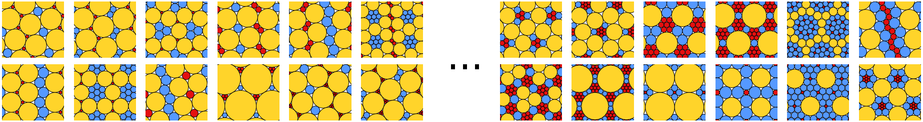
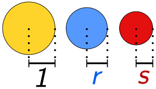
Theorem (Fernique, Hashemi, Sizova 2019)

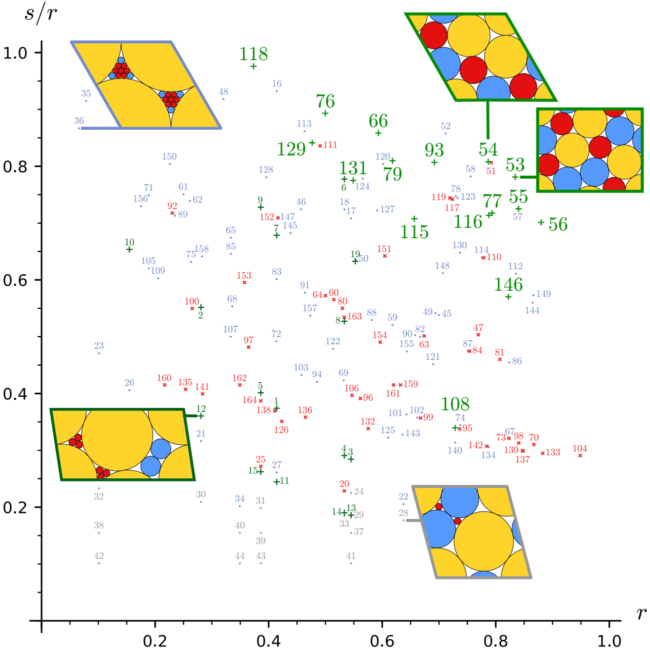
Discs of radii 1, r and s: there are 164 pairs (r, s) allowing triangulated packings.





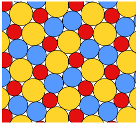
164 (r, s) allowing triangulated packings:

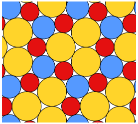
- 15 cases: non saturated
- 16+16 cases: a **ternary** or **binary** triangulated packing is densest
- 45 cases: a binary **non** triangulated packing is denser

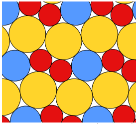


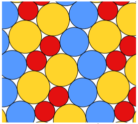
Theorem (Fernique, P 2023)

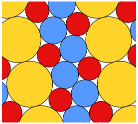
Each of the following packings is optimal for discs of radii 1, r and s:

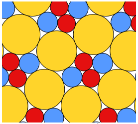

$r\approx0.8$ $s\approx0.7$ $\delta^*\approx90.9\%$

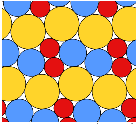

$r\approx0.8$ $s\approx0.6$ $\delta^*\approx90.9\%$

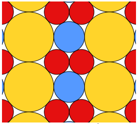

$r\approx0.8$ $s\approx0.6$ $\delta^*\approx91\%$

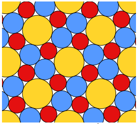

$r\approx0.8$ $s\approx0.6$ $\delta^*\approx90.9\%$

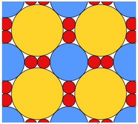

$r\approx0.6$ $s\approx0.5$ $\delta^*\approx91.2\%$

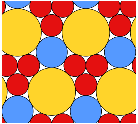

$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.5\%$

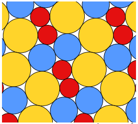

$r\approx0.8$ $s\approx0.6$ $\delta^*\approx91\%$

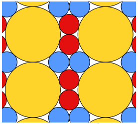

$r\approx0.6$ $s\approx0.5$ $\delta^*\approx91.3\%$

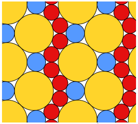

$r\approx0.7$ $s\approx0.5$ $\delta^*\approx91.1\%$

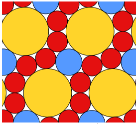

$r\approx0.7$ $s\approx0.2$ $\delta^*\approx93.2\%$

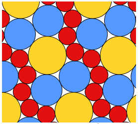

$r\approx0.7$ $s\approx0.5$ $\delta^*\approx91.5\%$


$r\approx0.8$ $s\approx0.6$ $\delta^*\approx91\%$


$r\approx0.4$ $s\approx0.4$ $\delta^*\approx92.2\%$

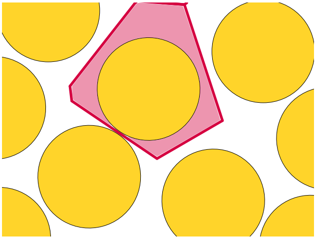

$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.8\%$


$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.7\%$

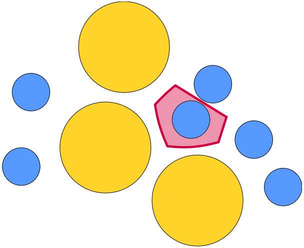

$r\approx0.8$ $s\approx0.5$ $\delta^*\approx91.2\%$

FM-triangulation

1-disc packing



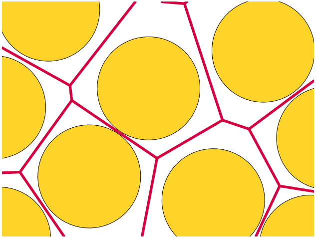
multi-size disc packing



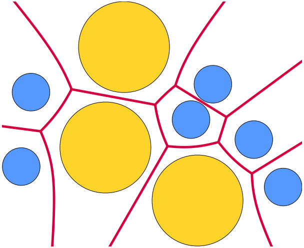
Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

FM-triangulation

1-disc packing



multi-size disc packing

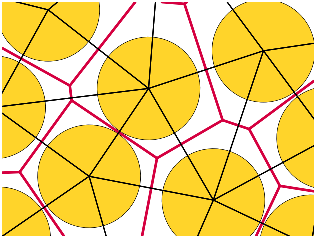


Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

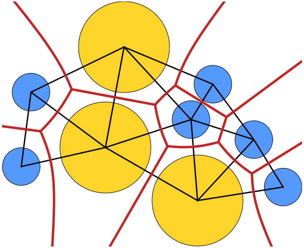
Voronoi diagram of a packing: partition of the plane into Voronoi cells

FM-triangulation

1-disc packing



multi-size disc packing



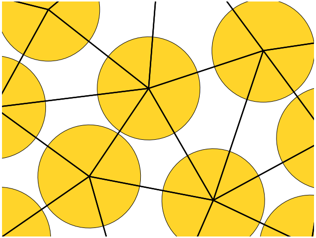
Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

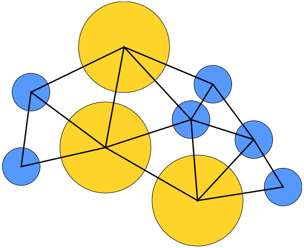
FM-triangulation of a packing: dual graph of the Voronoi diagram   Fejes Tóth, Mólnar

FM-triangulation

1-disc packing



multi-size disc packing



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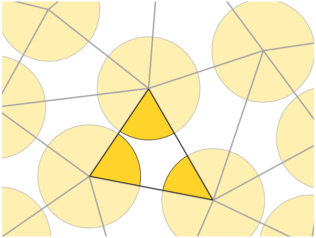
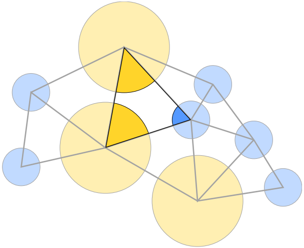
FM-triangulation

1-disc packing



multi-size disc packing



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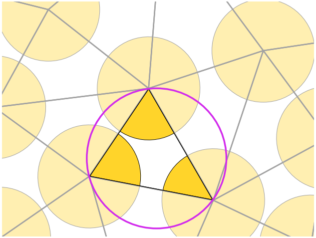
FM-triangulation of a packing: dual graph of the Voronoi diagram — Fejes Tóth, Mólnar

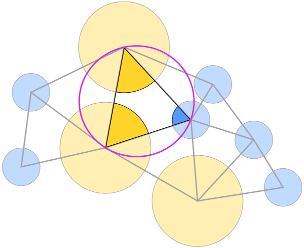
Density of a triangle Δ in a packing $=$ its proportion covered by discs $\delta_\Delta = \frac{area(\Delta \cap P)}{area(\Delta)}$

FM-triangulation

1-disc packing



multi-size disc packing



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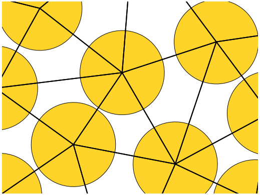
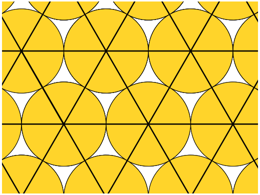
Density of a triangle Δ in a packing = its proportion covered by discs $\delta_\Delta = \frac{area(\Delta \cap P)}{area(\Delta)}$

Local density redistribution

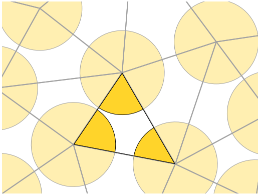


P of density $\delta(P)$



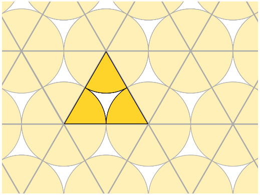
P^* of density δ^*

Local density redistribution



P of density $\delta(P)$

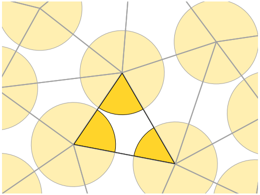
$\forall\Delta,\ \delta(\Delta) \leq \delta(\triangle) = \delta^*$



P^* of density δ^*

$\delta(\triangle) = \delta^*$

Local density redistribution



P of density $\delta(P)$

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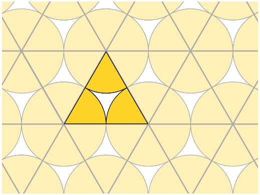
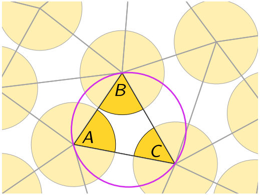
$\delta(P) \leq \delta^*$



P^* of density δ^*

$\delta($$) = \delta^*$

Local density redistribution



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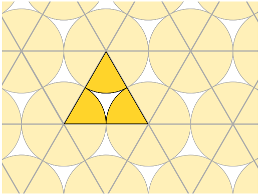


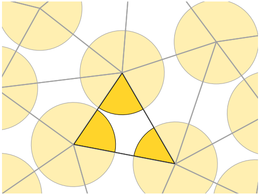
P^* of density δ^*

$\delta($$) = \delta^*$

Proof:

- the smallest angle of any Δ is at least $\frac{\pi}{6}$ $\qquad 2 > R = \frac{|AB|}{2\sin\widehat{C}} \geq \frac{1}{\sin\widehat{C}} \Longrightarrow \widehat{C} > \frac{\pi}{6}$
- thus the largest angle is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$
- density of a triangle Δ: $\delta(\Delta) = \frac{\pi/2}{area(\Delta)}$
- the area of a triangle ABC with the largest angle $\hat{A}$: $\frac{|AB|\cdot|AC|\cdot\sin\widehat{A}}{2} \geq \frac{2\cdot 2\cdot\frac{\sqrt{3}}{2}}{2} = \sqrt{3}$
- thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}} = \delta^*$

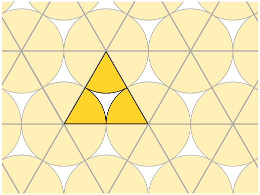
Local density redistribution



P of density $\delta(P)$

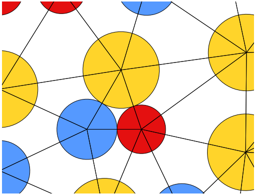
$\forall\Delta,\ \ \delta(\Delta) \leq \delta($$) = \delta^*$

$\delta(P) \leq \delta^*$

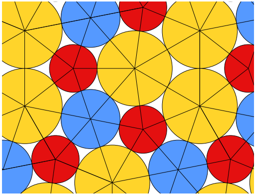


P^* of density δ^*

$\delta($$) = \delta^*$

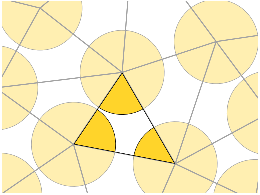


P of density $\delta(P)$



P^* of density δ^*

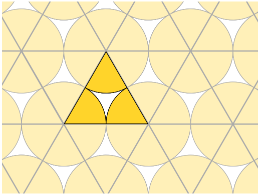
Local density redistribution



P of density $\delta(P)$

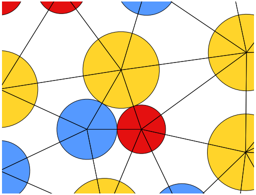
$\forall\Delta,\ \delta(\Delta) \leq \delta($$) = \delta^*$

$\delta(P) \leq \delta^*$



P^* of density δ^*

$\delta($$) = \delta^*$

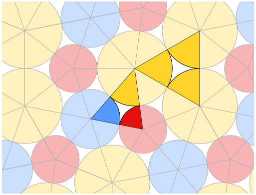


P of density $\delta(P)$

Triangles in P^* have different densities:

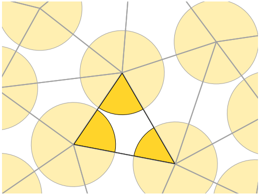
$\delta\Big($$\Big) < \delta^* < \delta\Big($$\Big)$

Hopeless to bound the density by δ^* in each triangle...



P^* of density δ^*

Local density redistribution



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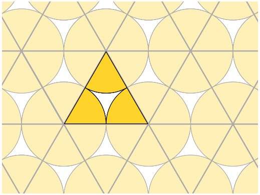
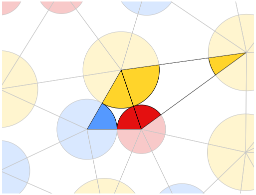


P^* of density δ^*

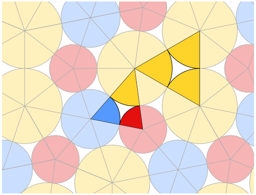
$\delta($$) = \delta^*$



P of density $\delta(P) \leq \delta'(P)$

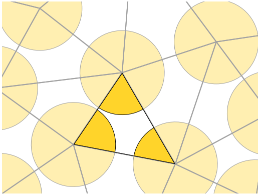
redistributed density δ':

dense triangles
share their density
with neighbors



P^* of density δ^*

Local density redistribution



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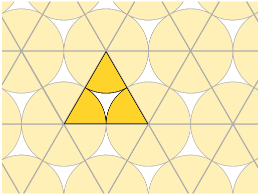
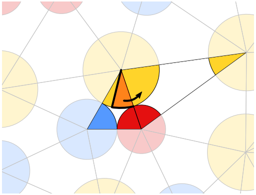


P^* of density δ^*

$\delta($$) = \delta^*$



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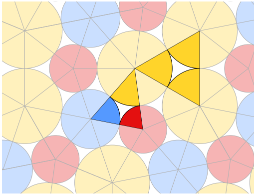
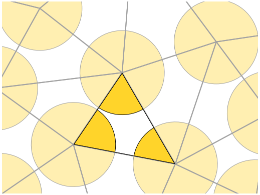


P^* of density δ^*

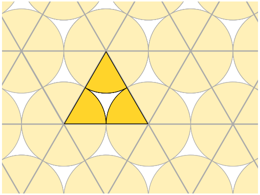
Local density redistribution



P of density $\delta(P)$

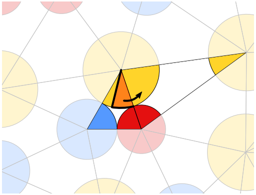
$\forall \Delta,\ \delta(\Delta)$ $\leq$ $\delta($$) = \delta^*$

$\delta(P) \leq \delta^*$



P^* of density δ^*

$\delta($$) = \delta^*$



P of density $\delta(P) \leq \delta'(P)$

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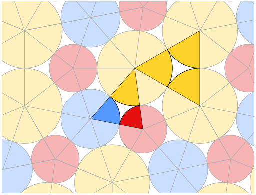


P^* of density δ^*

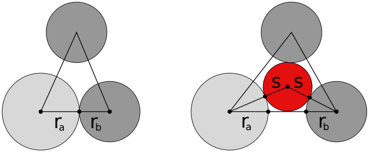
Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ? (there is a continuum of them)

Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ? (there is a continuum of them)

FM-triangulation properties + saturation $\Rightarrow$ uniform bound on edge length

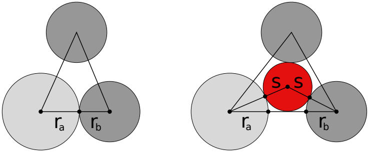


$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ? (there is a continuum of them)

FM-triangulation properties + saturation $\Rightarrow$ uniform bound on edge length

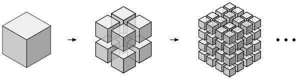


$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

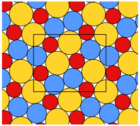
- Interval arithmetic: to verify $\delta'(\Delta_{a,b,c}) \leq \delta^*$ for all $(a, b, c) \in [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [\underline{c}, \overline{c}]$,

  we verify $[\underline{\delta}, \overline{\delta}] \leq \delta^*$ where $[\underline{\delta}, \overline{\delta}] = \delta'(\Delta_{[\underline{a},\overline{a}],[\underline{b},\overline{b}],[\underline{c},\overline{c}]})$

- If $\delta^* \in [\underline{\delta}, \overline{\delta}]$, recursive subdivision:



Our proof worked for these cases: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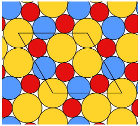
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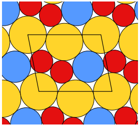
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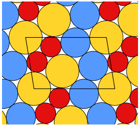
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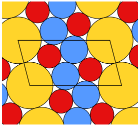
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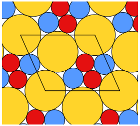
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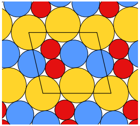
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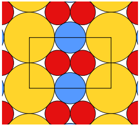
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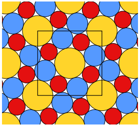
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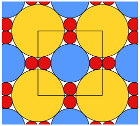
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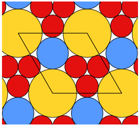
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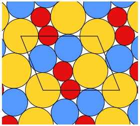
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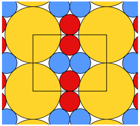
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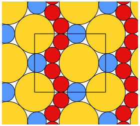
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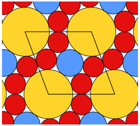
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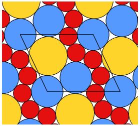
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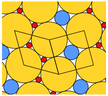
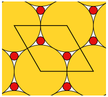
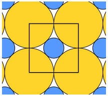
And these:

$\delta^* \approx 93\%$ 

$\delta^* \approx 95\%$ 

$\delta^* \approx 92\%$ 

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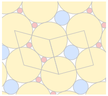
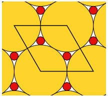
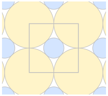
$\delta^* \approx 93\%$ 

$\delta^* \approx 95\%$ 

$\delta^* \approx 92\%$ 

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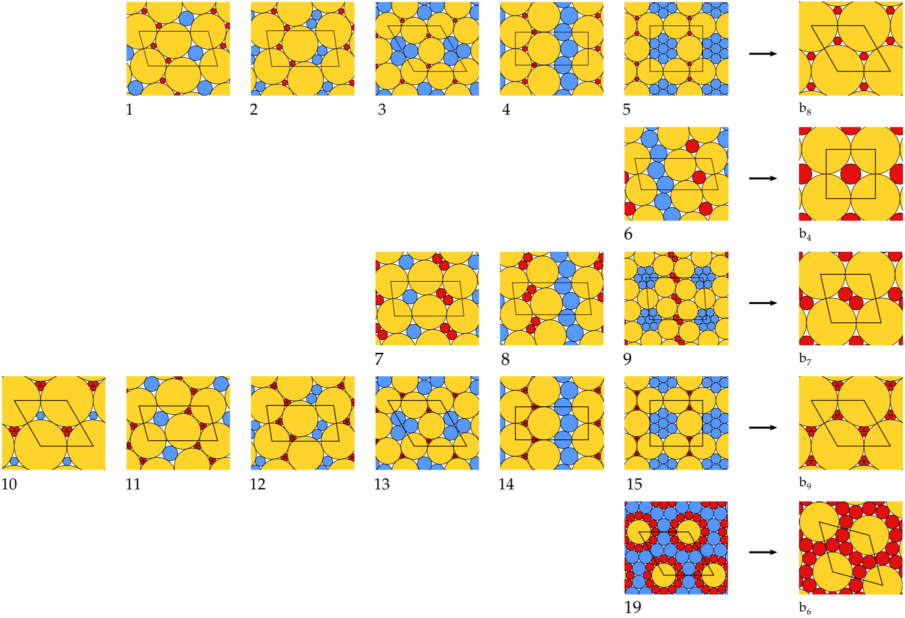
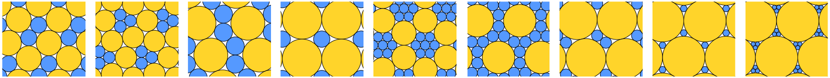
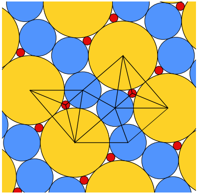


45 counter examples: *flip-and-flow* method



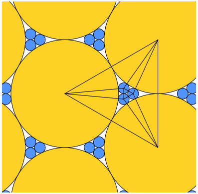
When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

triangulated ternary packing



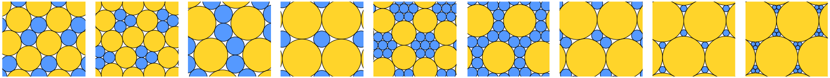
$\delta \leq 0.931369$ $s \approx 0.121445$

dense binary packing



$\delta \approx 0.962430$ $r \approx 0.101021$

45 counter examples: *flip-and-flow* method



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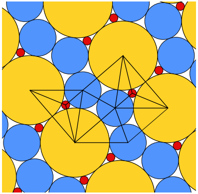
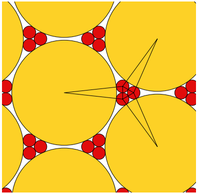


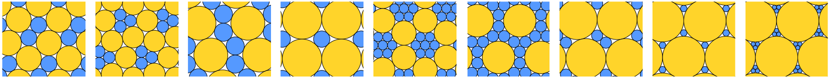
$\delta \leq 0.931369$ $s \approx 0.121445$

dense non-triangulated packing



$\delta \geq 0.937371$ $s \approx 0.121445$

45 counter examples: *flip-and-flow* method



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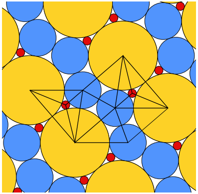
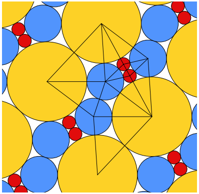
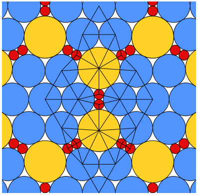
triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

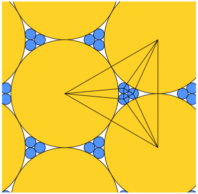


$\delta \leq 0.924522$ $s \approx 0.166169$



$\delta \leq 0.917352$ $s \approx 0.240205$

dense binary packing



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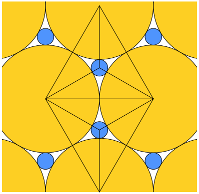
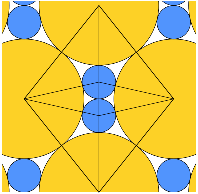
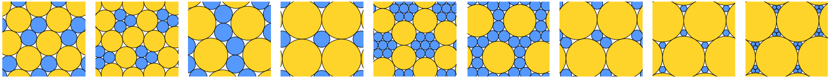


$\delta \approx 0.950308$ $r \approx 0.154701$



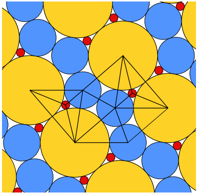
$\delta \approx 0.931901$ $r \approx 0.280776$

45 counter examples: *flip-and-flow* method

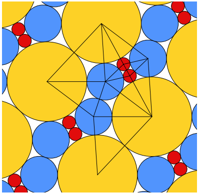


When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

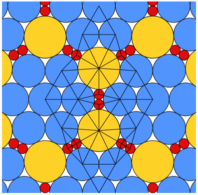
triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

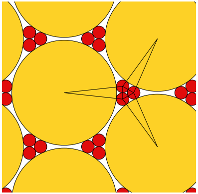


$\delta \leq 0.924522$ $s \approx 0.166169$

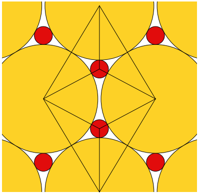


$\delta \leq 0.917352$ $s \approx 0.240205$

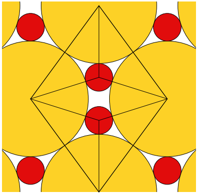
dense non-triangulated packing



$\delta \geq 0.937371$ $s \approx 0.121445$



$\delta \geq 0.939305$ $s \approx 0.166169$

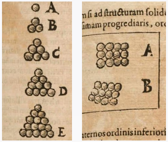


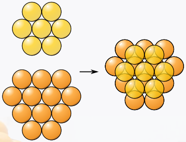
$\delta \geq 0.918420$ $s \approx 0.240205$

1. Introduction
2. Disc packings
3. Sphere packings
4. Conclusion

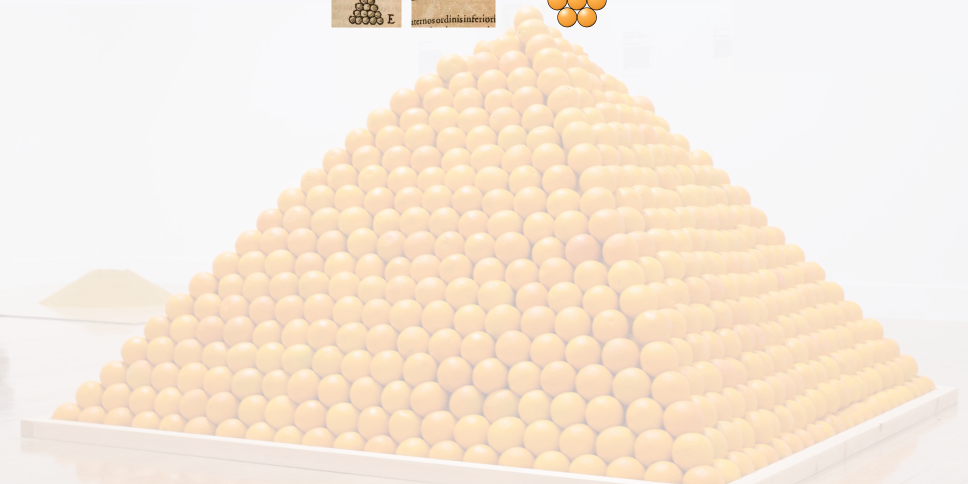
Kepler conjecture: -packings

3D close -packings:



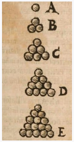


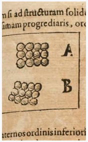
$\delta^* = \frac{\pi}{3\sqrt{2}} \approx 74\%$

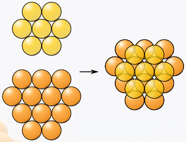


Kepler conjecture: ●-packings

3D close ●-packings:







$\delta^* = \frac{\pi}{3\sqrt{2}} \approx 74\%$

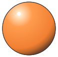
Hales, Ferguson, 1998–2014 (Conjectured by Kepler, 1611)

Close packings are optimal.

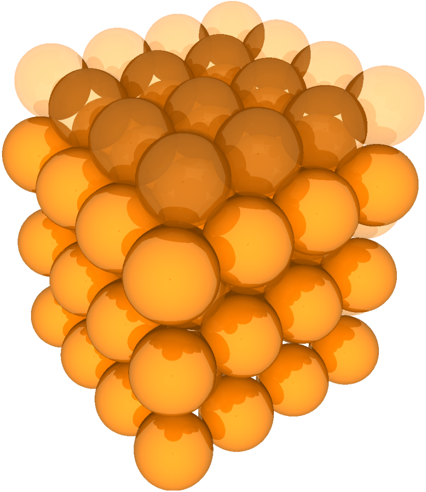
- close packings are optimal among lattice packings — Gauss, 1831
- 18th problem of the Hilbert's list — 1900
- 6 preprints by Hales and Ferguson — ArXiv 1998
  250 pages and $>$ 180000 lines of code
- reviewing: 13 reviewers, 4 years... "99% certain" — 1999–2003
- published proof: 300 pages, 3 computer programs — DCG 2006
- Flyspeck project: formal proof (HOL Light proof assistant) — 2003–2014
  Forum of Mathematatics, Pi 2017

Rock salt ●●-packings

sphere

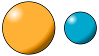


cannonball packing



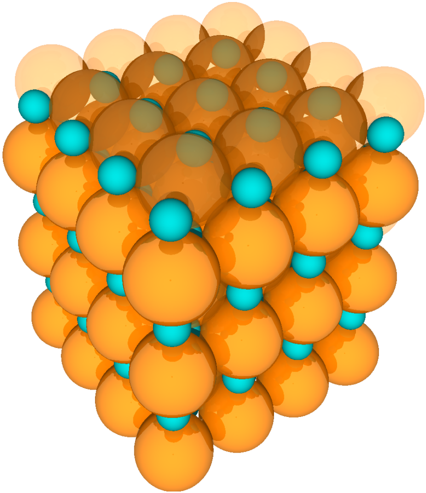
Rock salt -packings

rock salt spheres



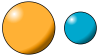
1 $r=\sqrt{2}-1$

rock salt packing



Rock salt ●●-packings

rock salt spheres



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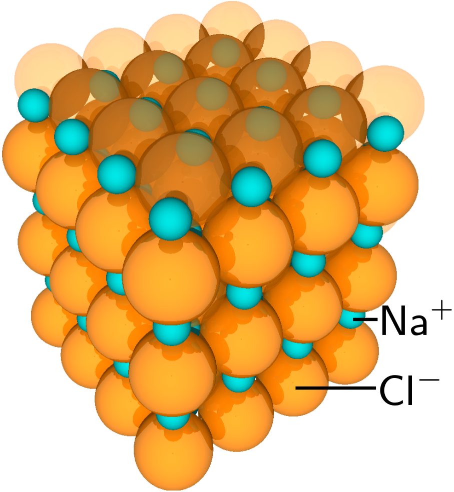
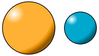
rock salt packing



Rock salt -packings

rock salt spheres



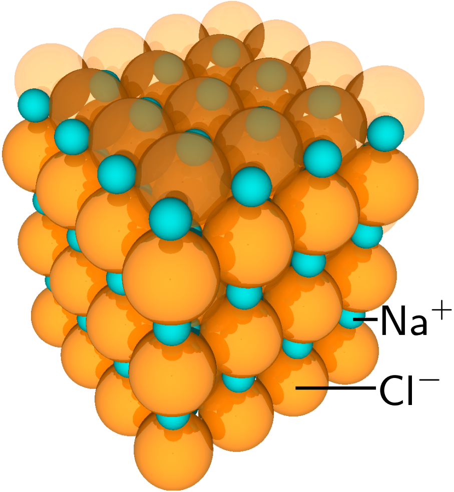
1 $r=\sqrt{2}-1$

triangulated → simplicial
(contact graph is a "tetrahedration")

Fernique, 2019

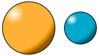
The only simplicial 2-sphere packings in 3D are rock salt packings.

rock salt packing



Rock salt -packings

rock salt spheres



1 $\quad r=\sqrt{2}-1$

triangulated $\rightarrow$ simplicial
(contact graph is a "tetrahedration")

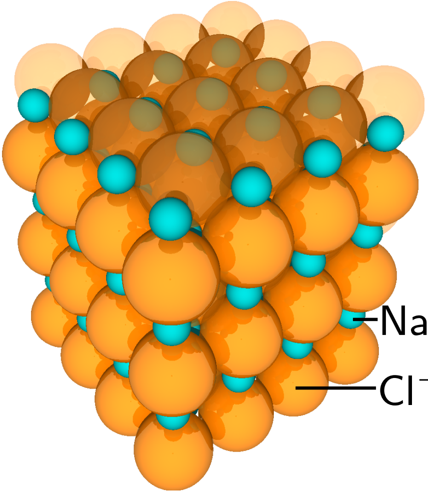
Fernique, 2019

The only simplicial 2-sphere packings in 3D are rock salt packings.

Salt conjecture **open problem**

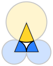
Rock salt packings are optimal
$\delta^* \approx 79\%$

rock salt packing



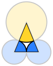
Upper density bound for -packings in 2D

Florian, 1960

The density of a packing never exceeds the density in the following triangle: 

Upper density bound for -packings in 2D

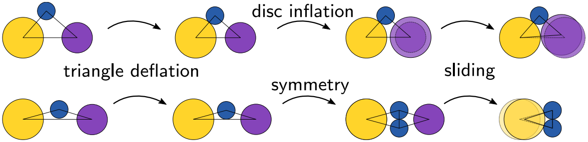
Florian, 1960

The density of a packing never exceeds the density in the following triangle: 

Proof:

- Reduce the dimension of the set of triangles $(3 \to 1)$ — Fejes Tóth, Mólnar, 1958

  For any triangle, there is a denser triangle with at least two contacts between discs.



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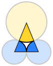
The density of a packing never exceeds the density in the following triangle: 

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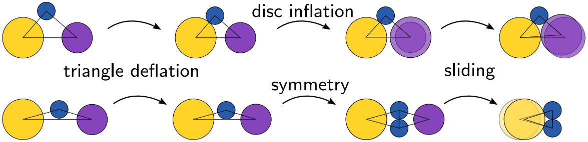
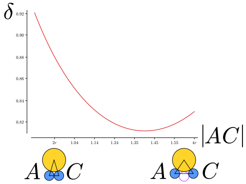
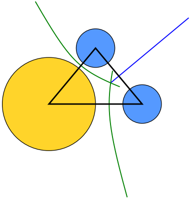


- Function analisys

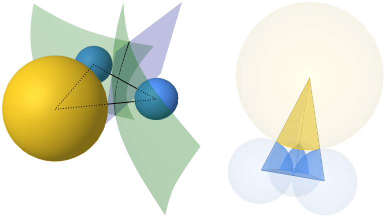


Upper density bound for ●●-packings

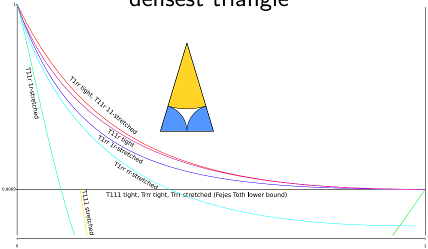
FM-triangulation (triangles) → FM-simplicial partition (tetrahedra)



→



densest triangle → densest tetrahedra



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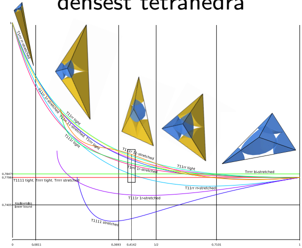
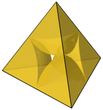


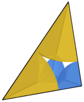
Upper density bound for -packings

Theorem, $r = \sqrt{2} - 1$ — in progress

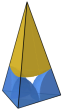
Each of the following tetrahedra is densest among the tetrahedra with the same spheres:



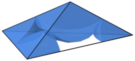
$\delta_{1111} \approx 0.7209$



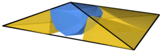
$\delta_{11rr} \approx 0.8105$



$\delta_{1rrr} \approx 0.8065$



$\delta_{rrrr} \approx 0.7847$



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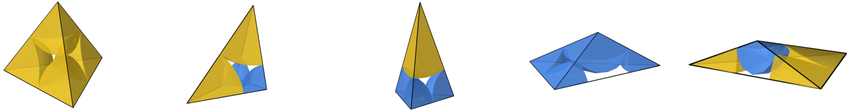

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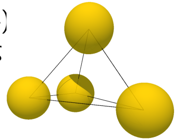
Proof:

- Reduce the dimension of the set $(6 \to 4)$ tetrahedron "deflation" + sphere sliding



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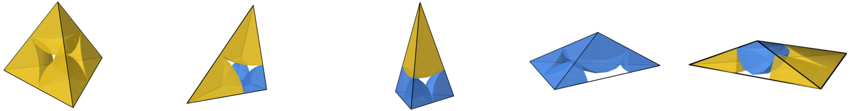

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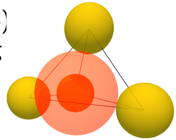

Upper density bound for ●●-packings

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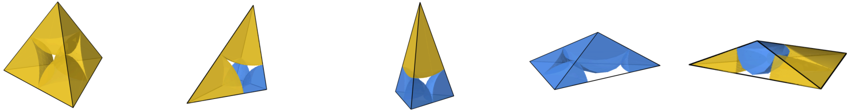

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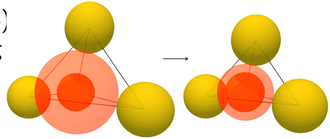

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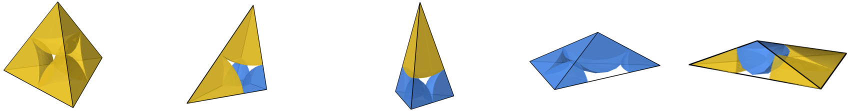

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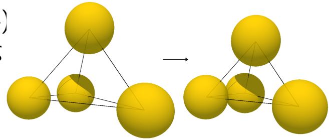

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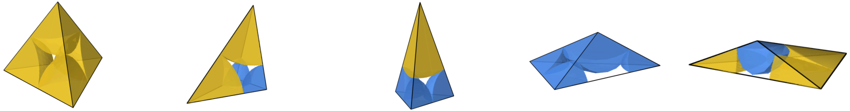

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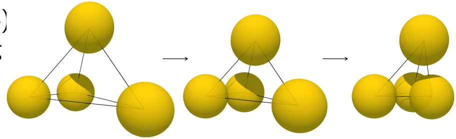

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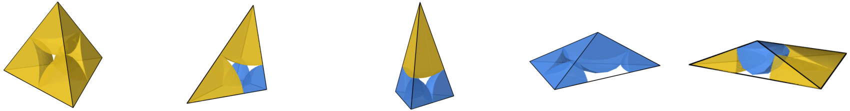

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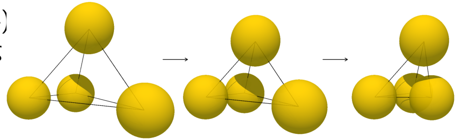

- Computer-assisted proof for tetrahedra with 2 contacts:
  recursive subdivision + interval arithmetic
  ≈ 1000 lines of code — 11h on 96 CPUs

Why the computations are so slow

interval arithmetic + huge formulas $\rightarrow$ loss of precision

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Example: to compute the support sphere radius, we need to solve $A\,r^2 + B\,r + C = 0$

$$
\begin{aligned}
A =& -4a^2b^2d^2+4a^2c^2d^2+4b^2c^2d^2-4c^4d^2-4c^2d^4+4a^2b^2e^2-4b^4e^2-4a^2c^2e^2+4b^2c^2e^2+4b^2d^2e^2+4c^2d^2e^2-4b^2e^4-4a^4f^2+4a^2b^2f^2+4a^2c^2f^2-4b^2c^2f^2+4a^2d^2f^2+4c^2d^2f^2+4a^2e^2f^2+4b^2e^2f^2-4d^2e^2f^2 \\
& -4a^2f^4+4d^4r_x^2-8d^2e^2r_x^2+4e^4r_x^2-8d^2f^2r_x^2-8e^2f^2r_x^2+4f^4r_x^2-8b^2d^2r_xr_y+8c^2d^2r_xr_y+8b^2e^2r_xr_y-8c^2e^2r_xr_y-16a^2f^2r_xr_y+8b^2f^2r_xr_y+8c^2f^2r_xr_y+8d^2f^2r_xr_y+8e^2f^2r_xr_y-8f^4r_xr_y+4b^4r_y^2-8b^2c^2r_y^2+4c^4r_y^2 \\
& -8b^2f^2r_y^2-8c^2f^2r_y^2+4f^4r_y^2-8a^2d^2r_xr_z+8c^2d^2r_xr_z+8a^2e^2r_xr_z-16b^2e^2r_xr_z+8c^2e^2r_xr_z+8d^2e^2r_xr_z-8e^4r_xr_z+8a^2f^2r_xr_z-8c^2f^2r_xr_z+8e^2f^2r_xr_z-8a^2b^2r_yr_z+8a^2c^2r_yr_z+8b^2c^2r_yr_z-8c^4r_yr_z-16c^2d^2r_yr_z \\
& +8b^2e^2r_yr_z+8c^2e^2r_yr_z+8a^2f^2r_yr_z+8c^2f^2r_yr_z-8e^2f^2r_yr_z+4a^4r_z^2-8a^2c^2r_z^2+4c^4r_z^2-8a^2e^2r_z^2-8c^2e^2r_z^2+4e^4r_z^2+8a^2d^2r_xr_w+8b^2d^2r_xr_w-16c^2d^2r_xr_w-8d^4r_xr_w-8a^2e^2r_xr_w+8b^2e^2r_xr_w+8d^2e^2r_xr_w \\
& +8a^2f^2r_xr_w-8b^2f^2r_xr_w+8d^2f^2r_xr_w+8a^2b^2r_yr_w-8b^4r_yr_w-8a^2c^2r_yr_w+8b^2c^2r_yr_w+8b^2d^2r_yr_w+8c^2d^2r_yr_w-16b^2e^2r_yr_w+8a^2f^2r_yr_w+8b^2f^2r_yr_w-8d^2f^2r_yr_w-8a^4r_zr_w+8a^2b^2r_zr_w+8a^2c^2r_zr_w-8b^2c^2r_zr_w \\
& +8a^2d^2r_zr_w+8c^2d^2r_zr_w+8a^2e^2r_zr_w+8b^2e^2r_zr_w-8d^2e^2r_zr_w-16a^2f^2r_zr_w+4a^4r_w^2-8a^2b^2r_w^2+4b^4r_w^2-8a^2d^2r_w^2-8b^2d^2r_w^2+4d^4r_w^2
\end{aligned}
$$

$$
\begin{aligned}
B =& -4c^2d^4r_x+4b^2d^2e^2r_x+4c^2d^2e^2r_x-4b^2e^4r_x+4a^2d^2f^2r_x+4c^2d^2f^2r_x+4a^2e^2f^2r_x+4b^2e^2f^2r_x-8d^2e^2f^2r_x-4a^2f^4r_x+4d^4r_x^3-8d^2e^2r_x^3+4e^4r_x^3-8d^2f^2r_x^3-8e^2f^2r_x^3+4f^4r_x^3+4b^2c^2d^2r_y-4c^4d^2r_y-4b^4e^2r_y \\
& +4b^2c^2e^2r_y+4a^2b^2f^2r_y+4a^2c^2f^2r_y-8b^2c^2f^2r_y+4c^2d^2f^2r_y+4b^2e^2f^2r_y-4a^2f^4r_y-4b^2d^2r_x^2r_y+4c^2d^2r_x^2r_y+4b^2e^2r_x^2r_y-4c^2e^2r_x^2r_y-8a^2f^2r_x^2r_y+4b^2f^2r_x^2r_y+4c^2f^2r_x^2r_y+4d^2f^2r_x^2r_y+4e^2f^2r_x^2r_y-4f^4r_x^2r_y \\
& -4b^2d^2r_xr_y^2+4c^2d^2r_xr_y^2+4b^2e^2r_xr_y^2-4c^2e^2r_xr_y^2-8a^2f^2r_xr_y^2+4b^2f^2r_xr_y^2+4c^2f^2r_xr_y^2+4d^2f^2r_xr_y^2+4e^2f^2r_xr_y^2-4f^4r_xr_y^2+4b^4r_y^3-8b^2c^2r_y^3+4c^4r_y^3-8b^2f^2r_y^3-8c^2f^2r_y^3+4f^4r_y^3+4a^2c^2d^2r_z-4c^4d^2r_z+4a^2b^2e^2r_z \\
& -8a^2c^2e^2r_z+4b^2c^2e^2r_z+4c^2d^2e^2r_z-4b^2e^4r_z-4a^4f^2r_z+4a^2c^2f^2r_z+4a^2e^2f^2r_z-4a^2d^2r_x^2r_z+4c^2d^2r_x^2r_z+4a^2e^2r_x^2r_z-8b^2e^2r_x^2r_z+4c^2e^2r_x^2r_z+4d^2e^2r_x^2r_z-4e^4r_x^2r_z+4a^2f^2r_x^2r_z-4c^2f^2r_x^2r_z+4e^2f^2r_x^2r_z \\
& -4a^2b^2r_y^2r_z+4a^2c^2r_y^2r_z+4b^2c^2r_y^2r_z-4c^4r_y^2r_z-8c^2d^2r_y^2r_z+4b^2e^2r_y^2r_z+4c^2e^2r_y^2r_z+4a^2f^2r_y^2r_z+4c^2f^2r_y^2r_z-4e^2f^2r_y^2r_z-4a^2d^2r_xr_z^2+4c^2d^2r_xr_z^2+4a^2e^2r_xr_z^2-8b^2e^2r_xr_z^2+4c^2e^2r_xr_z^2+4d^2e^2r_xr_z^2-4e^4r_xr_z^2 \\
& +4a^2f^2r_xr_z^2-4c^2f^2r_xr_z^2+4e^2f^2r_xr_z^2-4a^2b^2r_yr_z^2+4a^2c^2r_yr_z^2+4b^2c^2r_yr_z^2-4c^4r_yr_z^2-8c^2d^2r_yr_z^2+4b^2e^2r_yr_z^2+4c^2e^2r_yr_z^2+4a^2f^2r_yr_z^2+4c^2f^2r_yr_z^2-4e^2f^2r_yr_z^2+4a^4r_z^3-8a^2c^2r_z^3+4c^4r_z^3-8a^2e^2r_z^3-8c^2e^2r_z^3 \\
& +4e^4r_z^3-8a^2b^2d^2r_w+4a^2c^2d^2r_w+4b^2c^2d^2r_w-4c^2d^4r_w+4a^2b^2e^2r_w-4b^4e^2r_w+4b^2d^2e^2r_w-4a^4f^2r_w+4a^2b^2f^2r_w+4a^2d^2f^2r_w+4a^2d^2r_x^2r_w+4b^2d^2r_x^2r_w-8c^2d^2r_x^2r_w-4d^4r_x^2r_w-4a^2e^2r_x^2r_w+4b^2e^2r_x^2r_w \\
& +4d^2e^2r_x^2r_w+4a^2f^2r_x^2r_w-4b^2f^2r_x^2r_w+4d^2f^2r_x^2r_w+4a^2b^2r_y^2r_w-4b^4r_y^2r_w-4a^2c^2r_y^2r_w+4b^2c^2r_y^2r_w+4b^2d^2r_y^2r_w+4c^2d^2r_y^2r_w-8b^2e^2r_y^2r_w+4a^2f^2r_y^2r_w+4b^2f^2r_y^2r_w-4d^2f^2r_y^2r_w-4a^4r_z^2r_w+4a^2b^2r_z^2r_w+4a^2c^2r_z^2r_w \\
& -4b^2c^2r_z^2r_w+4a^2d^2r_z^2r_w+4c^2d^2r_z^2r_w+4a^2e^2r_z^2r_w+4b^2e^2r_z^2r_w-4d^2e^2r_z^2r_w-8a^2f^2r_z^2r_w+4a^2d^2r_xr_w^2+4b^2d^2r_xr_w^2-8c^2d^2r_xr_w^2-4d^4r_xr_w^2-4a^2e^2r_xr_w^2+4b^2e^2r_xr_w^2+4d^2e^2r_xr_w^2+4a^2f^2r_xr_w^2-4b^2f^2r_xr_w^2 \\
& +4d^2f^2r_xr_w^2+4a^2b^2r_yr_w^2-4b^4r_yr_w^2-4a^2c^2r_yr_w^2+4b^2c^2r_yr_w^2+4b^2d^2r_yr_w^2+4c^2d^2r_yr_w^2-8b^2e^2r_yr_w^2+4a^2f^2r_yr_w^2+4b^2f^2r_yr_w^2-4d^2f^2r_yr_w^2-4a^4r_zr_w^2+4a^2b^2r_zr_w^2+4a^2c^2r_zr_w^2 \\
& -4b^2c^2r_zr_w^2+4a^2d^2r_zr_w^2+4c^2d^2r_zr_w^2+4a^2e^2r_zr_w^2+4b^2e^2r_zr_w^2-4d^2e^2r_zr_w^2-8a^2f^2r_zr_w^2+4a^4r_w^3-8a^2b^2r_w^3+4b^4r_w^3-8a^2d^2r_w^3-8b^2d^2r_w^3+4d^4r_w^3
\end{aligned}
$$

$$
\begin{aligned}
C =& c^4d^4-2b^2c^2d^2e^2+b^4e^4-2a^2c^2d^2f^2-2a^2b^2e^2f^2+a^4f^4-2c^2d^4r_x^2+2b^2d^2e^2r_x^2+2c^2d^2e^2r_x^2-2b^2e^4r_x^2+2a^2d^2f^2r_x^2+2c^2d^2f^2r_x^2+2a^2e^2f^2r_x^2+2b^2e^2f^2r_x^2-4d^2e^2f^2r_x^2-2a^2f^4r_x^2+d^4r_x^4-2d^2e^2r_x^4+e^4r_x^4 \\
& -2d^2f^2r_x^4-2e^2f^2r_x^4+f^4r_x^4+2b^2c^2d^2r_y^2-2c^4d^2r_y^2-2b^4e^2r_y^2+2b^2c^2e^2r_y^2+2a^2b^2f^2r_y^2+2a^2c^2f^2r_y^2-4b^2c^2f^2r_y^2+2c^2d^2f^2r_y^2+2b^2e^2f^2r_y^2-2a^2f^4r_y^2-2b^2d^2r_x^2r_y^2+2c^2d^2r_x^2r_y^2+2b^2e^2r_x^2r_y^2-2c^2e^2r_x^2r_y^2-4a^2f^2r_x^2r_y^2 \\
& +2b^2f^2r_x^2r_y^2+2c^2f^2r_x^2r_y^2+2d^2f^2r_x^2r_y^2+2e^2f^2r_x^2r_y^2-2f^4r_x^2r_y^2+b^4r_y^4-2b^2c^2r_y^4+c^4r_y^4-2b^2f^2r_y^4-2c^2f^2r_y^4+f^4r_y^4+2a^2c^2d^2r_z^2-2c^4d^2r_z^2+2a^2b^2e^2r_z^2-4a^2c^2e^2r_z^2+2b^2c^2e^2r_z^2+2c^2d^2e^2r_z^2-2b^2e^4r_z^2 \\
& -2a^4f^2r_z^2+2a^2c^2f^2r_z^2+2a^2e^2f^2r_z^2-2a^2d^2r_x^2r_z^2+2c^2d^2r_x^2r_z^2+2a^2e^2r_x^2r_z^2-4b^2e^2r_x^2r_z^2+2c^2e^2r_x^2r_z^2+2d^2e^2r_x^2r_z^2-2e^4r_x^2r_z^2+2a^2f^2r_x^2r_z^2-2c^2f^2r_x^2r_z^2+2e^2f^2r_x^2r_z^2-2a^2b^2r_y^2r_z^2+2a^2c^2r_y^2r_z^2+2b^2c^2r_y^2r_z^2 \\
& -2c^4r_y^2r_z^2-4c^2d^2r_y^2r_z^2+2b^2e^2r_y^2r_z^2+2c^2e^2r_y^2r_z^2+2a^2f^2r_y^2r_z^2+2c^2f^2r_y^2r_z^2-2e^2f^2r_y^2r_z^2+a^4r_z^4-2a^2c^2r_z^4+c^4r_z^4-2a^2e^2r_z^4-2c^2e^2r_z^4+e^4r_z^4-4a^2b^2d^2r_w^2+2a^2c^2d^2r_w^2+2b^2c^2d^2r_w^2-2c^2d^4r_w^2+2a^2b^2e^2r_w^2 \\
& -2b^4e^2r_w^2+2b^2d^2e^2r_w^2-2a^4f^2r_w^2+2a^2b^2f^2r_w^2+2a^2d^2f^2r_w^2+2a^2d^2r_x^2r_w^2+2b^2d^2r_x^2r_w^2-4c^2d^2r_x^2r_w^2-2d^4r_x^2r_w^2-2a^2e^2r_x^2r_w^2+2b^2e^2r_x^2r_w^2+2d^2e^2r_x^2r_w^2+2a^2f^2r_x^2r_w^2-2b^2f^2r_x^2r_w^2+2d^2f^2r_x^2r_w^2+2a^2b^2r_y^2r_w^2 \\
& -2b^4r_y^2r_w^2-2a^2c^2r_y^2r_w^2+2b^2c^2r_y^2r_w^2+2b^2d^2r_y^2r_w^2+2c^2d^2r_y^2r_w^2-4b^2e^2r_y^2r_w^2+2a^2f^2r_y^2r_w^2+2b^2f^2r_y^2r_w^2-2d^2f^2r_y^2r_w^2-2a^4r_z^2r_w^2+2a^2b^2r_z^2r_w^2+2a^2c^2r_z^2r_w^2-2b^2c^2r_z^2r_w^2+2a^2d^2r_z^2r_w^2+2c^2d^2r_z^2r_w^2 \\
& +2a^2e^2r_z^2r_w^2+2b^2e^2r_z^2r_w^2-2d^2e^2r_z^2r_w^2-4a^2f^2r_z^2r_w^2+a^4r_w^4-2a^2b^2r_w^4+b^4r_w^4-2a^2d^2r_w^4-2b^2d^2r_w^4+d^4r_w^4
\end{aligned}
$$

Why the computations are so slow

interval arithmetic + huge formulas → loss of precision

Example: to compute the support sphere radius, we need to solve $A\,r^2 + B\,r + C = 0$

Thanks to dimension reduction:

compute with fixed radii and edge lengths, then "simplify"

$r_x = r_y = r_z = r_w = 1, \;\; a = b = c = 2:$

$A_{1111} = 4\left(d^2 - e^2\right)^2 + 4\,f^4 + \left(\left(d^2 - 8\right)e^2 - 8\,d^2\right)f^2$

$B_{1111} = 8\left(d^2 - e^2\right)^2 + 8\,f^4 + 2\left(\left(d^2 - 8\right)e^2 - 8\,d^2\right)f^2$

$C_{1111} = d^2 e^2 f^2$

1 Introduction

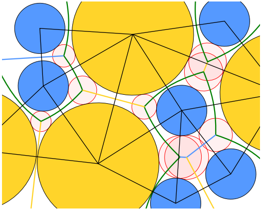
2 Disc packings

3 Sphere packings

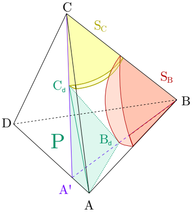
4 Conclusion

Techniques

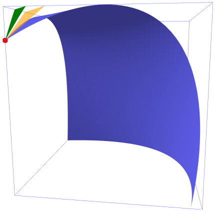
Geometry:



properties of triangulations

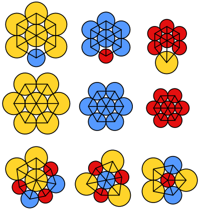


... and "tetrahedrizations"



differential geometry

Computer assistance:



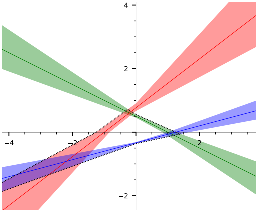
case enumeration
Python, C++

$$\mathcal{A}_{1111} = 4\,(d^2-e^2)^2 + 4\,f^4 + ((d^2-8)e^2 - 8\,d^2)f^2$$
$$\mathcal{B}_{1111} = 8\,(d^2-e^2)^2 + 8\,f^4 + 2\,((d^2-8)e^2 - 8\,d^2)f^2$$
$$\mathcal{C}_{1111} = d^2e^2f^2$$

$$\delta_{1111} = \frac{8\left(\begin{array}{r} \arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(d+2)e^2+2\,d^2+(d^2+4\,d)e-2\,f^2}\right) \\ +\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(d+2)f^2+2\,d^2-2\,e^2+(d^2+4\,d)f}\right) \\ +\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(e+2)f^2-2\,d^2+2\,e^2+(e^2+4\,e)f}\right) \\ -\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{2\,(d^2+e^2+f^2-32)}\right) \end{array}\right)}{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}.$$

symbolic calculus
SageMath

$\Longleftrightarrow$



interval arithmetic
MPFI (RIF SageMath)
Boost (C++)

Thank you for your attention!