Experimental Takes: Circle Packing and Discrete Schwarzians

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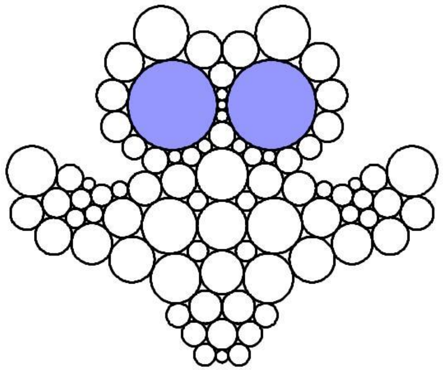
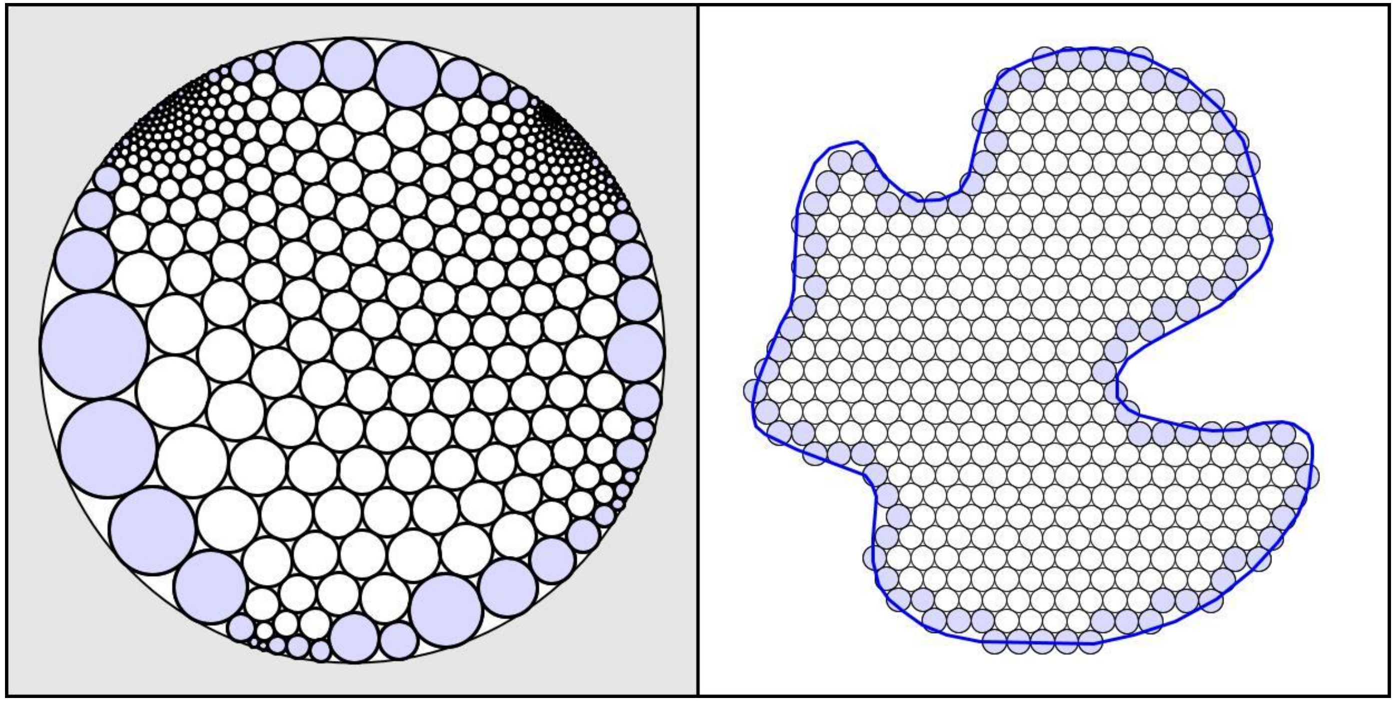
ICERM 2025



Outline

- Circle packing and discrete analytic functions: Missing in action: discrete rational functions
- Schwarzian derivatives, classical and discrete
- “Intrinsic” schwarzians
- Normalized flower layouts
- Special flowers

Discrete Analytic Functions:



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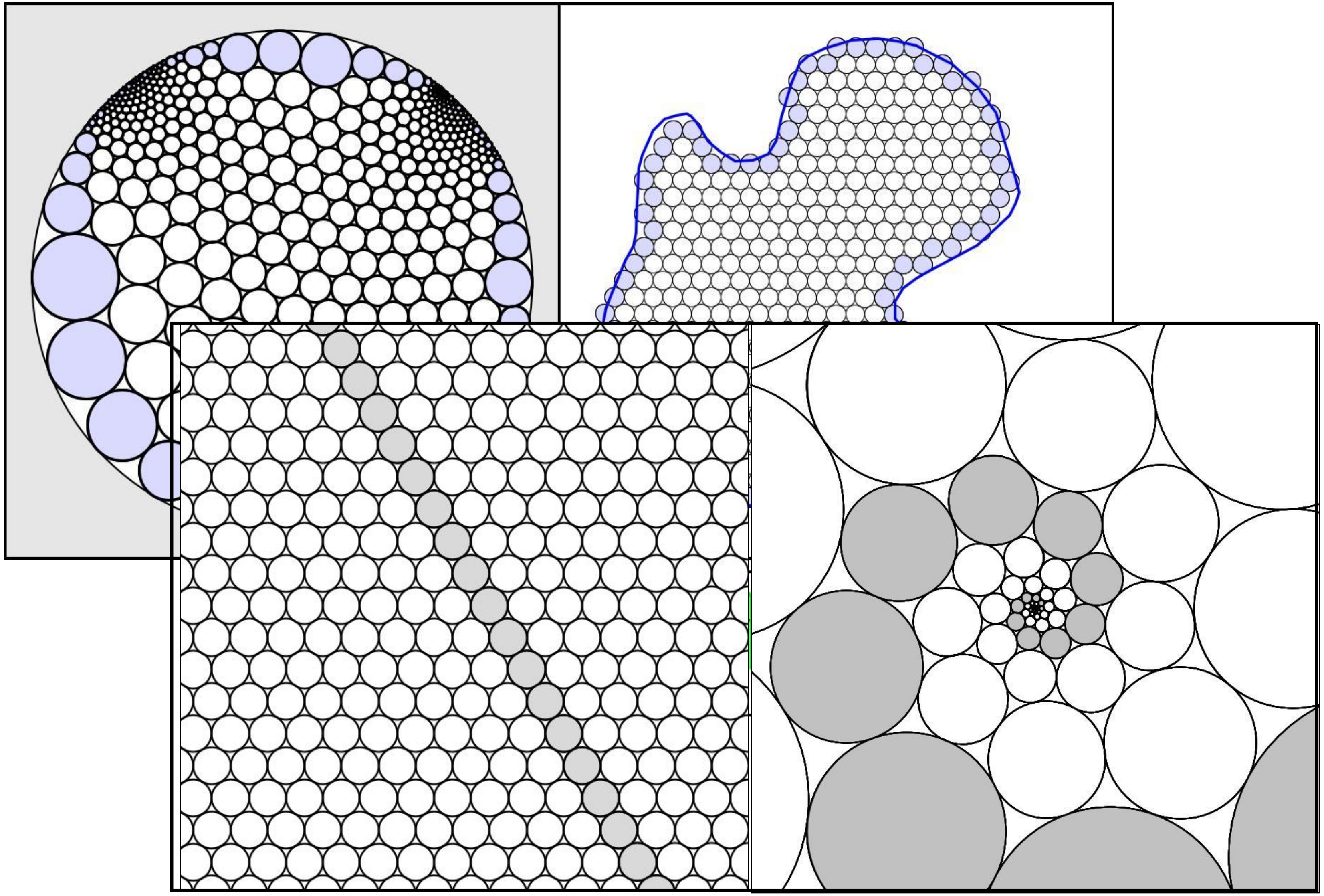

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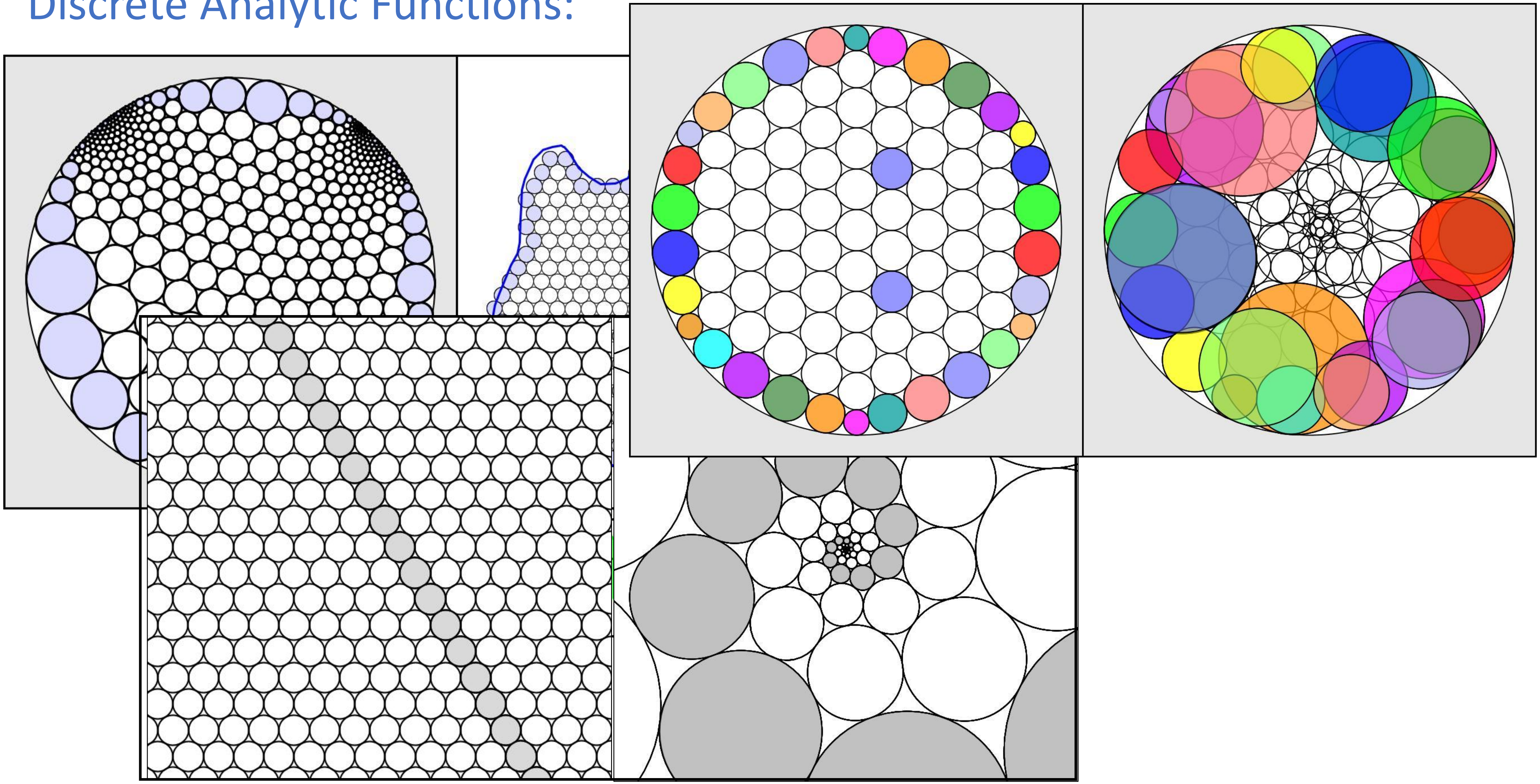

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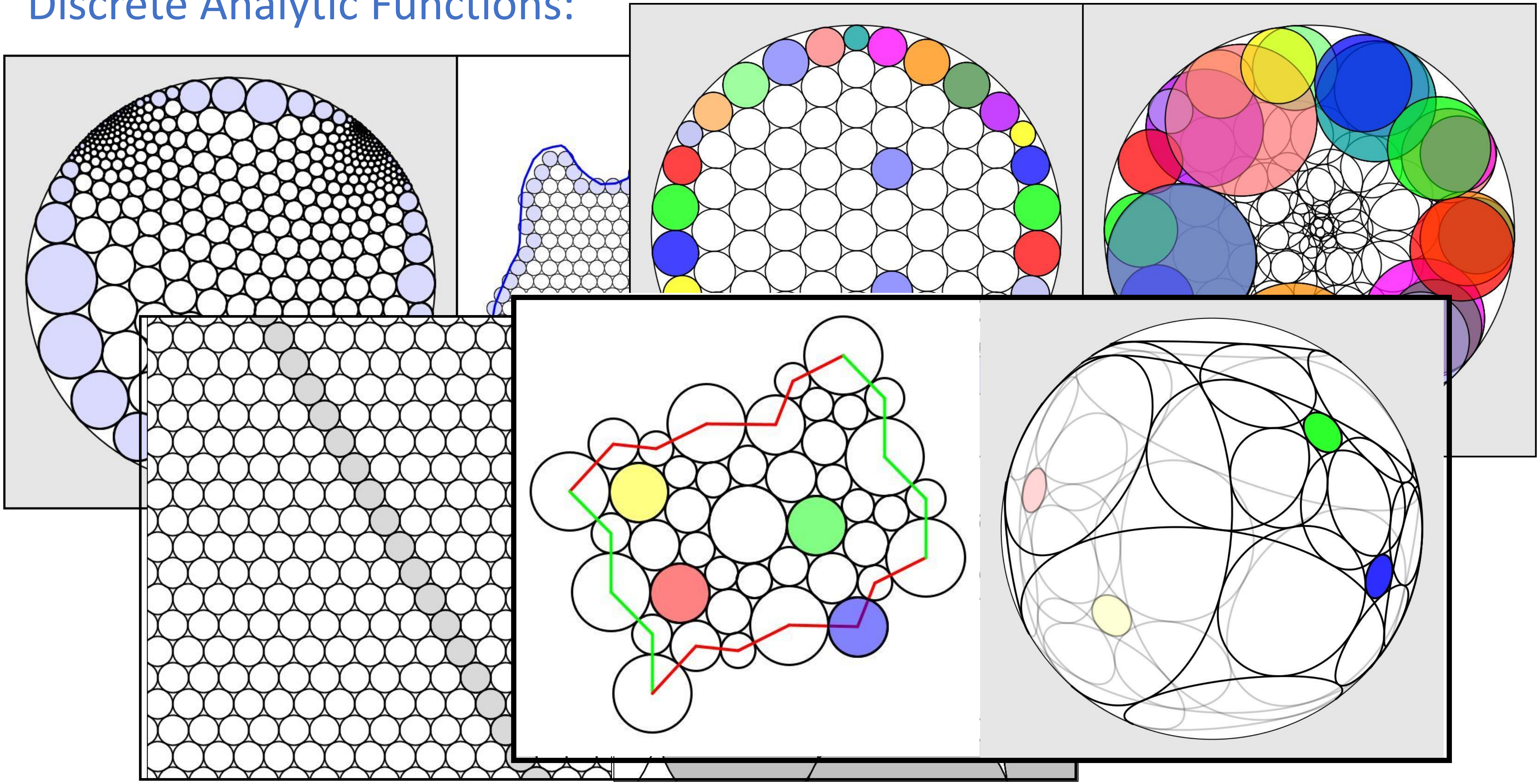

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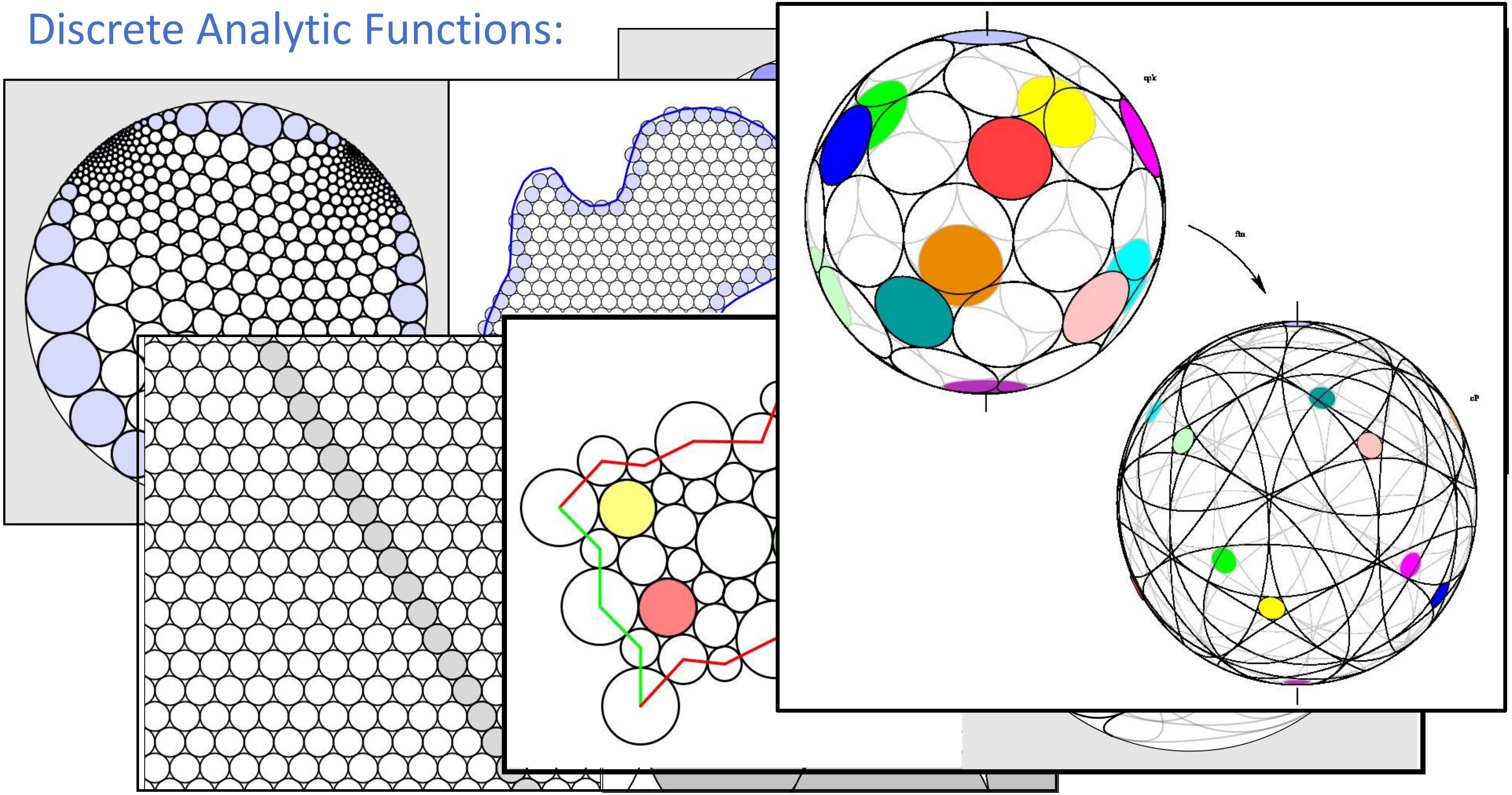
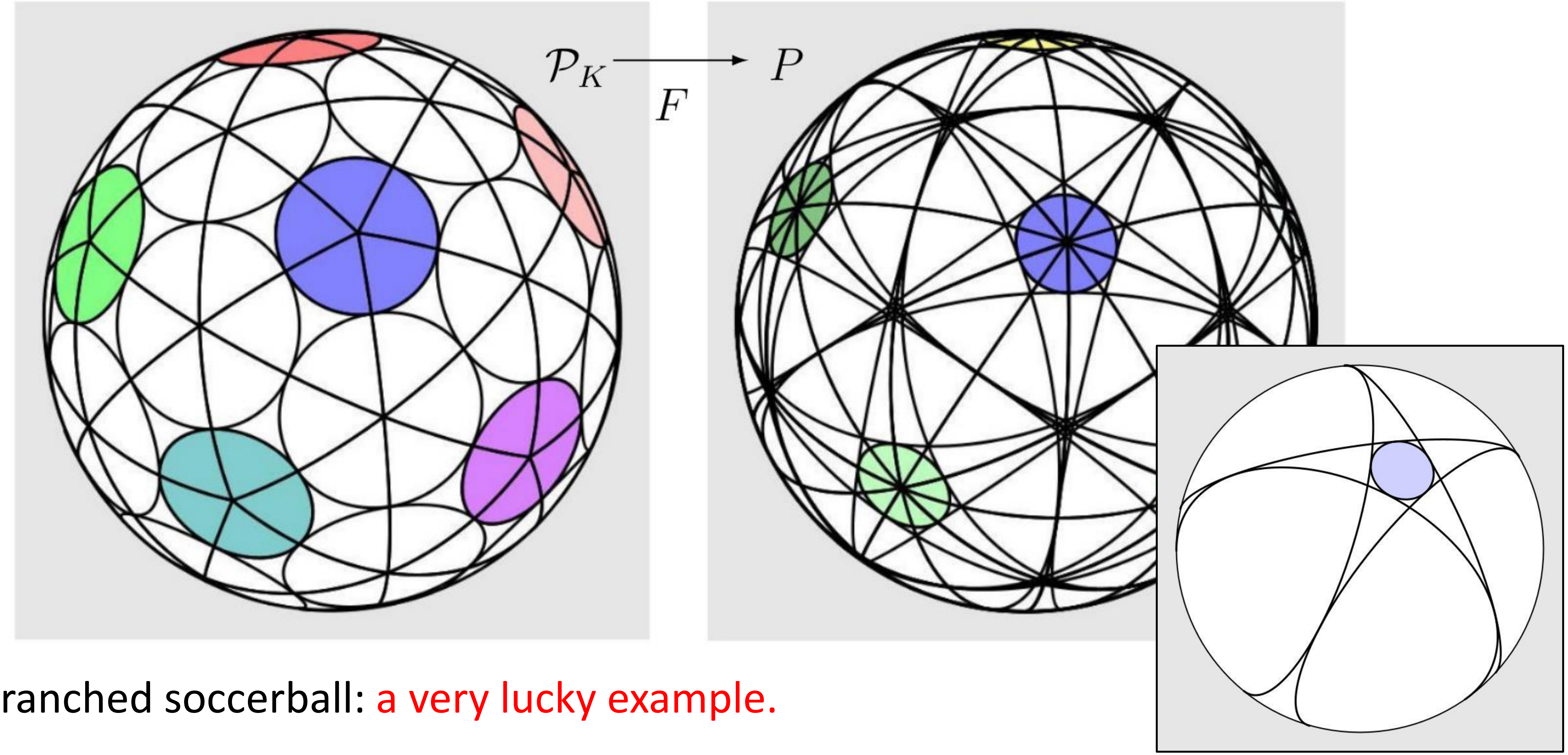


A fly in the Ointment:

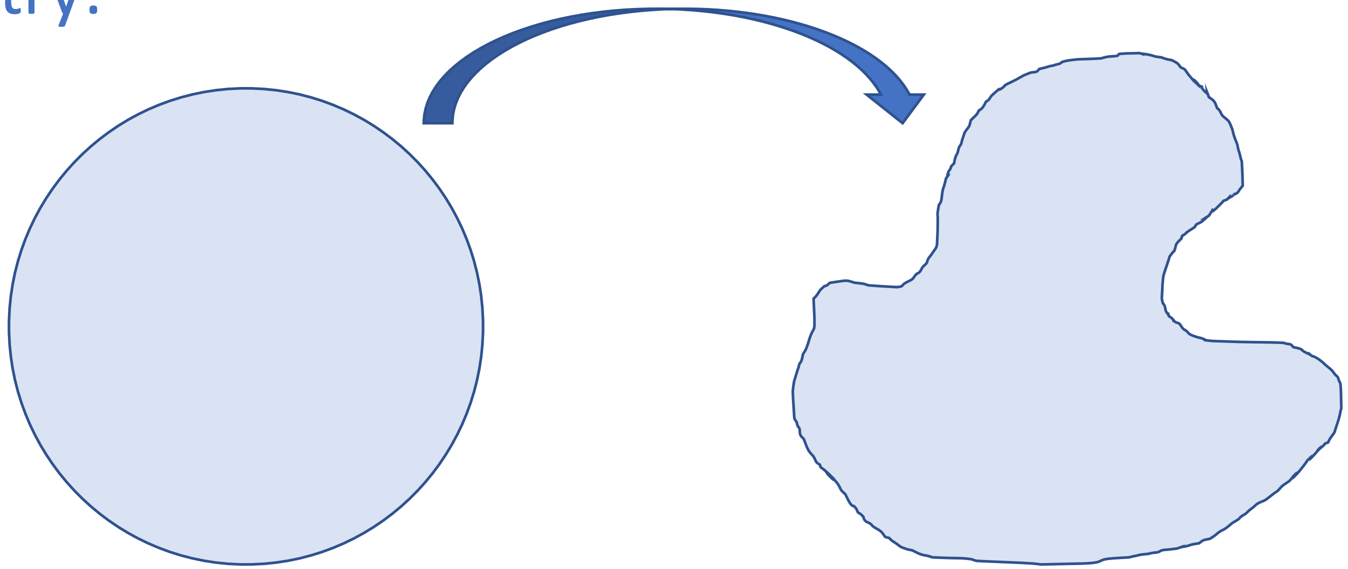
Creating/manipulating discrete rational functions



The branched soccerball: a very lucky example.

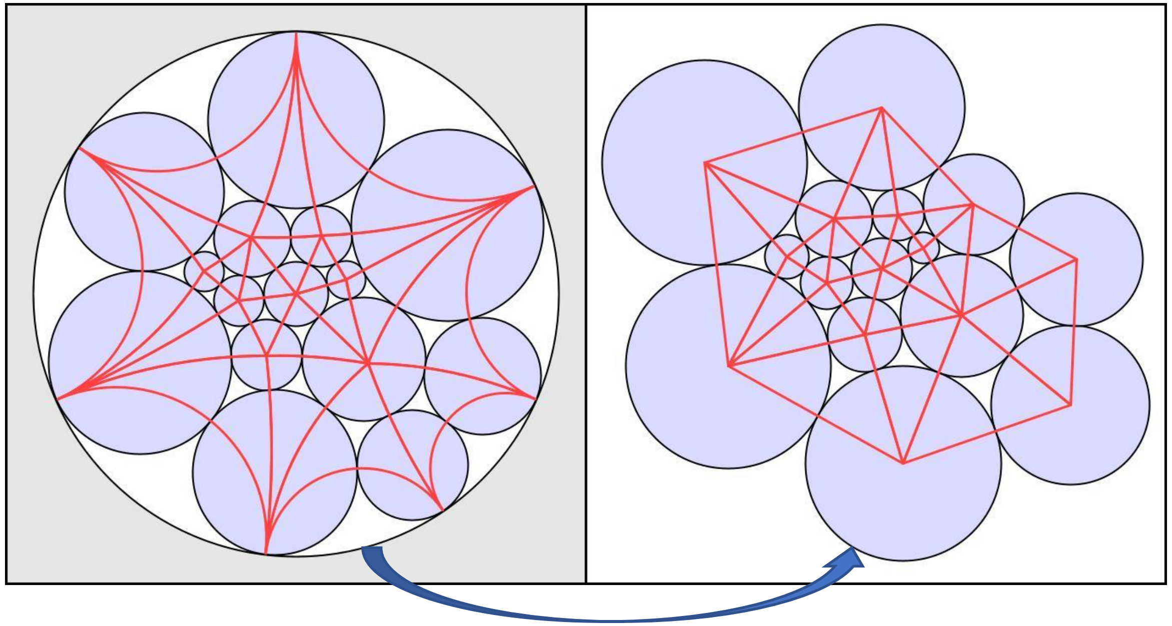
Discrete Conformal Geometry:

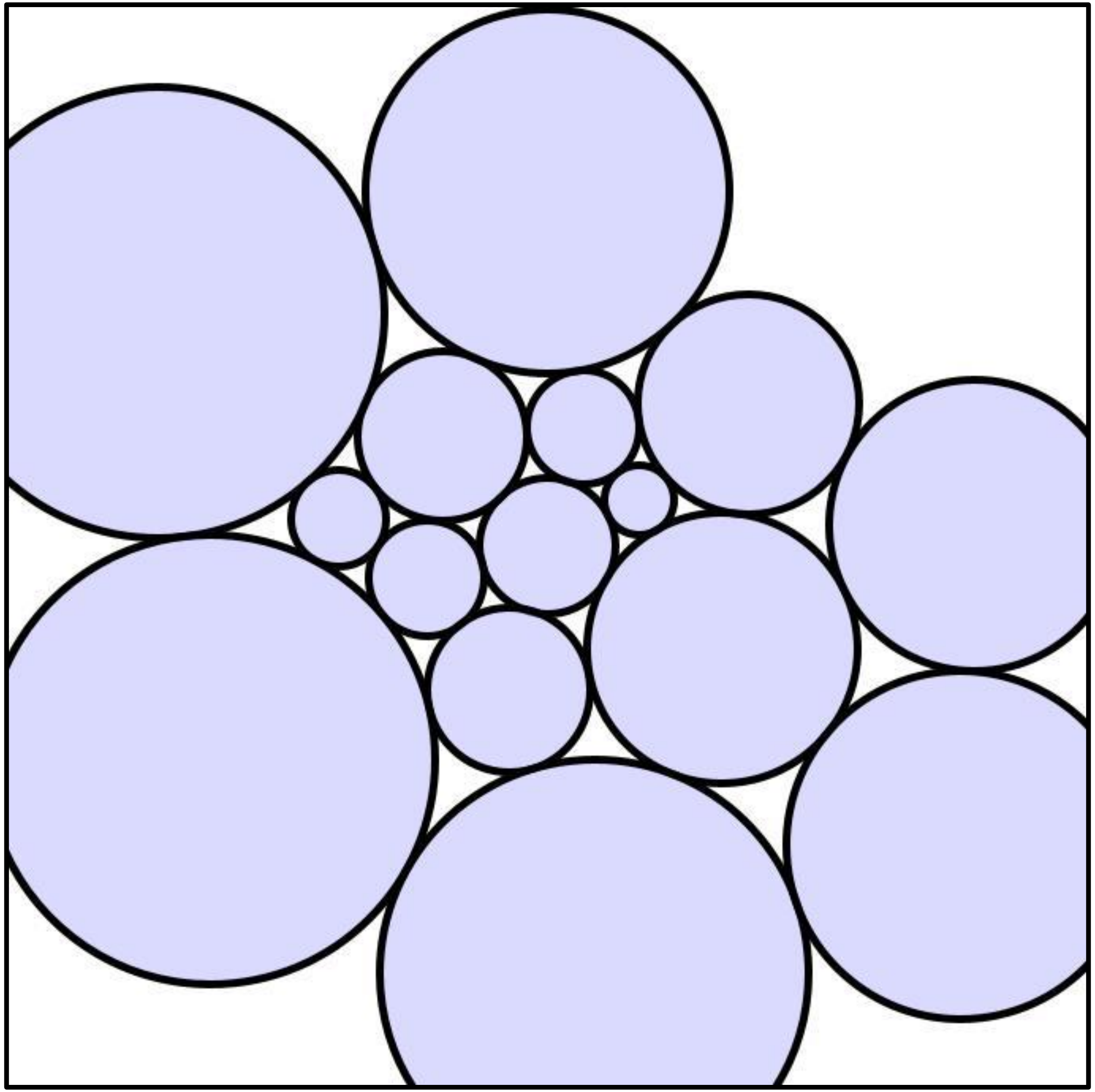
Classical analytic function



Discrete analytic function

Circle packing provides a fairly comprehensive and faithful parallel to the classical world.

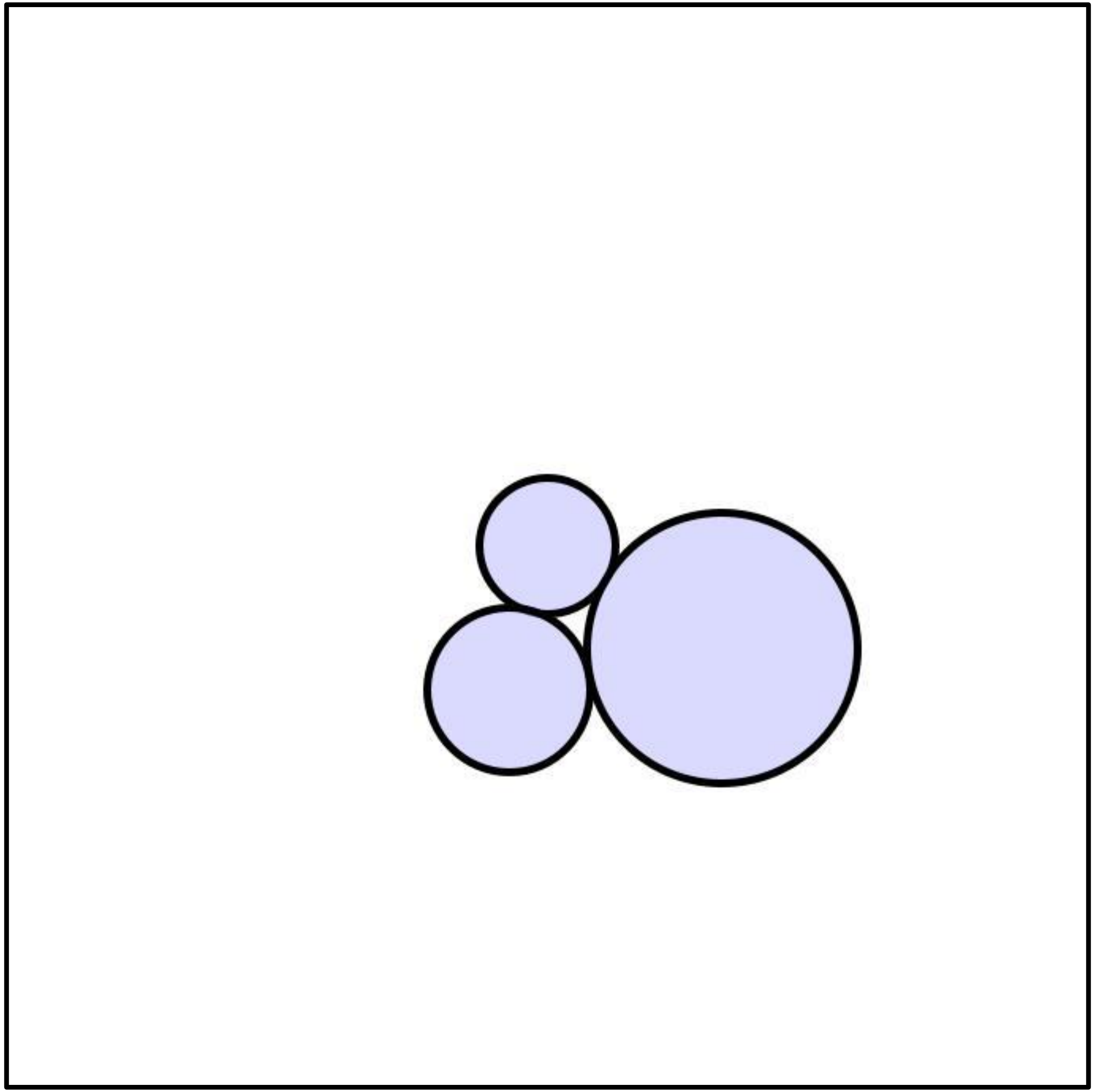


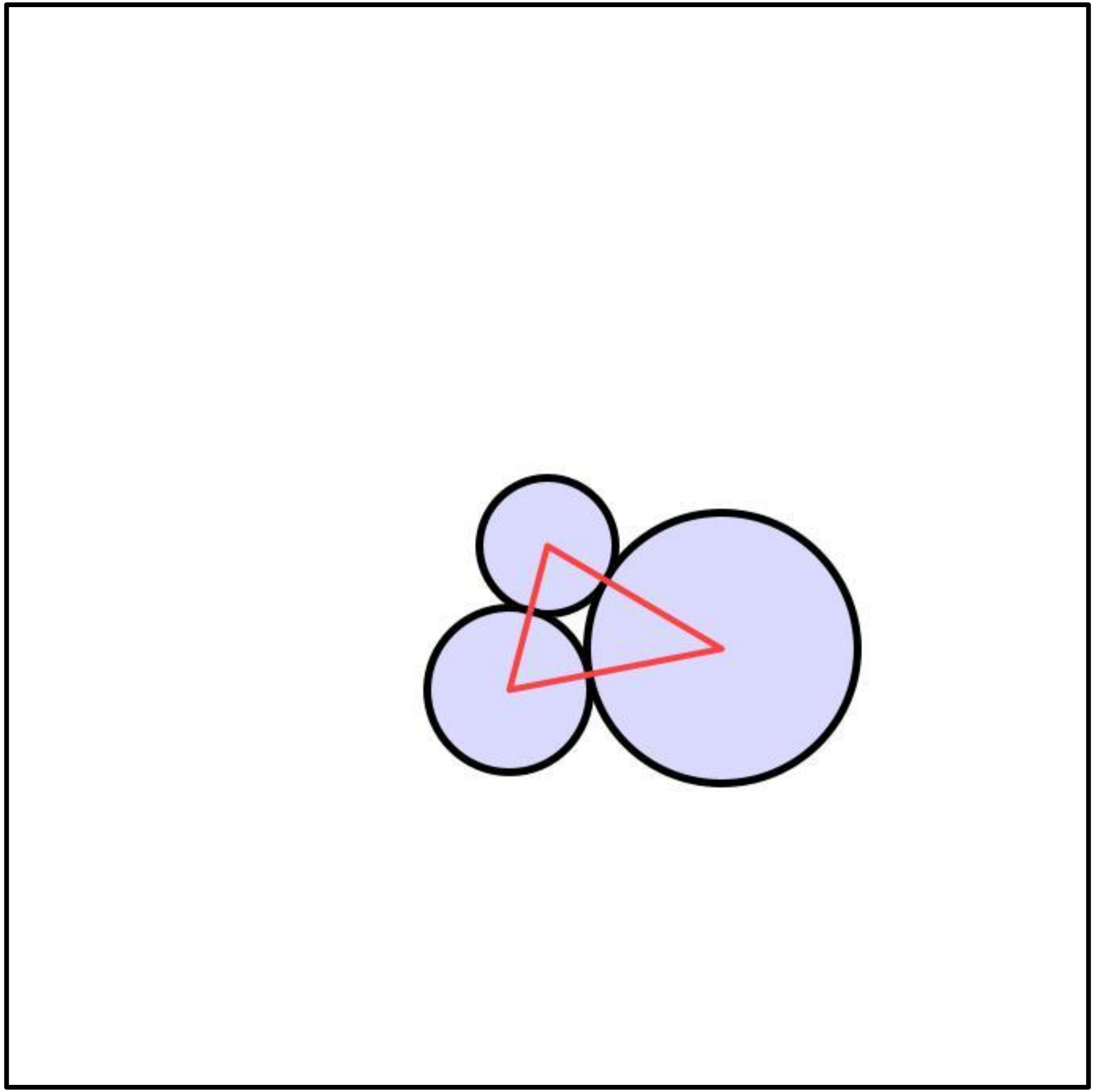


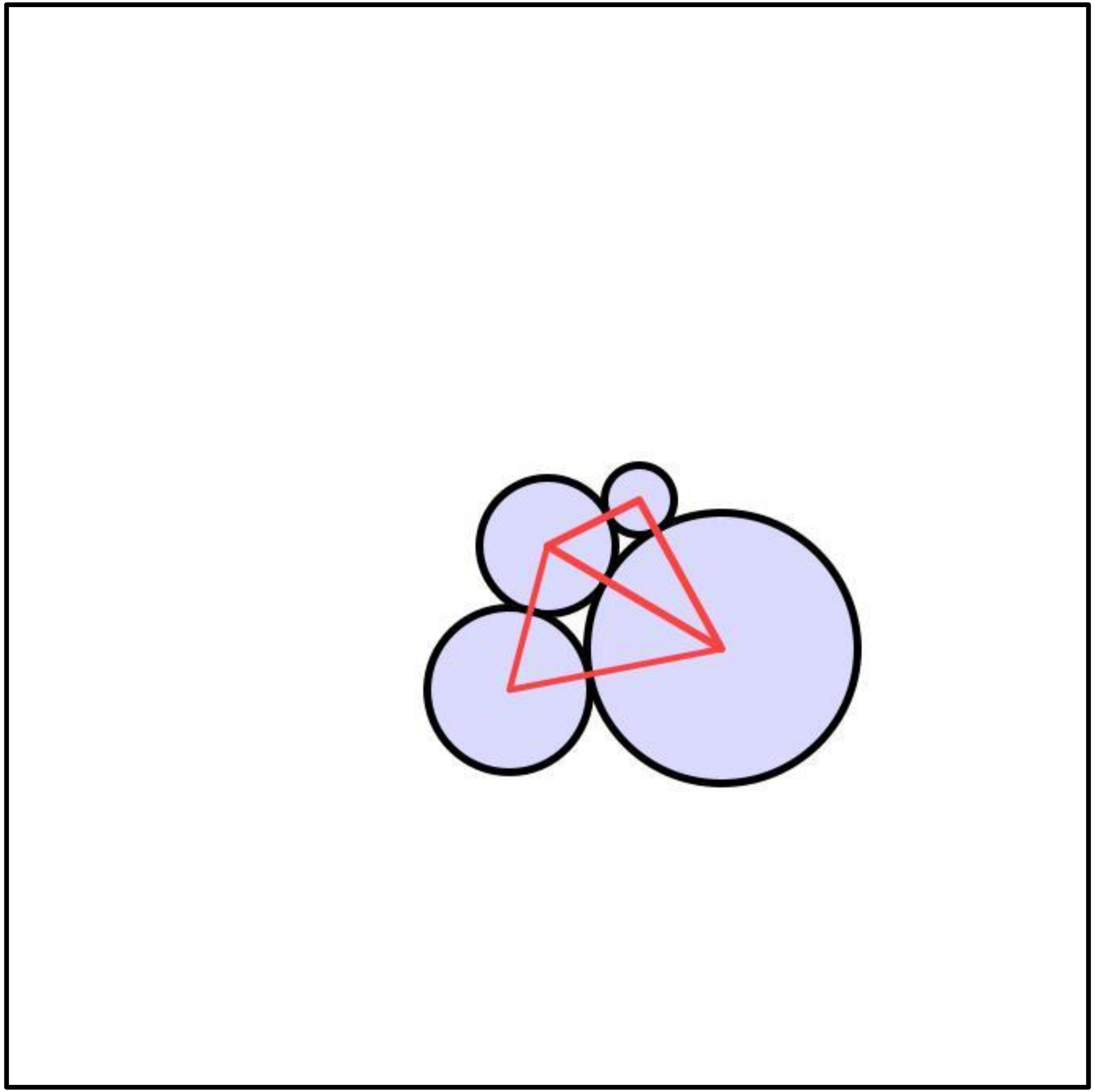
Traditional circle packings are created and manipulated using radii as the parameters.

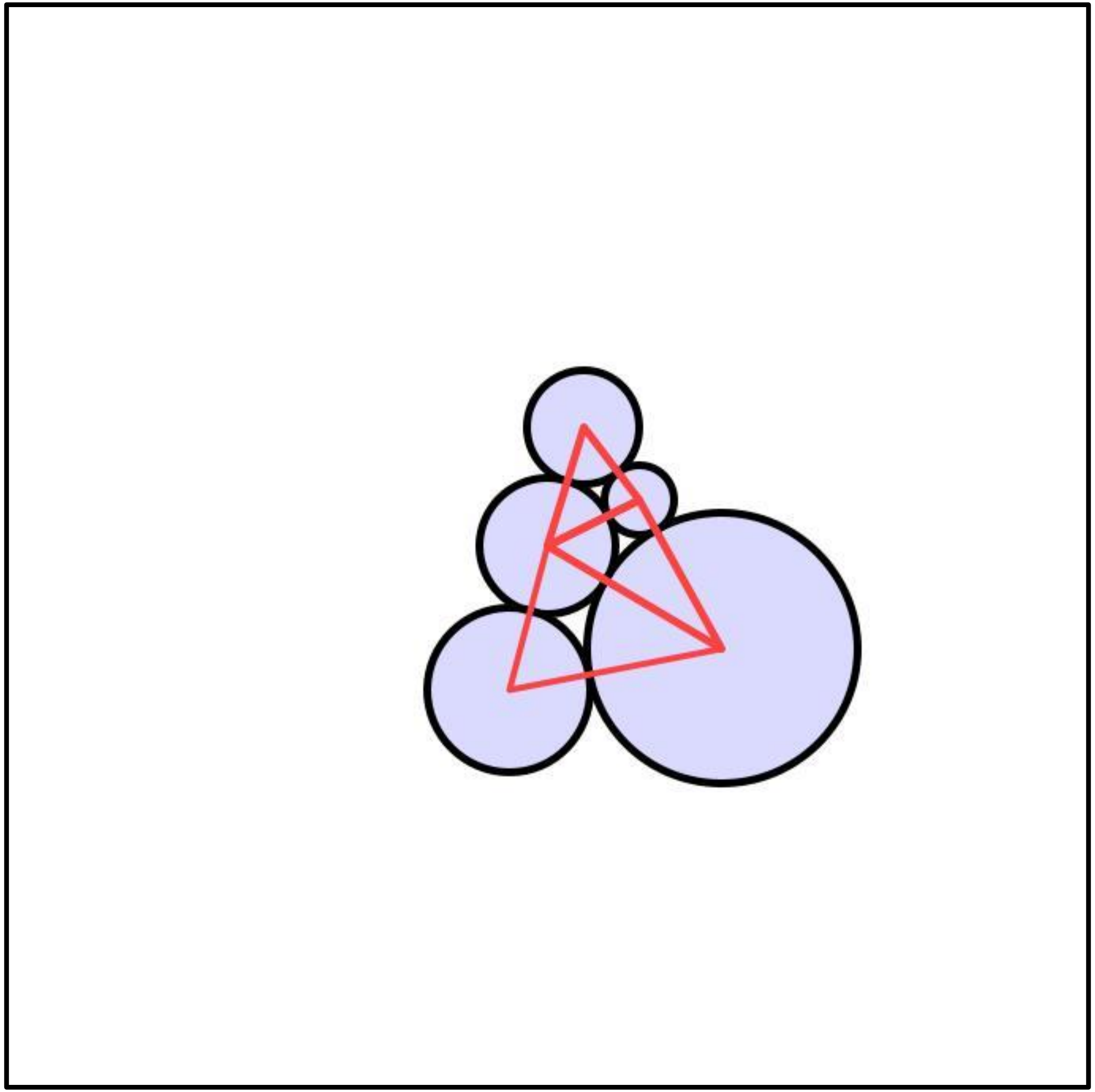
For Example: Given complex K

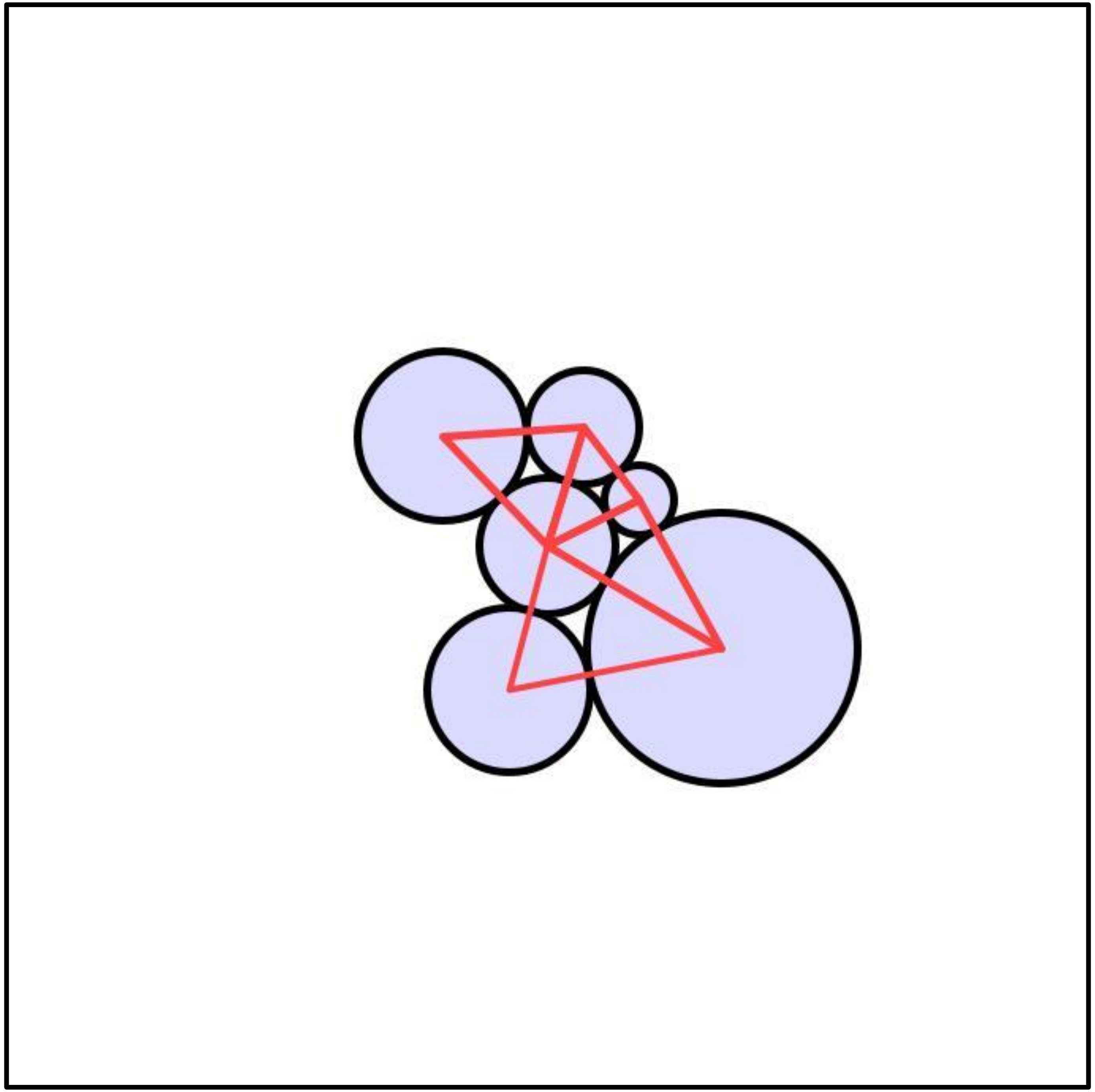
- Set the **boundary** radii
- Interatively **adjust** internal radii until all internal vertices have **angle sum** a multiple of pi.
- **Lay out** an initial triple of circles
- Successively **lay out** the remaining circles.

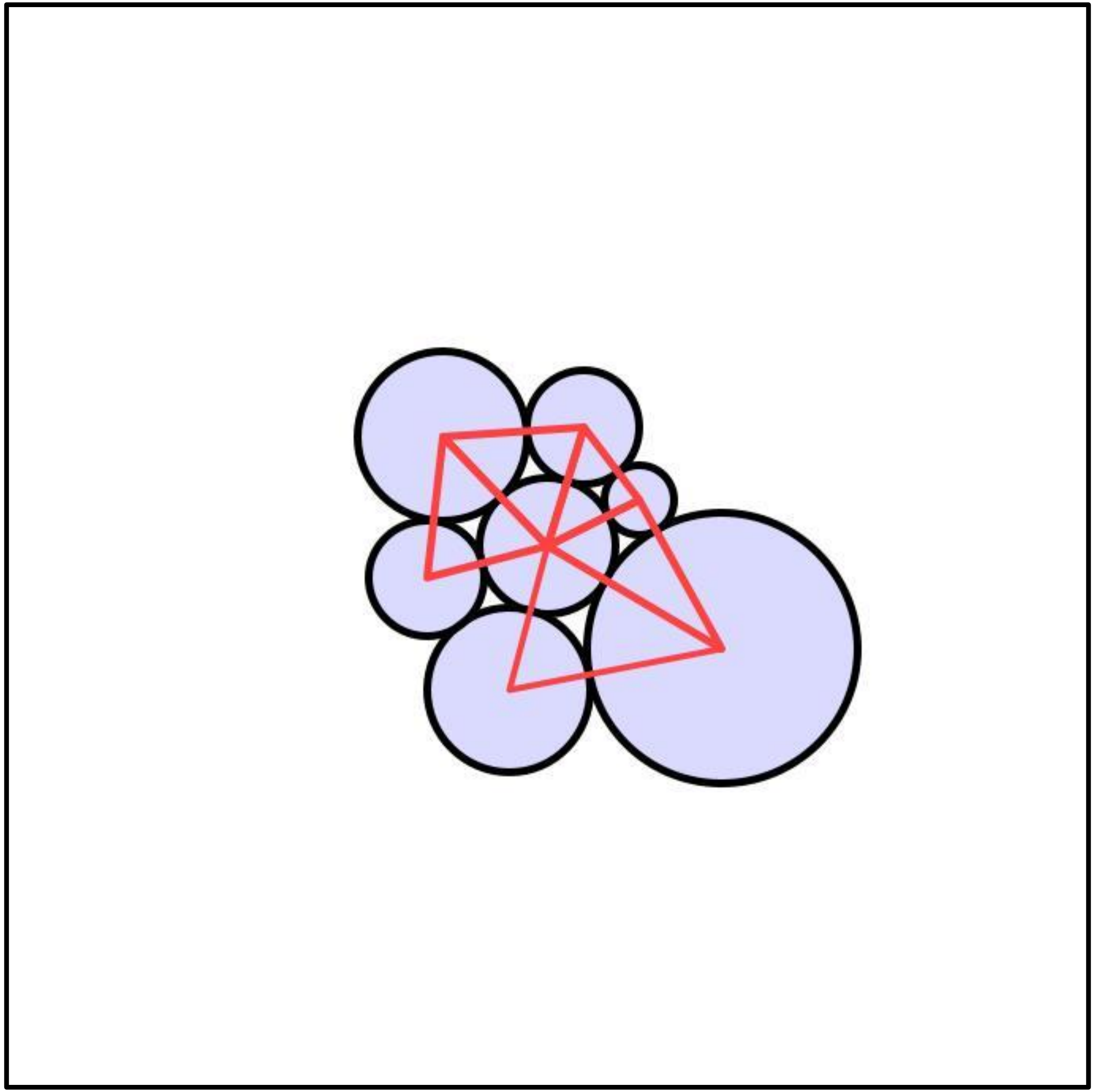


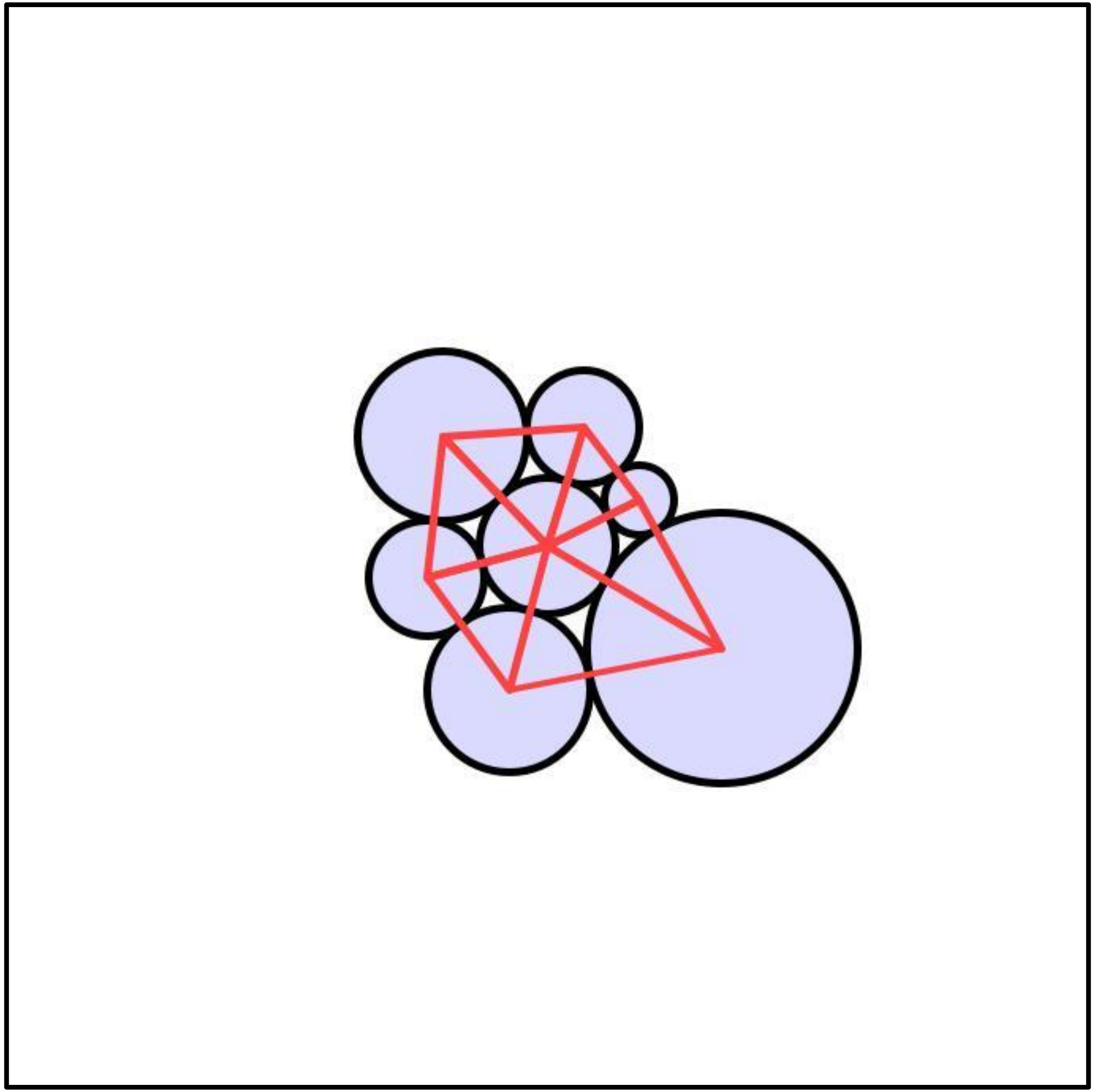


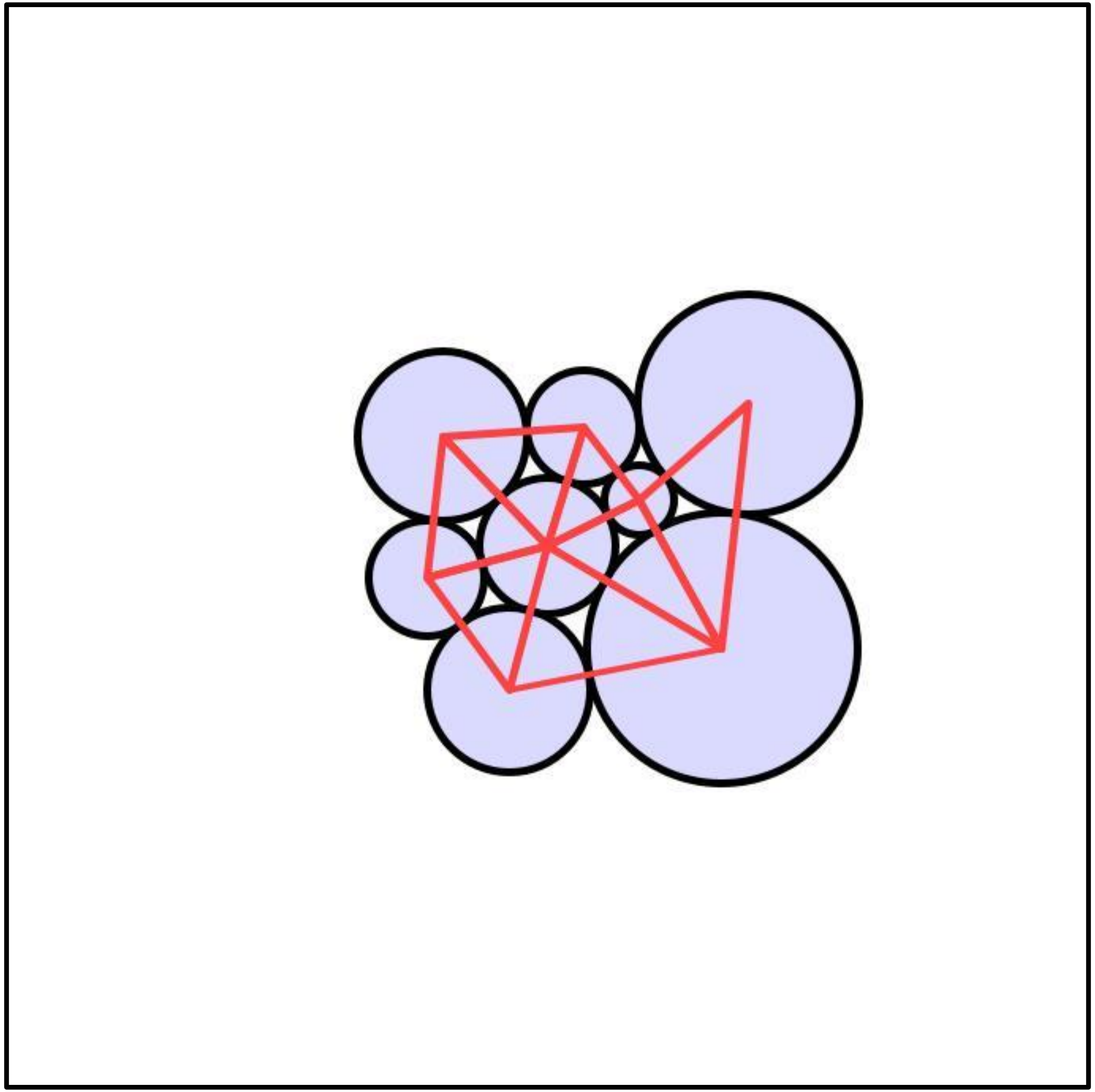


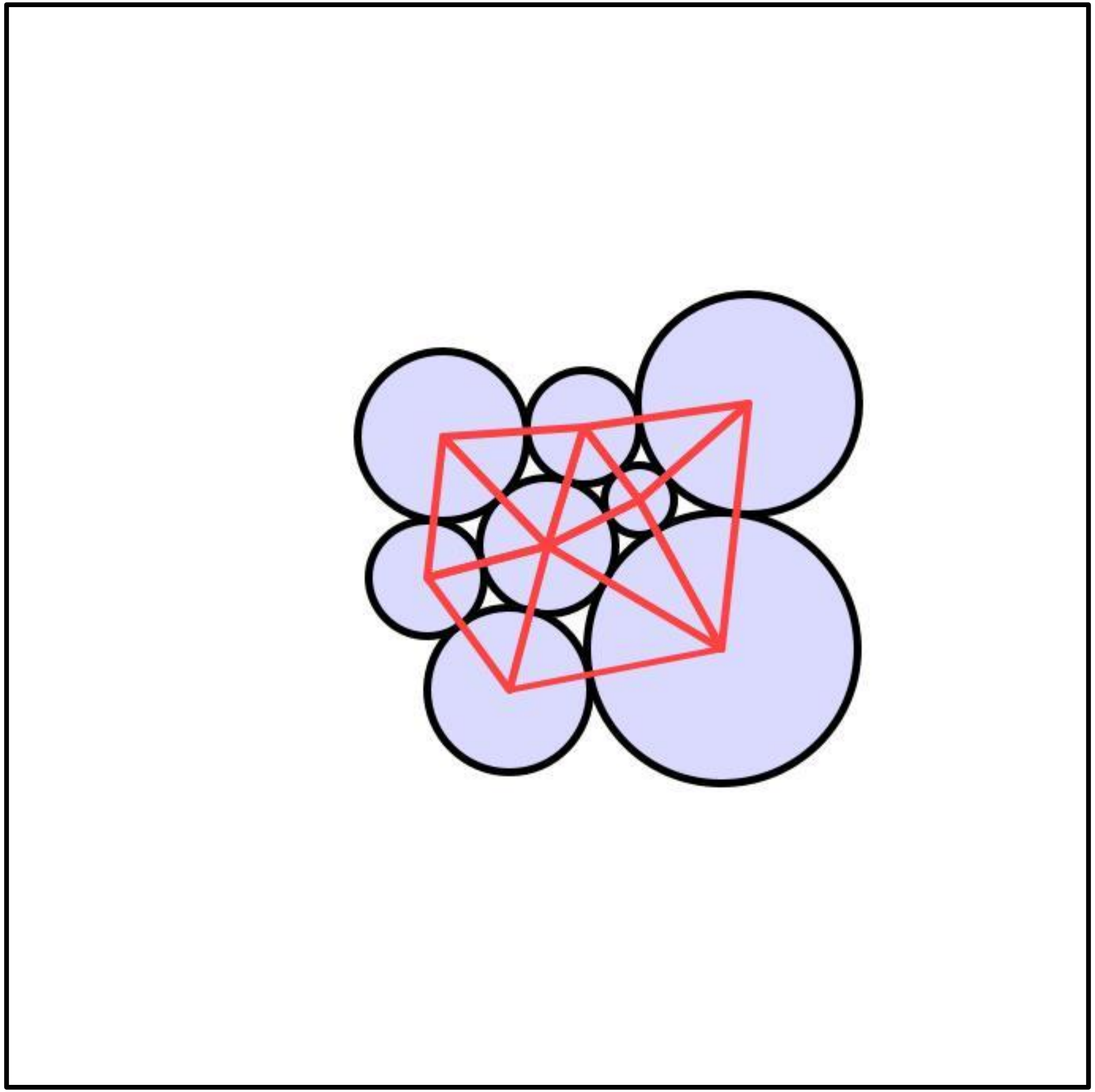


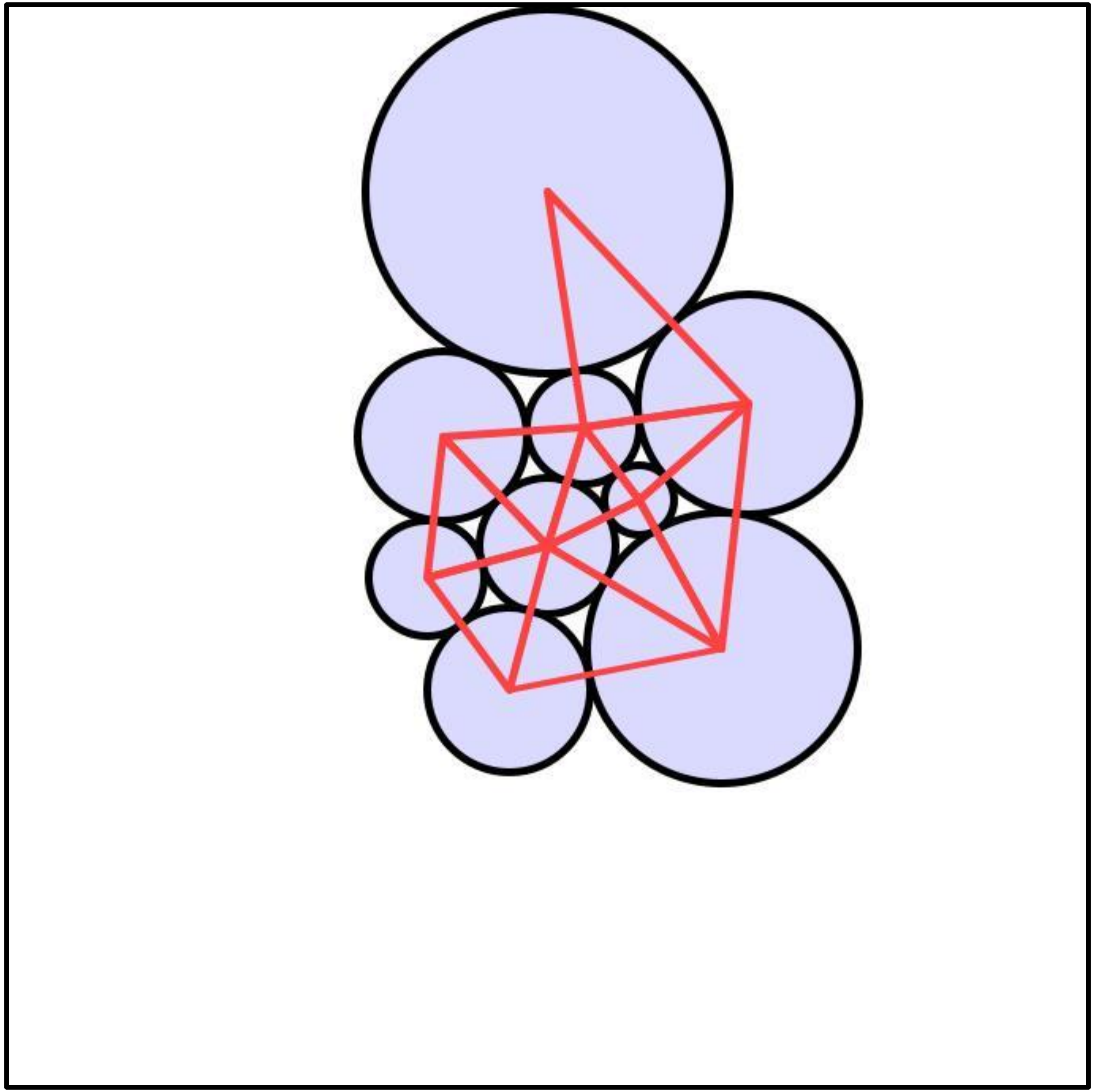


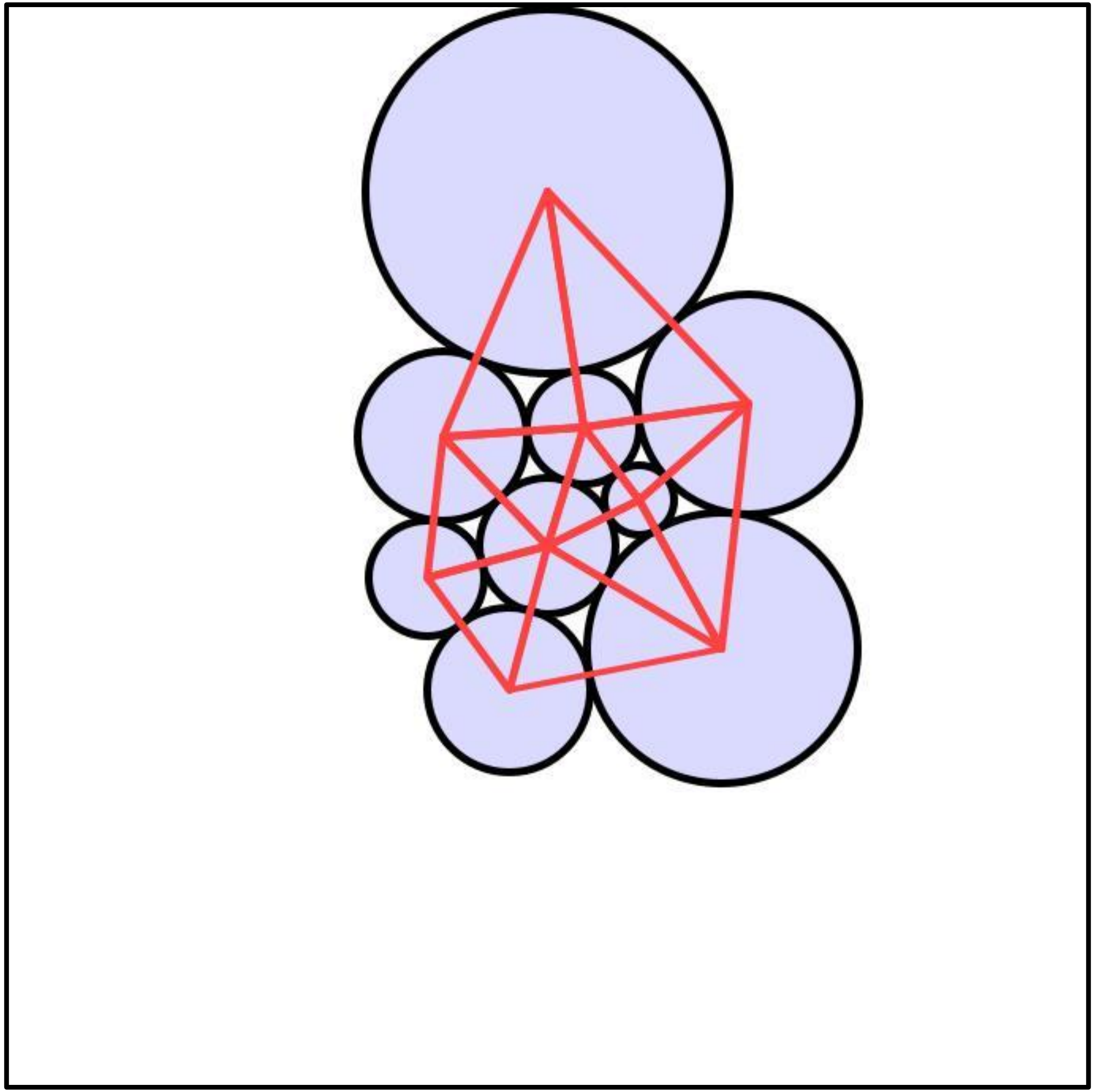


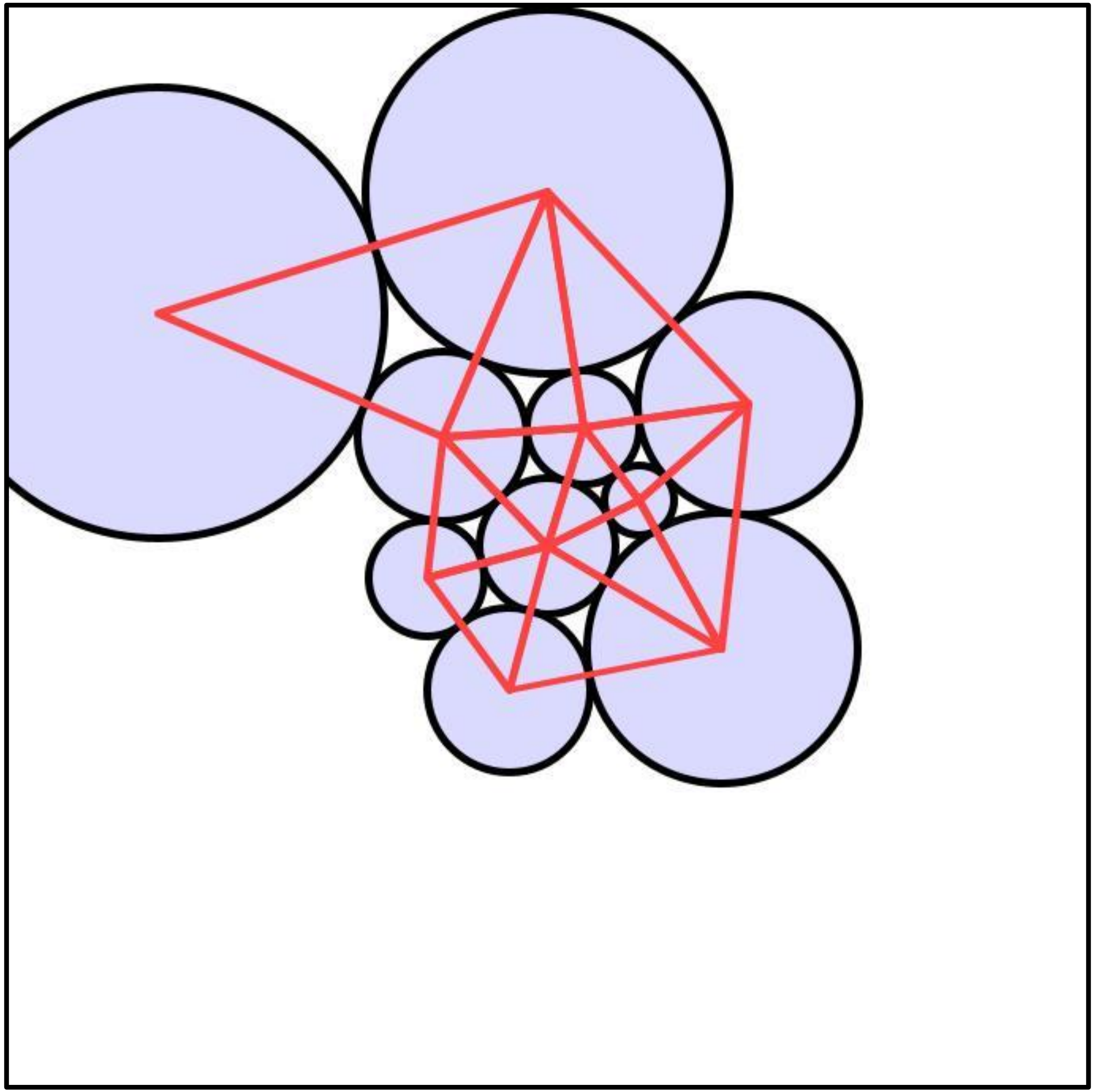


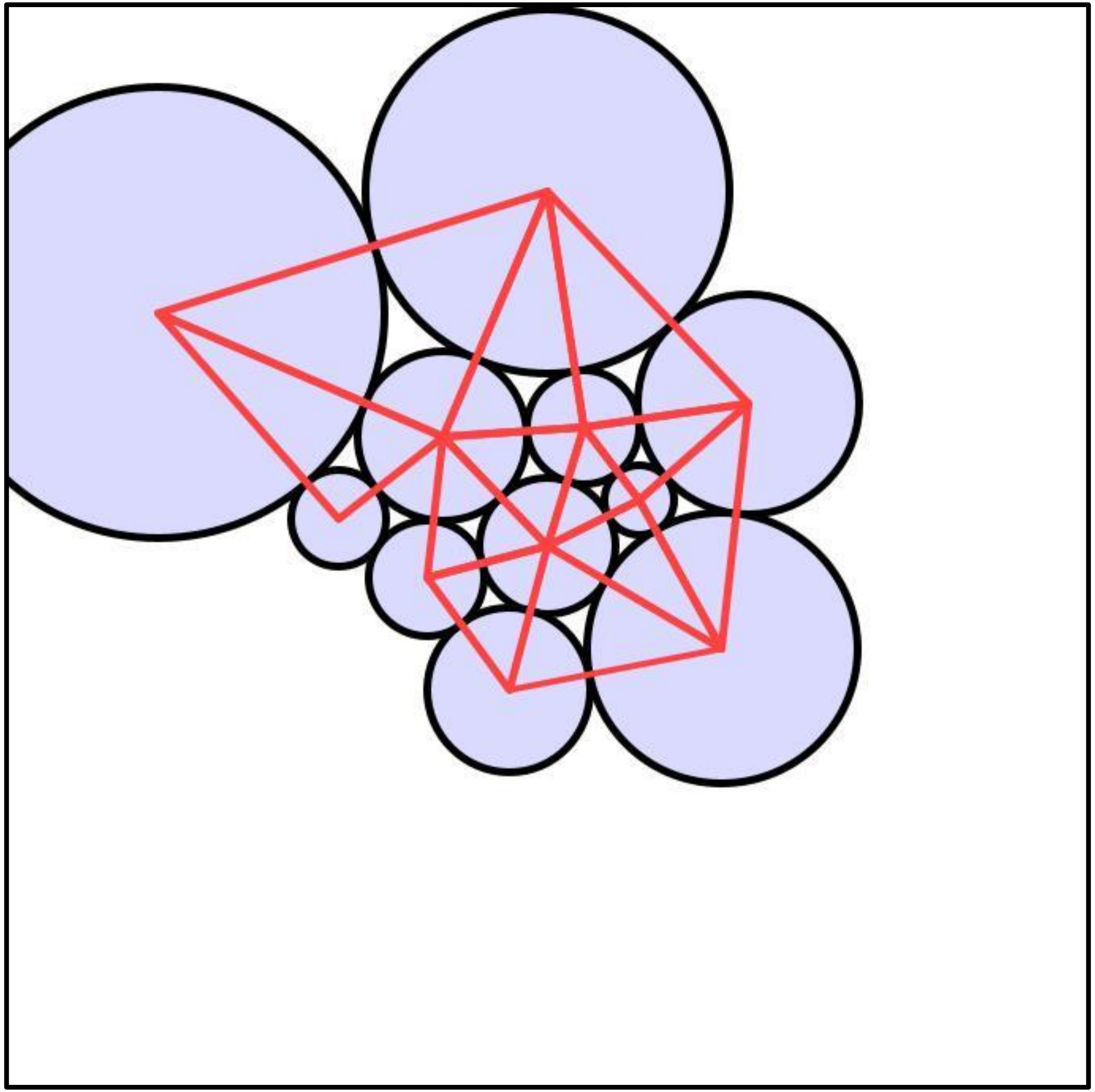


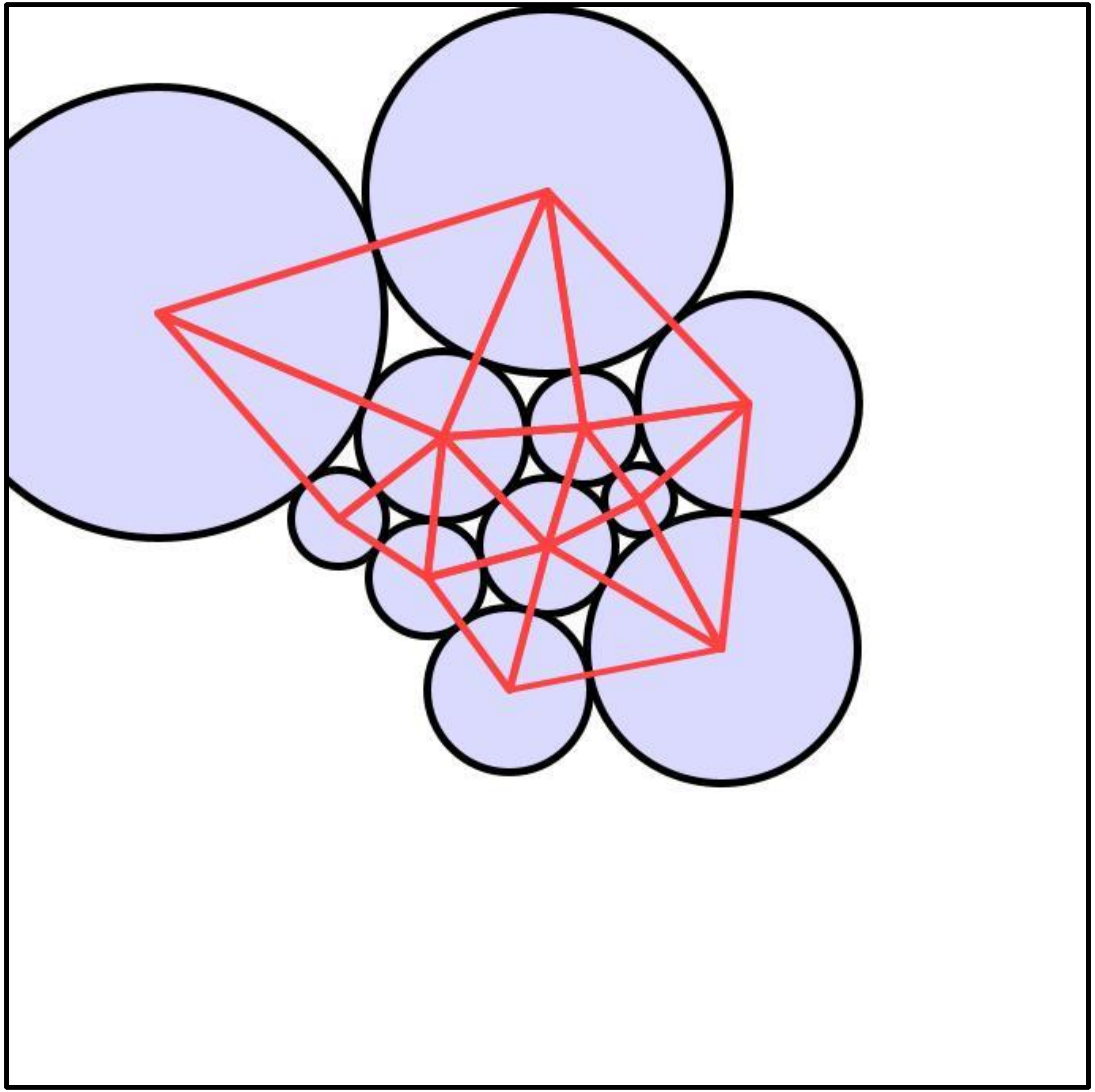


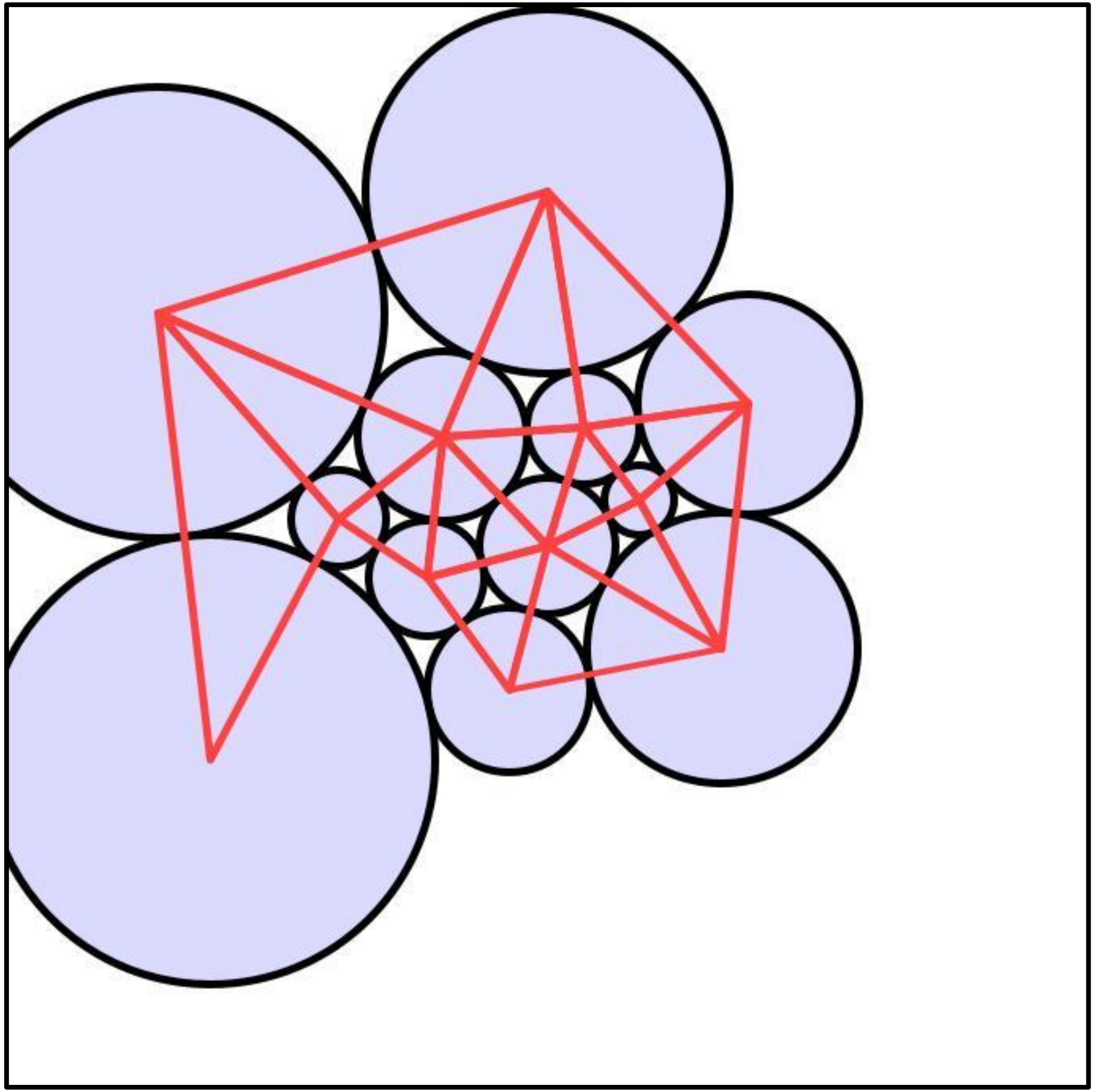


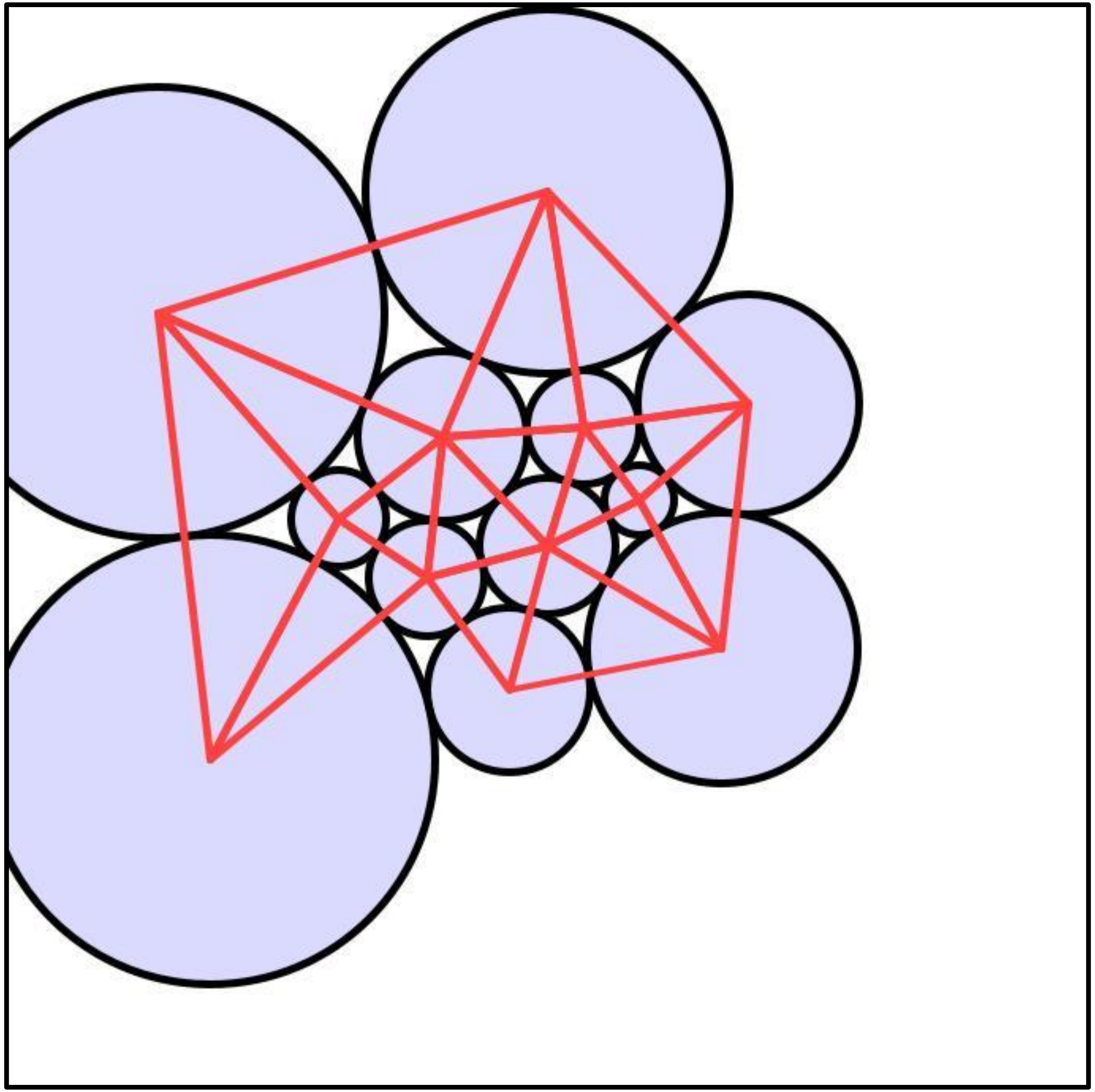


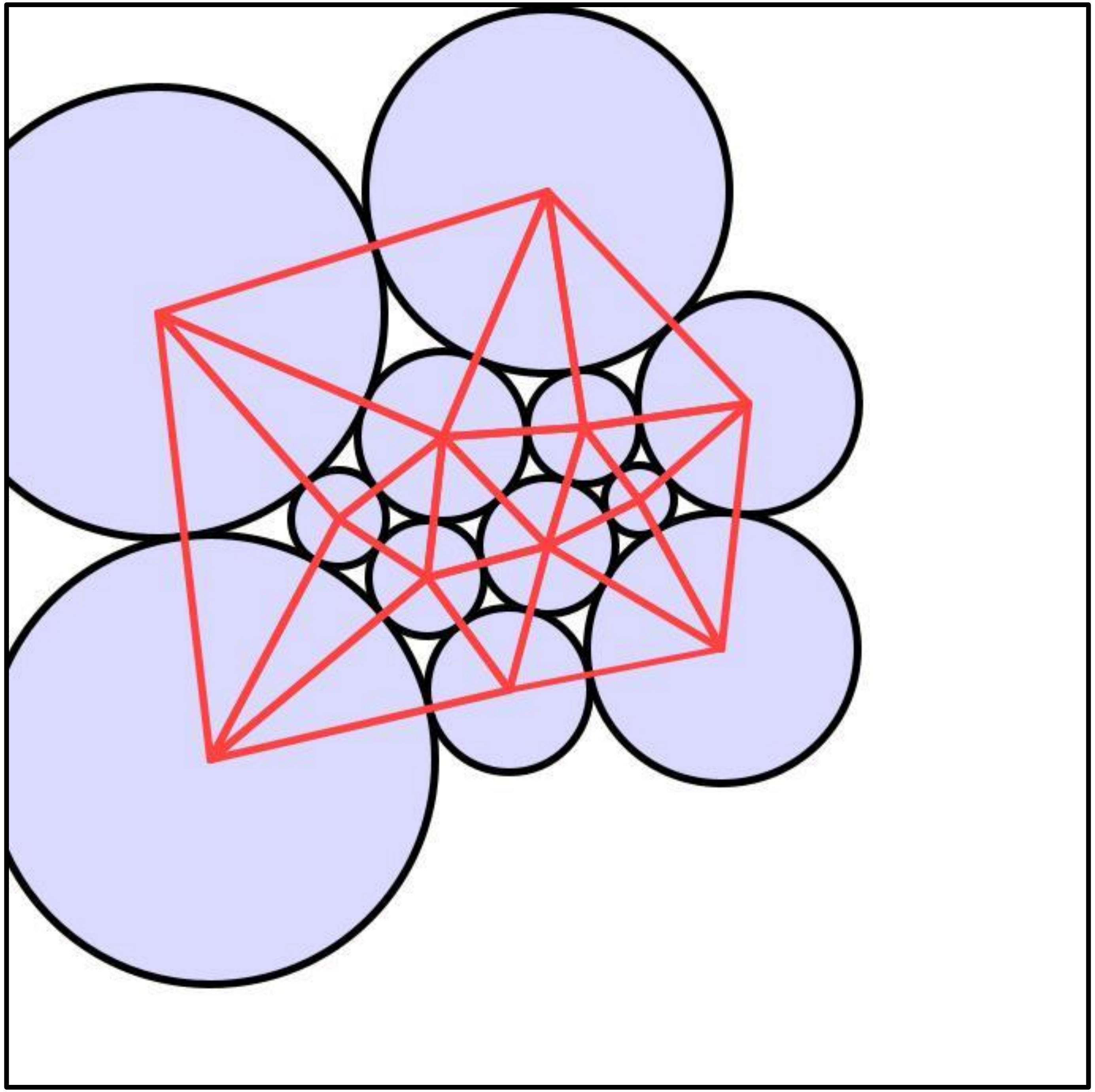


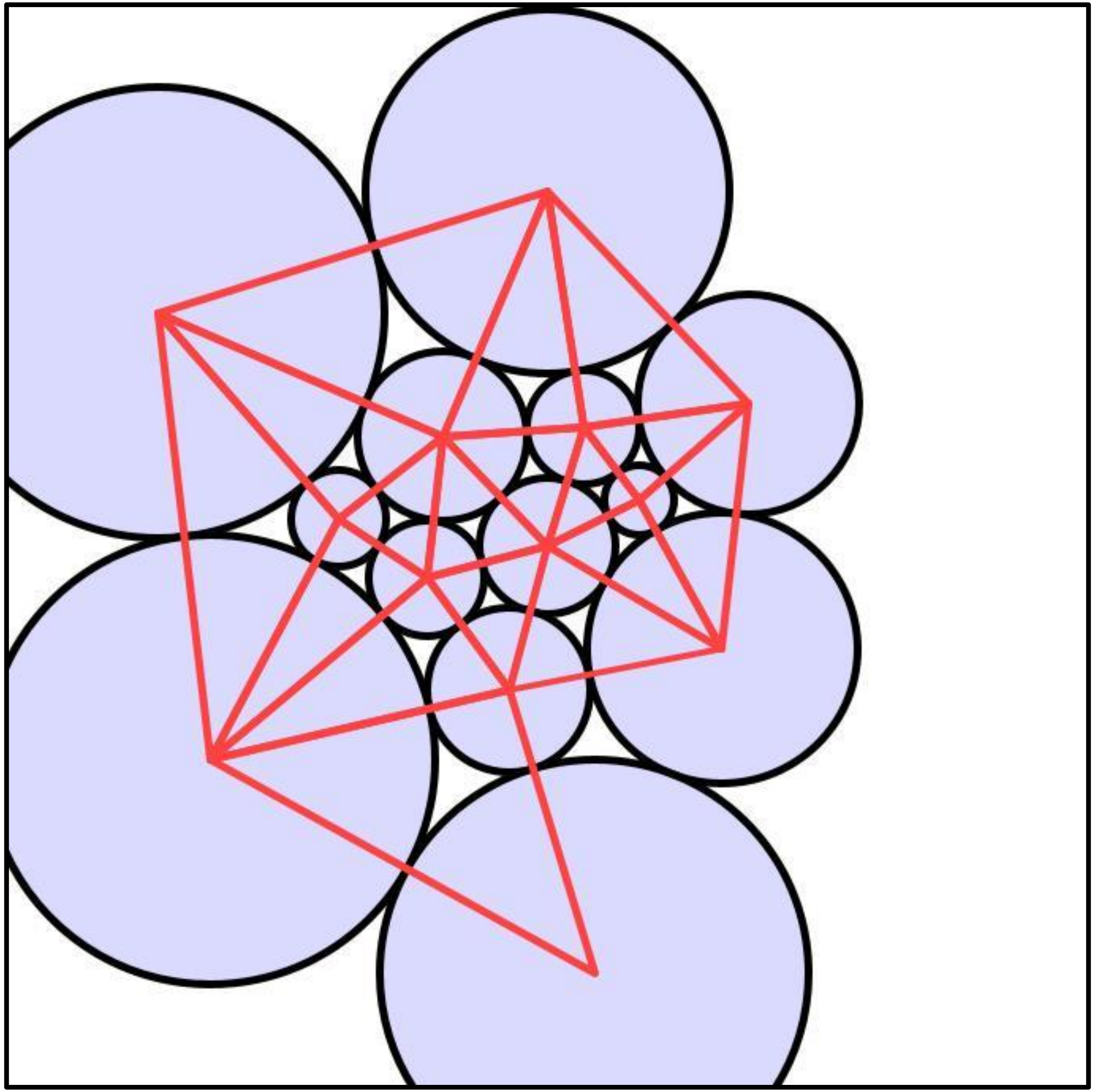


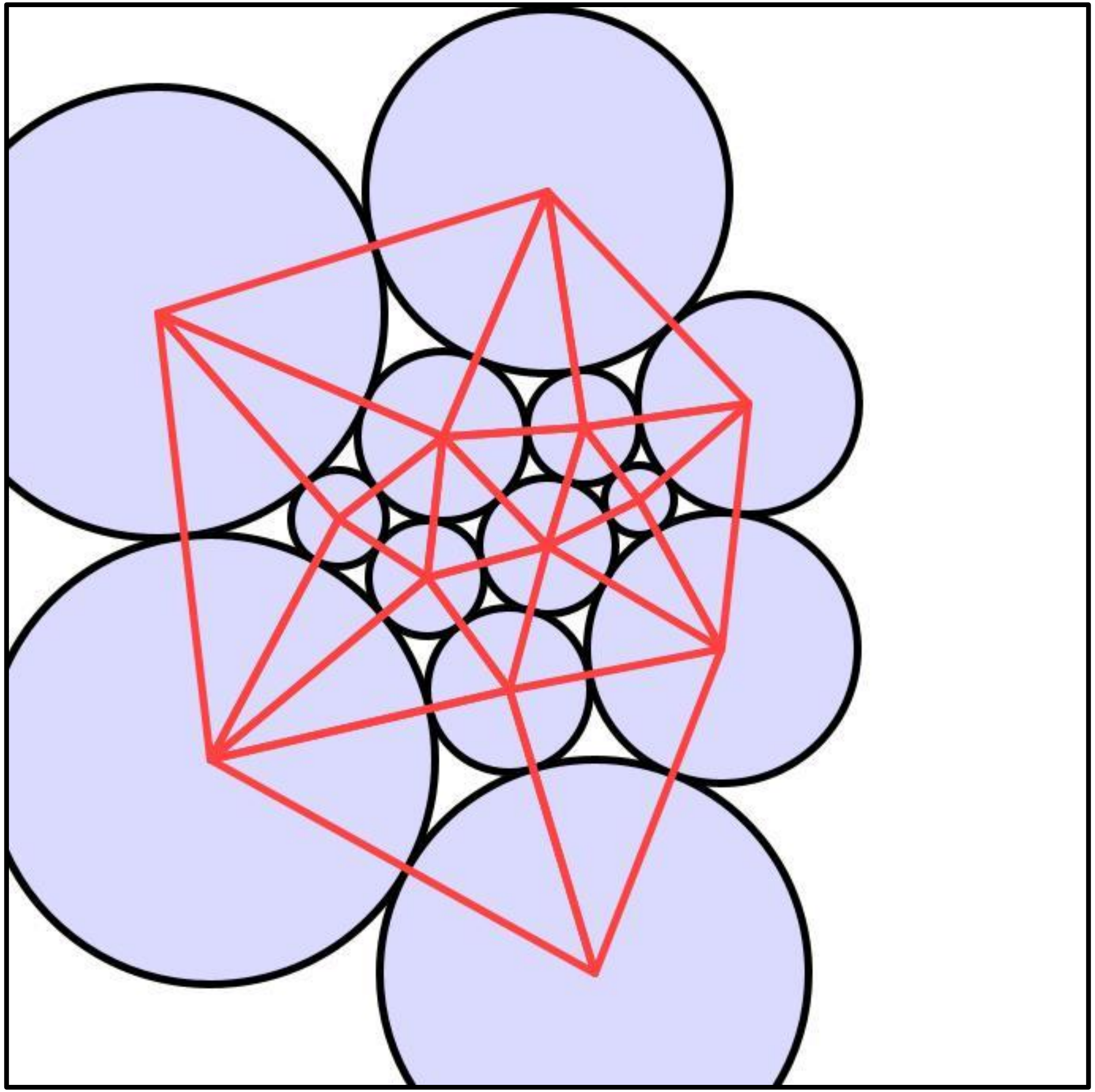


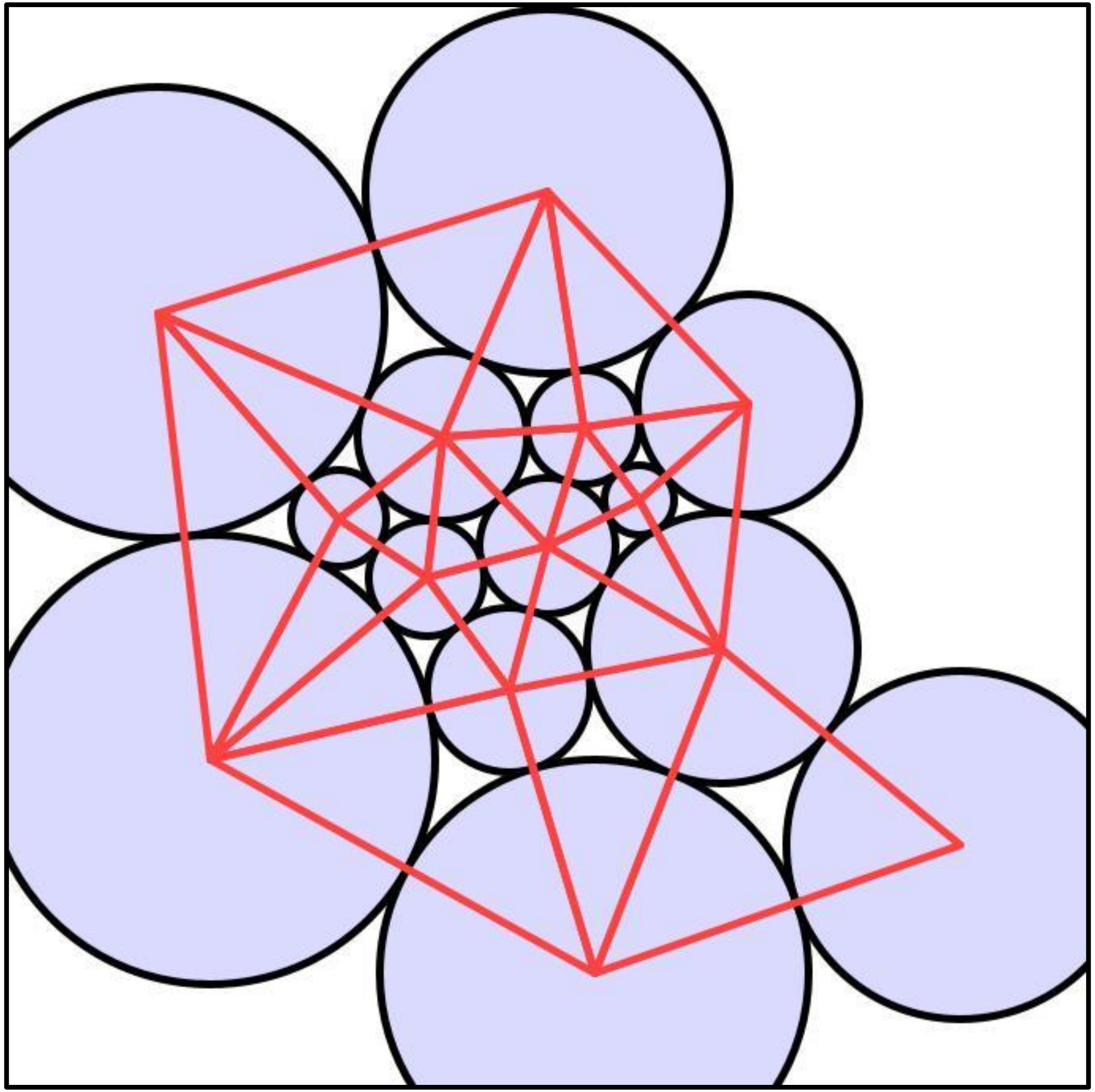


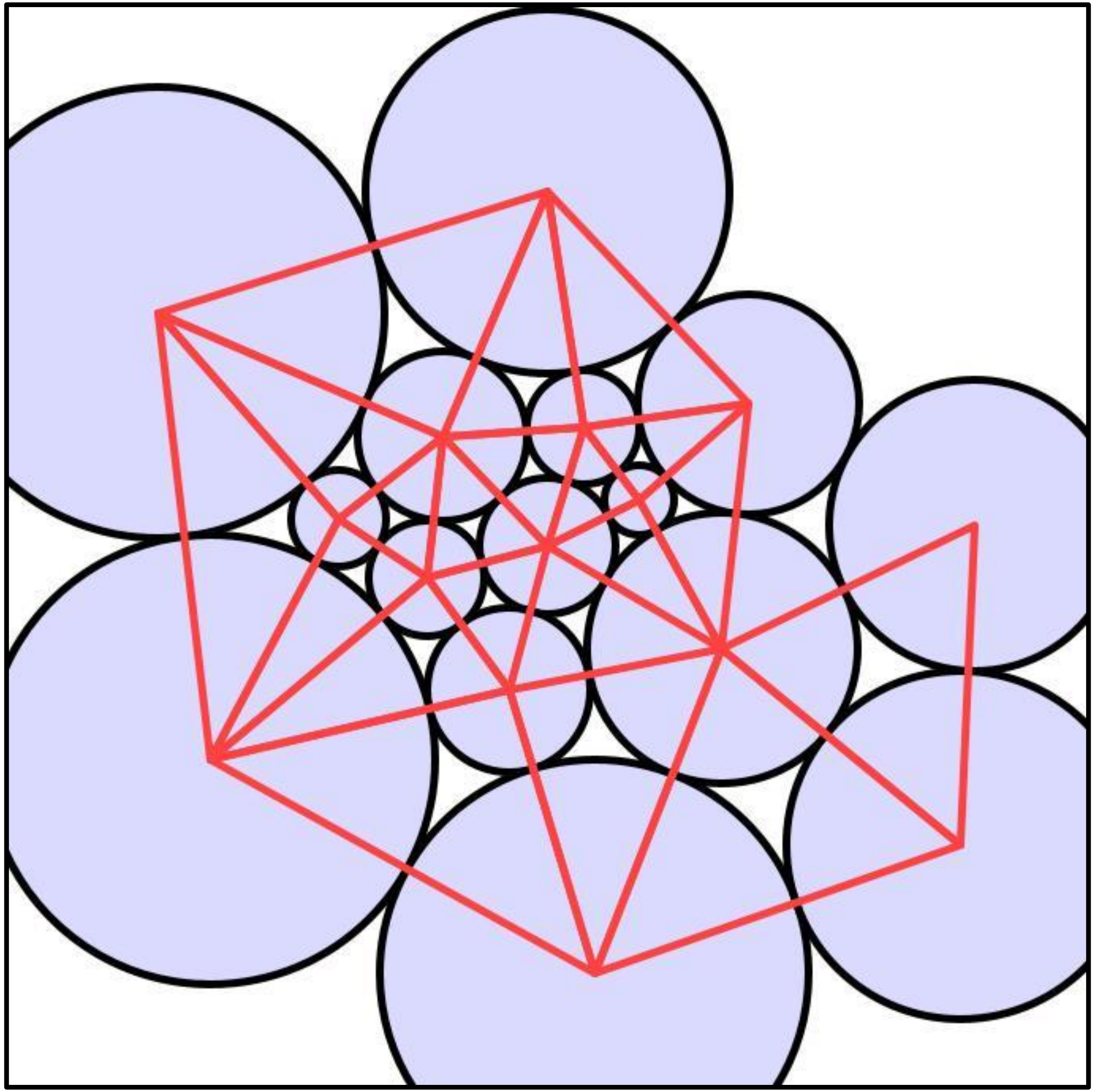


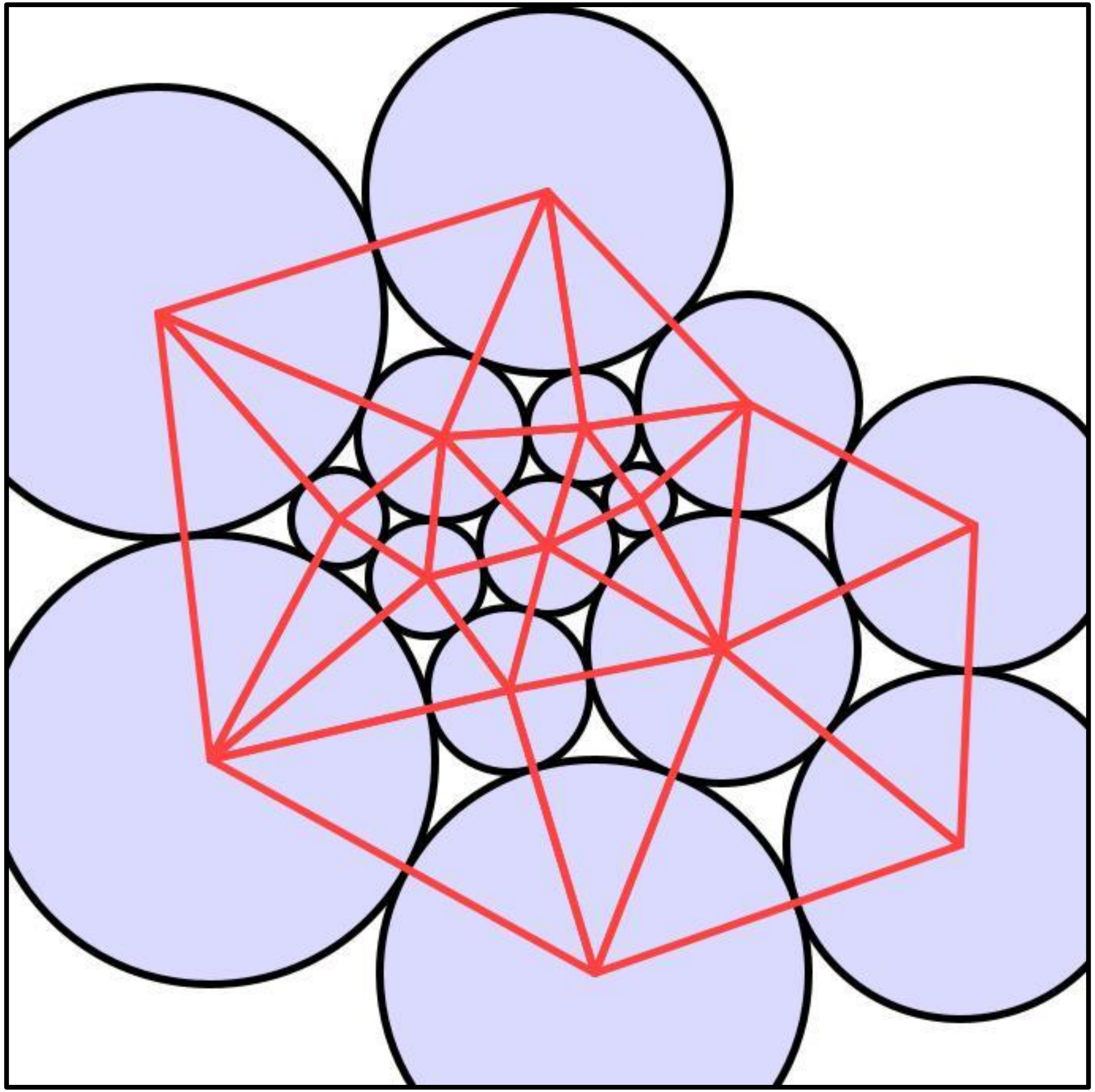












Done!

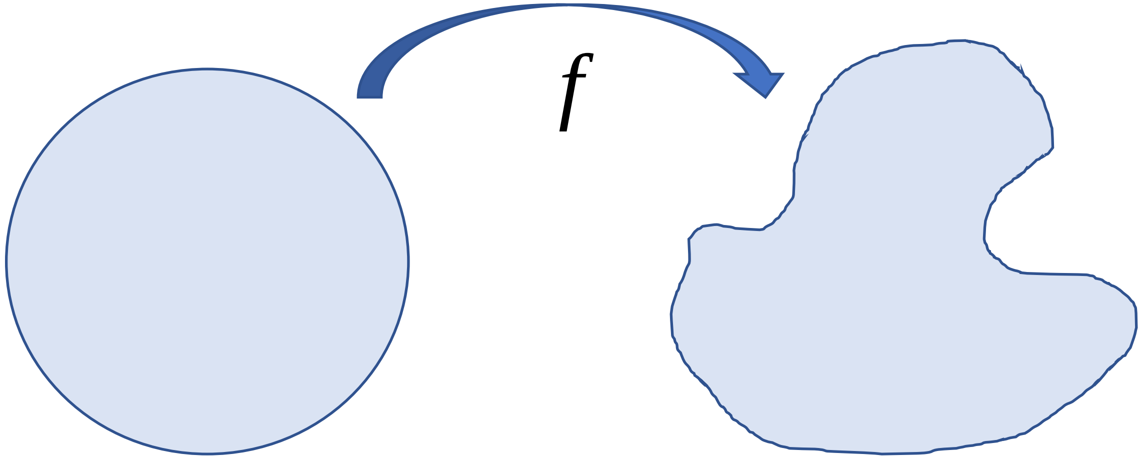
Algorithms:

- A “packing algorithm” refers to some systematic method for computing the data associated with a circle packing P for a given complex K.
- Bill Thurston introduced circle packing and blessed us with a wonderful iterative algorithm based on radii. This works amazingly well in **euclidean** and **hyperbolic** geometry; the latest algorithms will pack complexes involving several million circles.
- However, there is as yet no algorithm which works directly in **spherical** geometry.
- Nearly all circle packings on the sphere have been stereographically projected from the plane or the disc. Except in certain rare exceptional situations, stereographic projection **does not work** for circle packings with **branching**.
- This talk is about opening a new approach, replacing radii as parameters by something Mobius invariant; namely, discrete Schwarzian data associate with edges.

I am nowhere near an algorithm, so I can use some help!

A Discrete
Schwarzian Derivative

A Discrete Schwarzian Derivative:

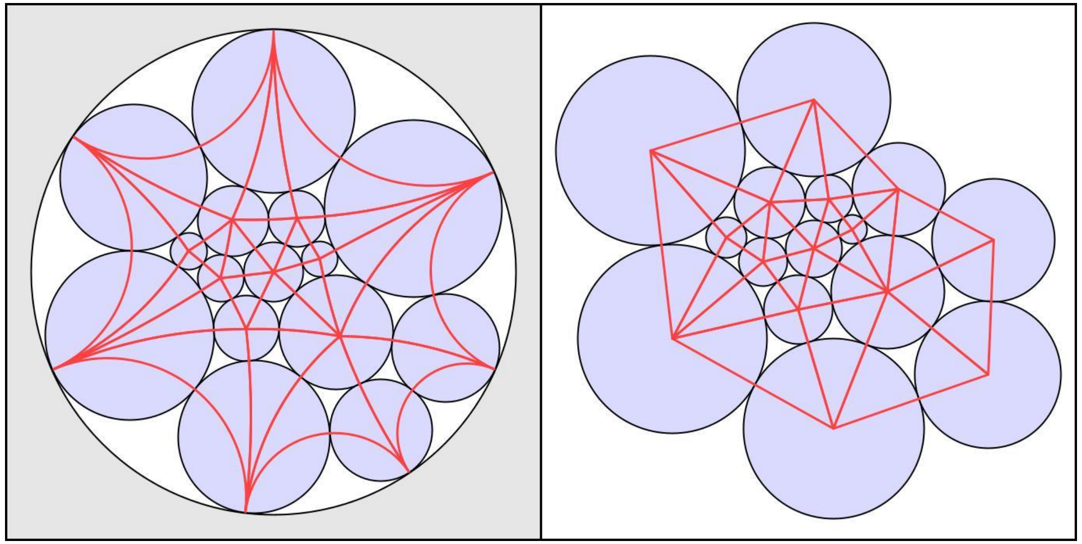


Classical:

* $S(f) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$
* $f \equiv 0$ iff f Is Mobius
* Invariance: $S(\phi(f)) \equiv S(f)$ If ϕ is Mobius
* Measures the “distance” of f from the closest Mobius.

Discrete: ????

- Combinatoric
- Face-by-face
- Conformally faithful



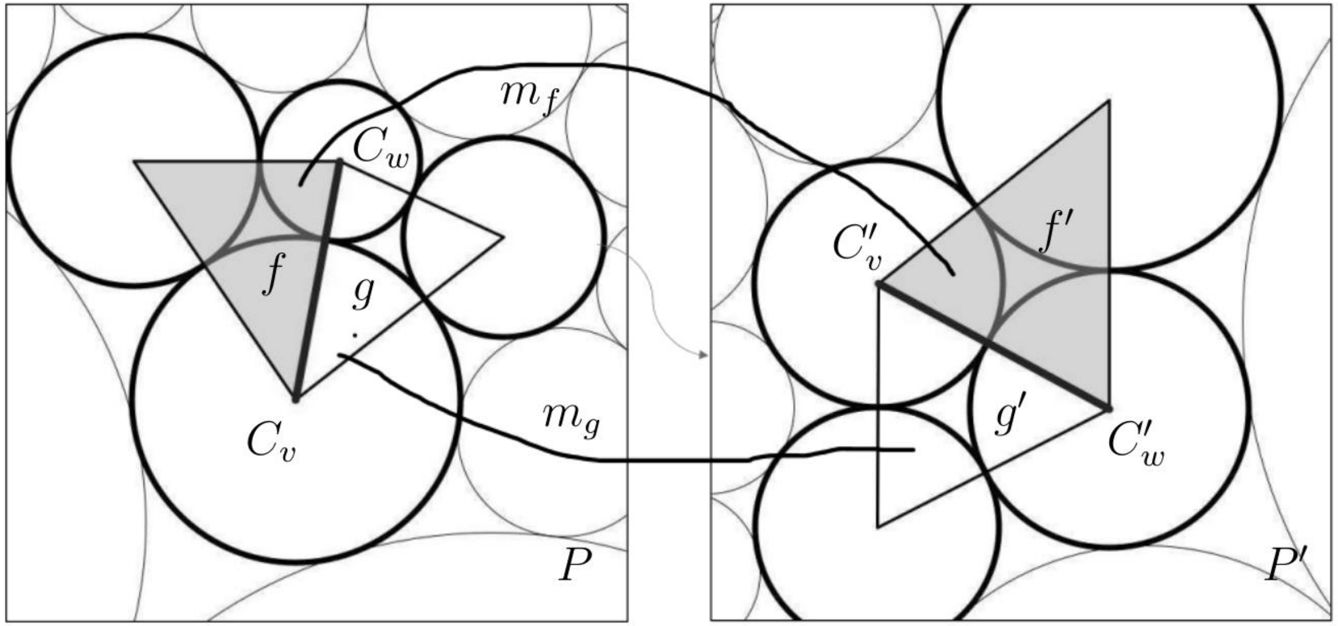
Defining the discrete Schwarzian Derivative: (Following Gerald Orick)

Given discrete analytic function

$$F : P \longrightarrow P'$$

and edge e between faces f and g, define Möbius transformations:

$$m_f : f \longrightarrow f'$$
$$m_g : g \longrightarrow g'$$



The discrete Schwarzian derivative of F on edge e is given by

$$\Sigma_F(e) = \sigma$$

where $$m_g^{-1} \circ m_f = \mathbb{I} + \sigma \begin{bmatrix} t & -t^2 \\ 1 & -t \end{bmatrix} = \begin{bmatrix} 1+\sigma t & 1-\sigma t^2 \\ \sigma & 1-\sigma t \end{bmatrix}$$

Sadly, Schwarzian Derivatives are <u>not a good setting for experiments</u> (2 packings, complex numbers)

The Intrinsic Schwarzian for interior edges

Plan B: intrinsic schwarzians:

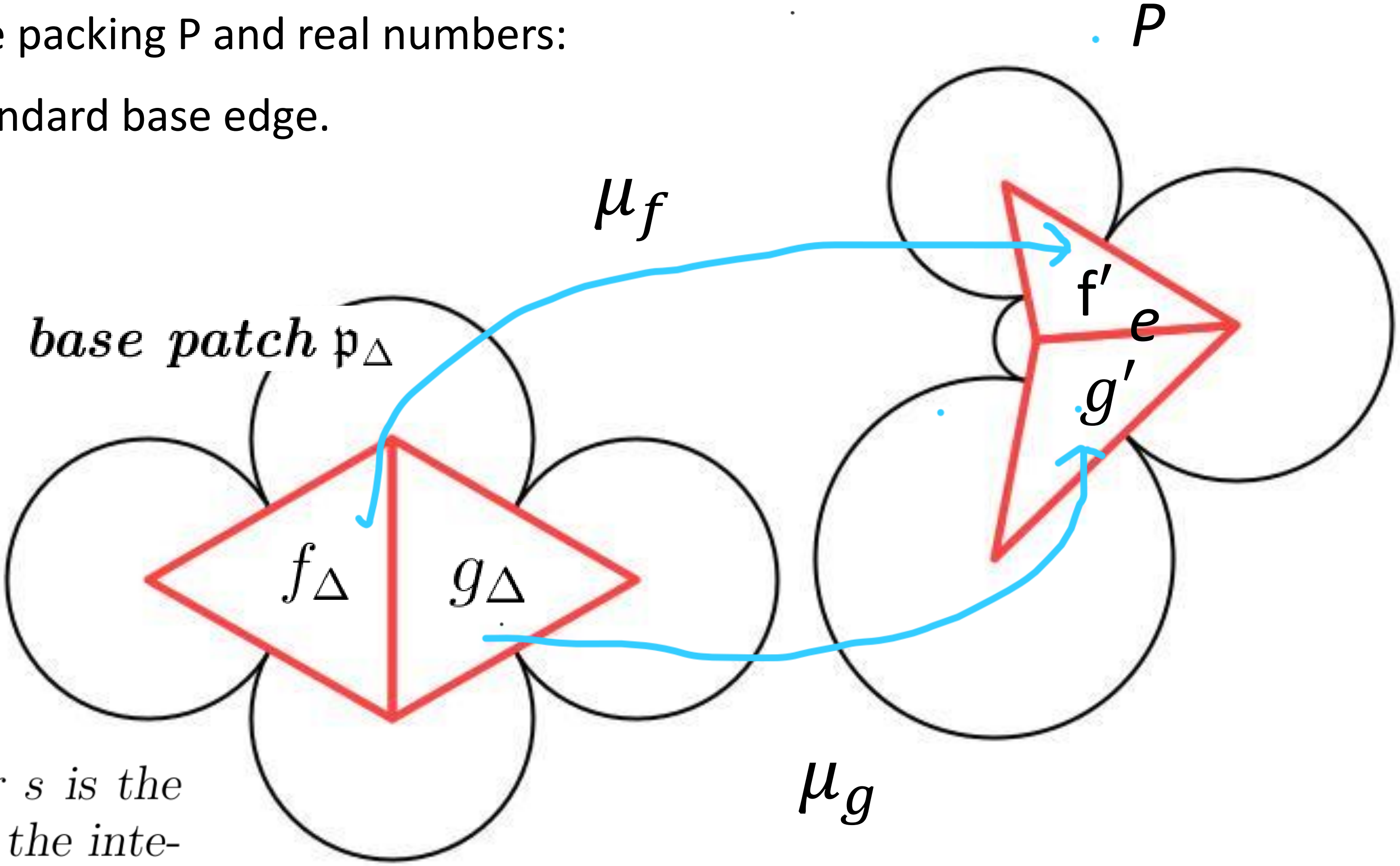
Reformulate for use with a single packing P and real numbers:

Compare every edge *e* to the standard base edge.

$$m_e = \mu_g^{-1} \circ \mu_f$$

$$m_e = \begin{bmatrix} 1+s & -s \\ s & 1-s \end{bmatrix}$$

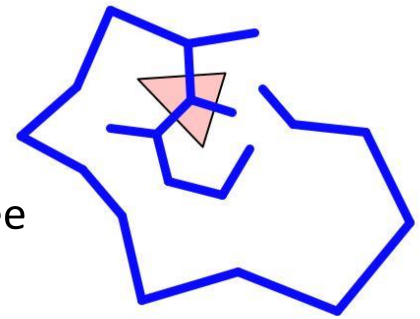
$$m_e = I + \begin{bmatrix} s & -s \\ s & -s \end{bmatrix}$$



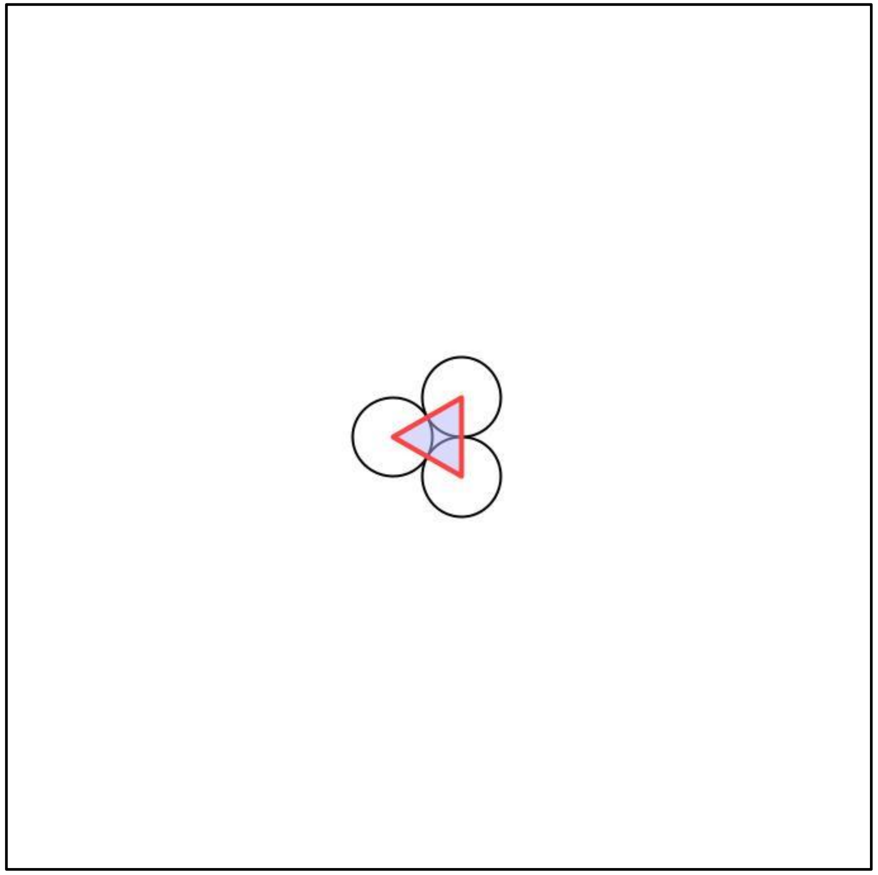
Definition: *The real number s is the* ***(intrinsic) schwarzian*** *for the interior edge e of P.*

Using known discrete Schwarzians:

Dual spanning tree



On the plane



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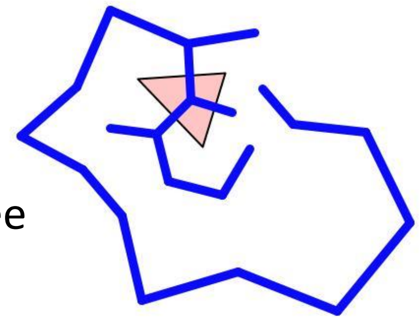

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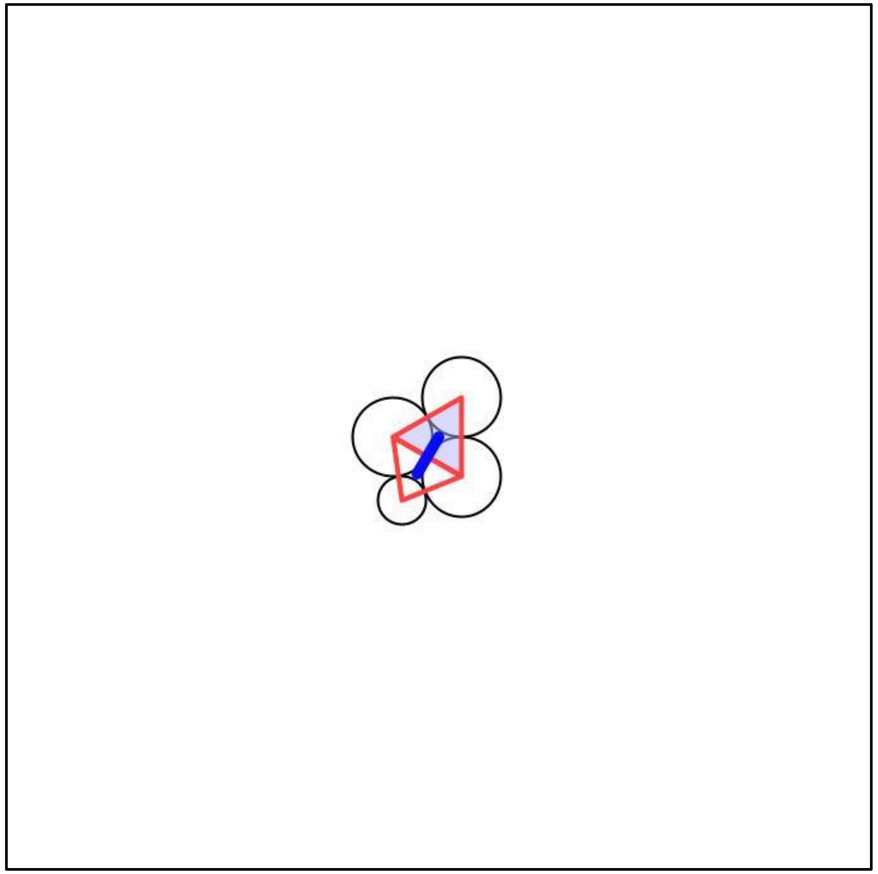

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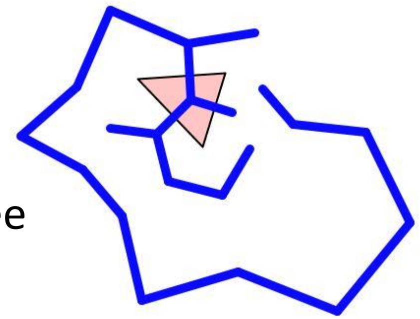

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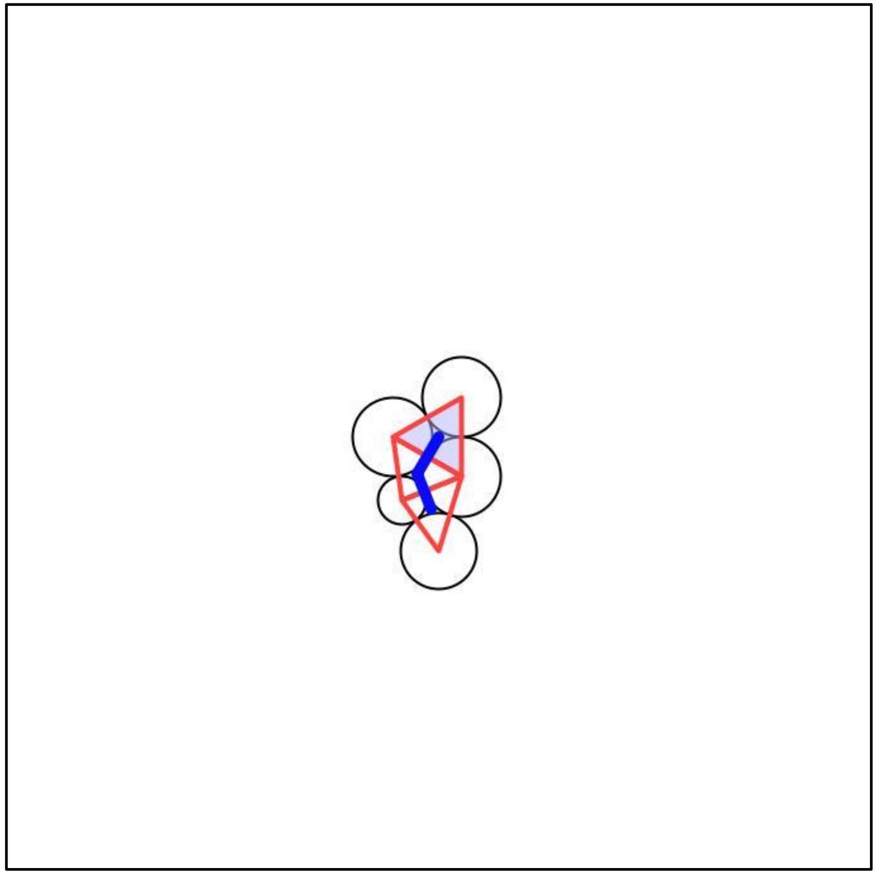

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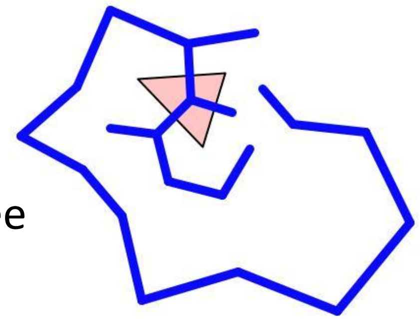

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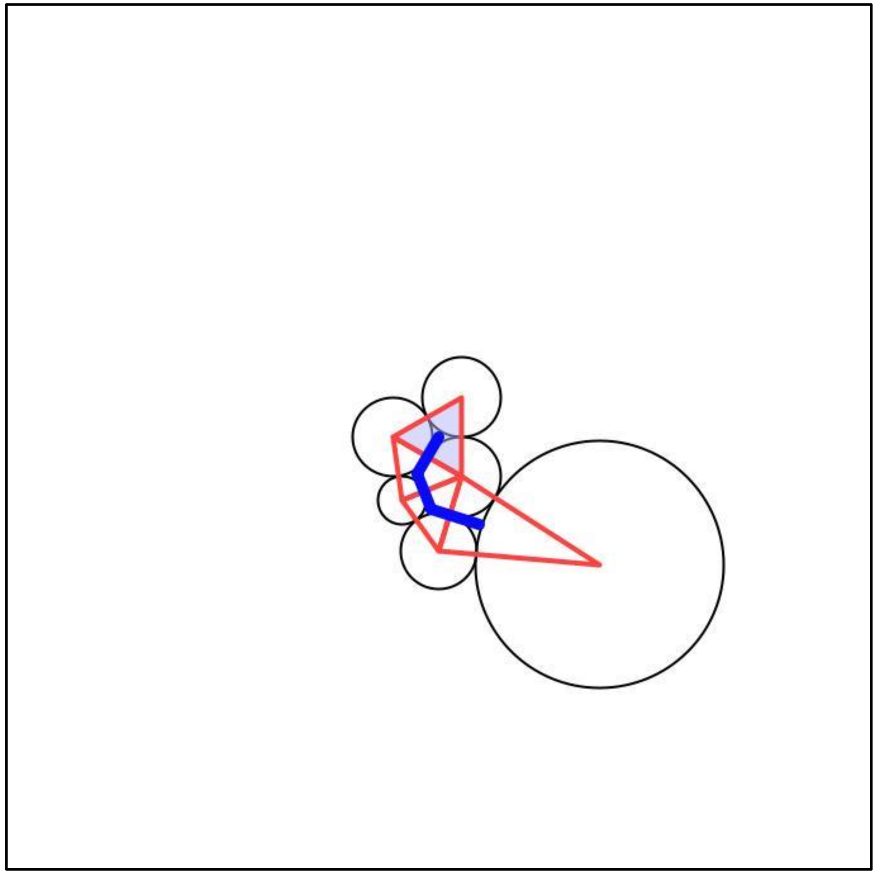

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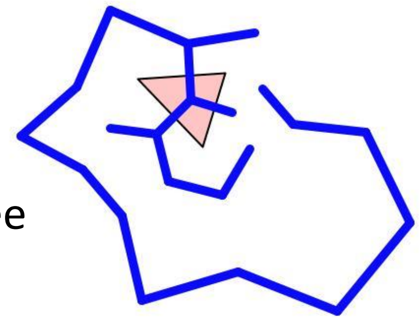

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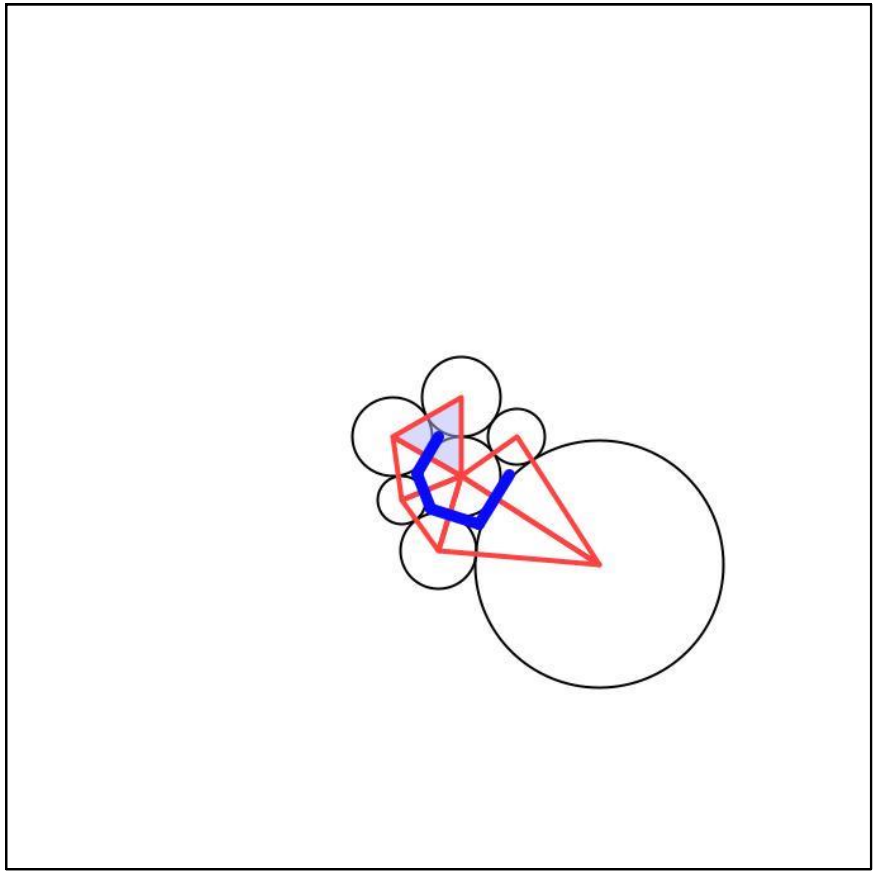

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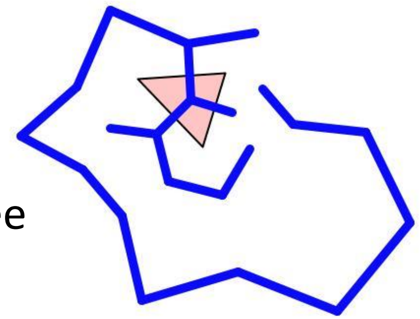

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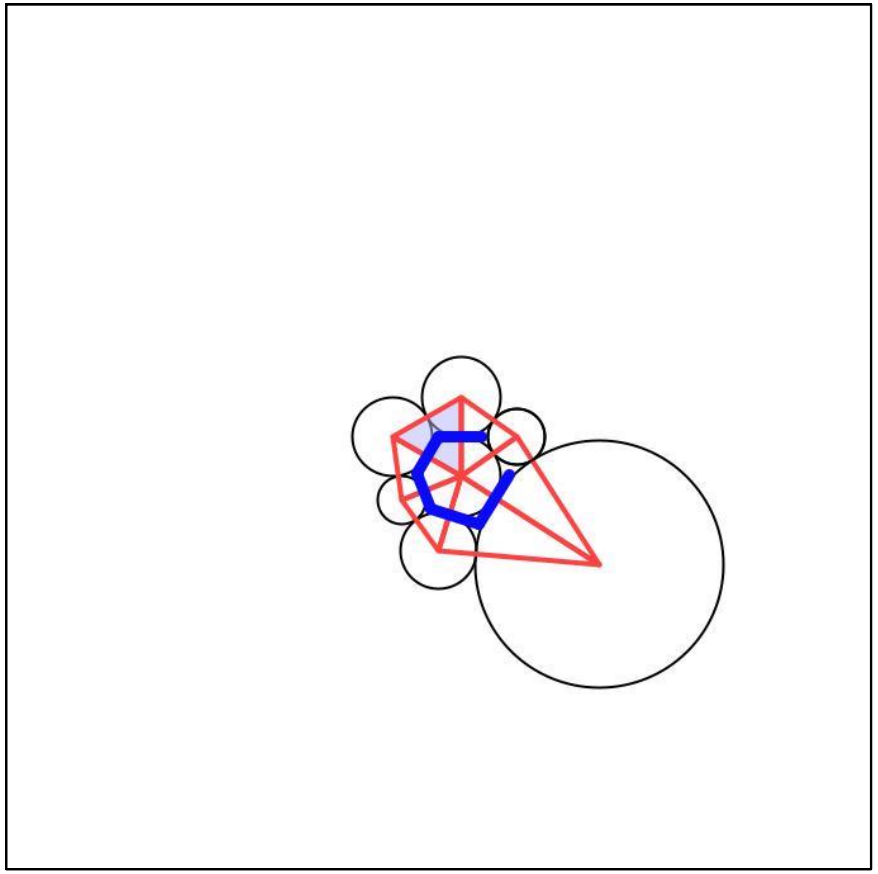

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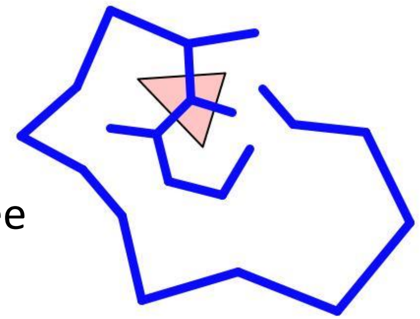

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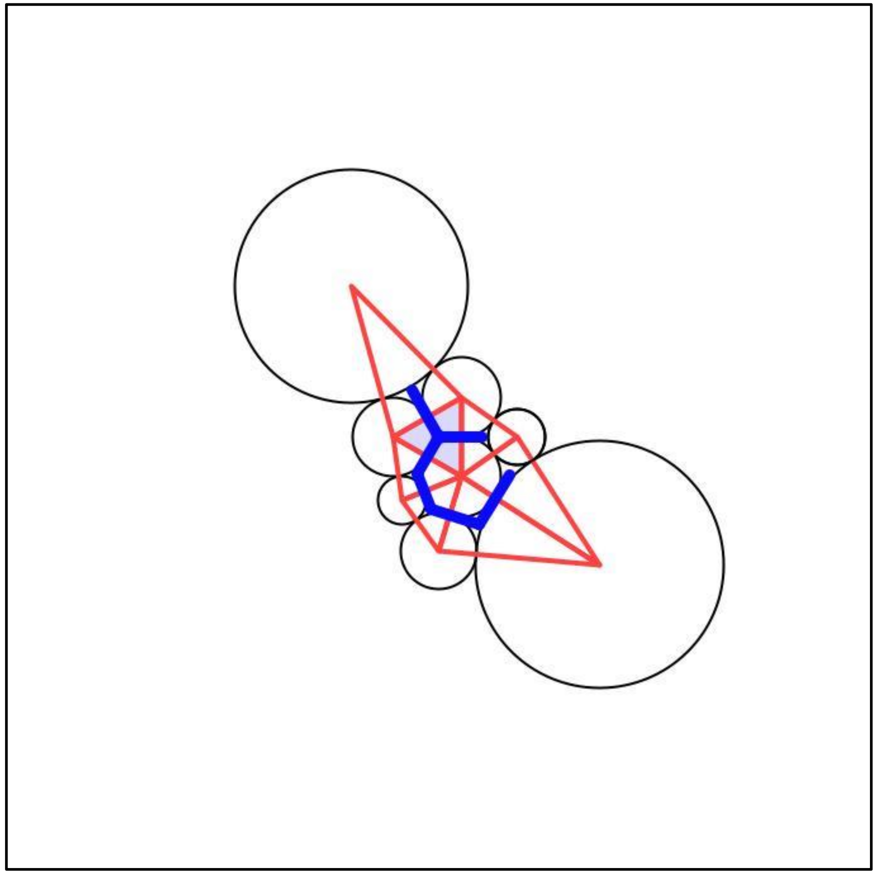

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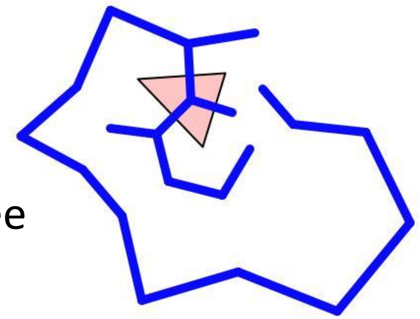

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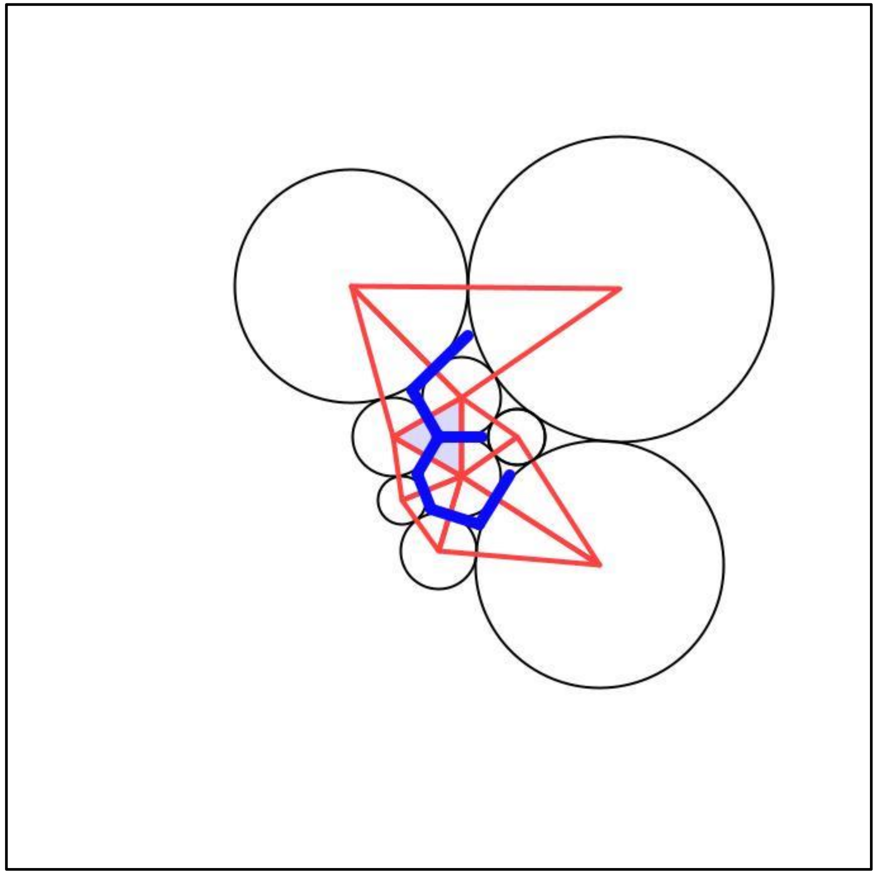

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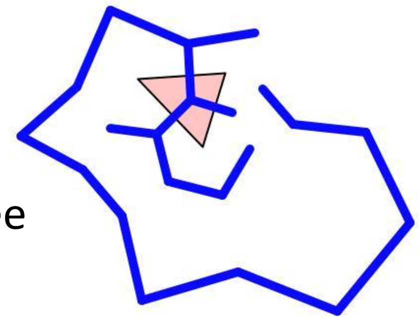

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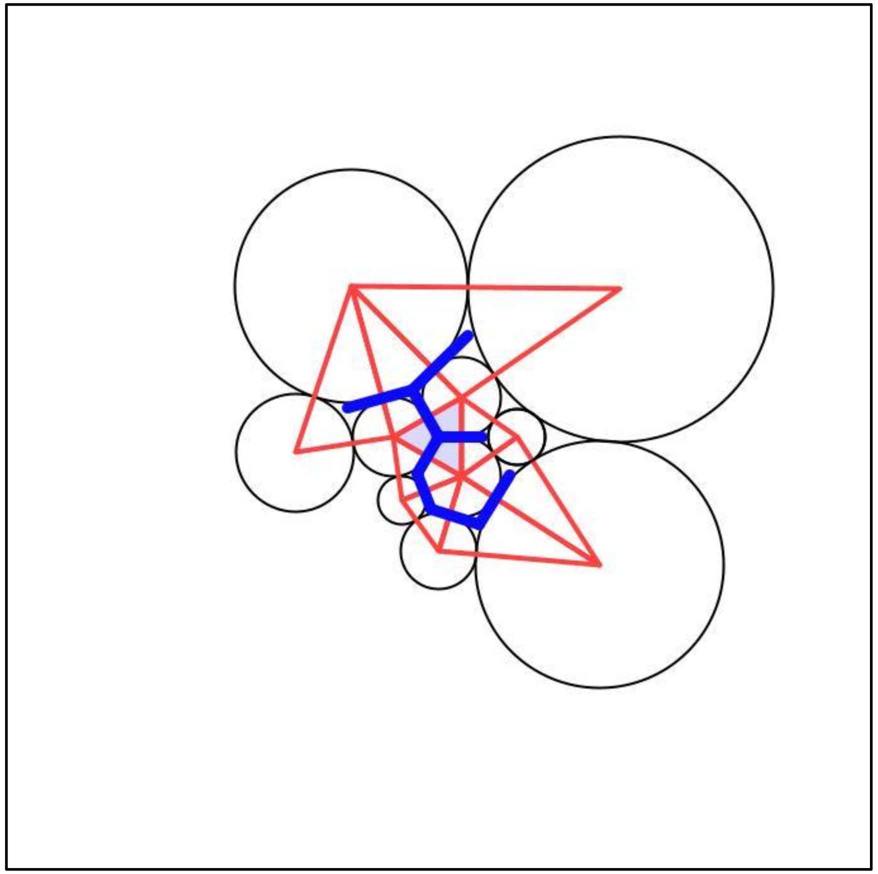

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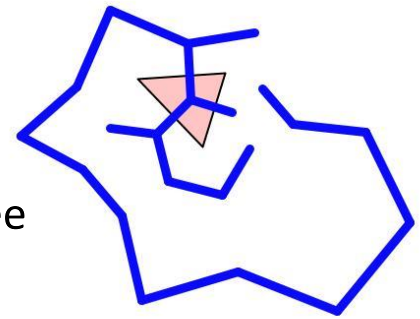

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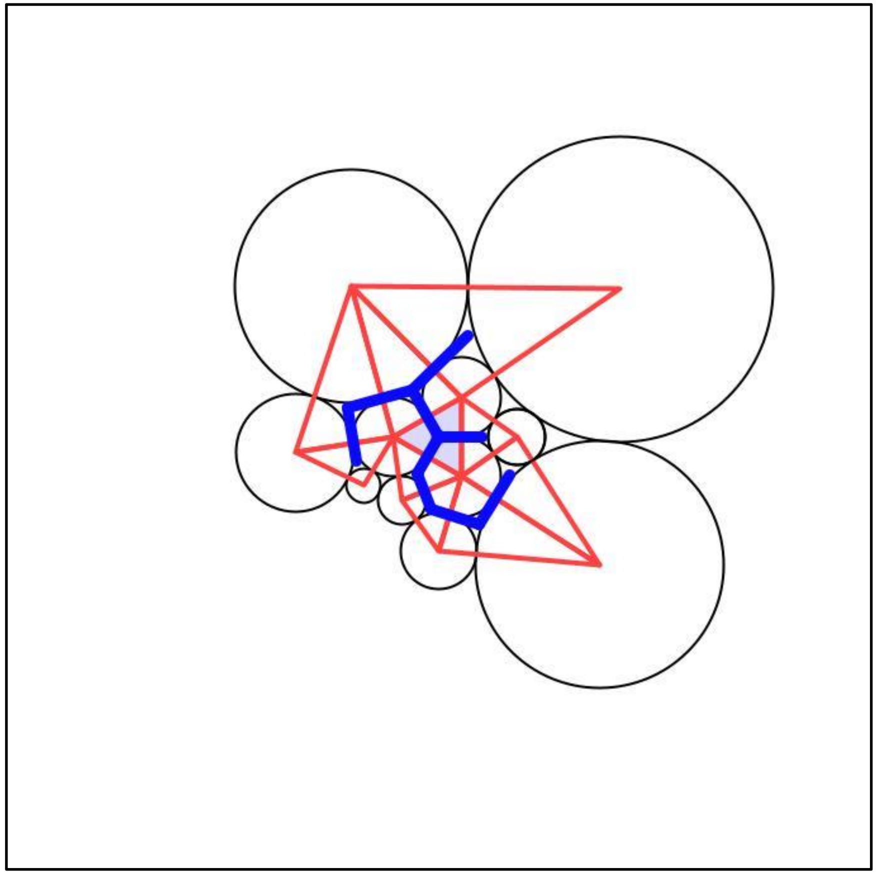

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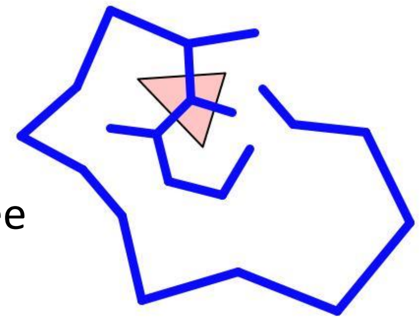

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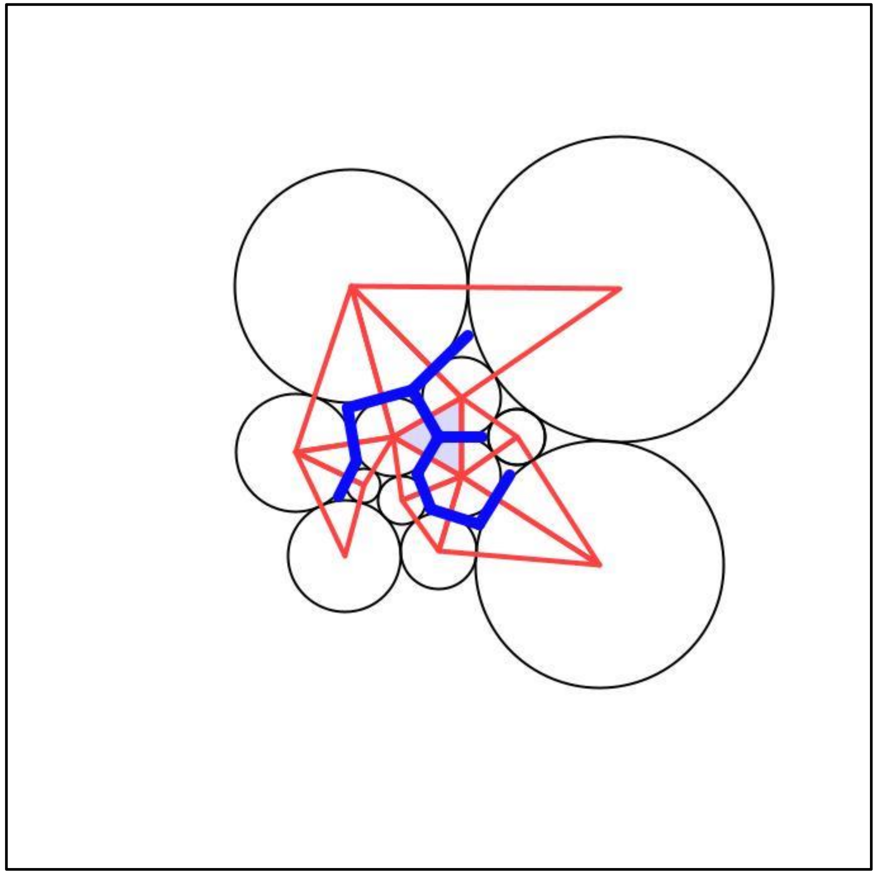

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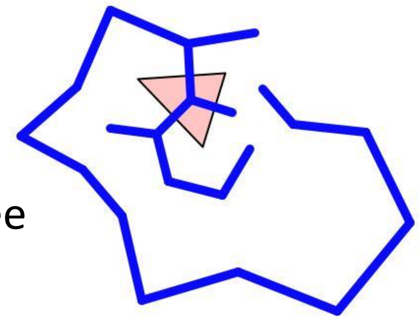

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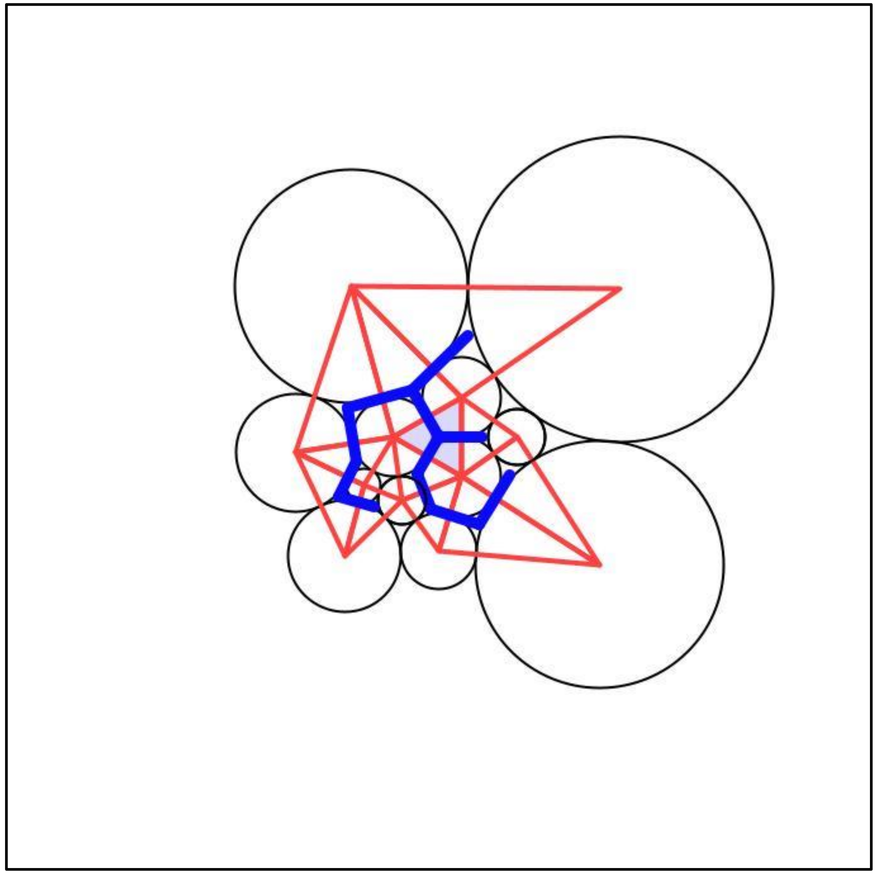

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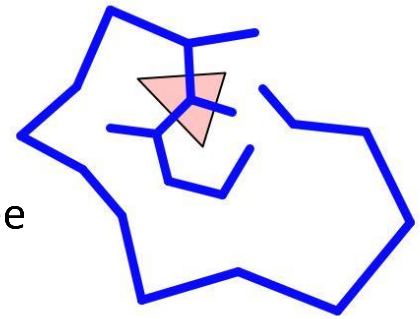

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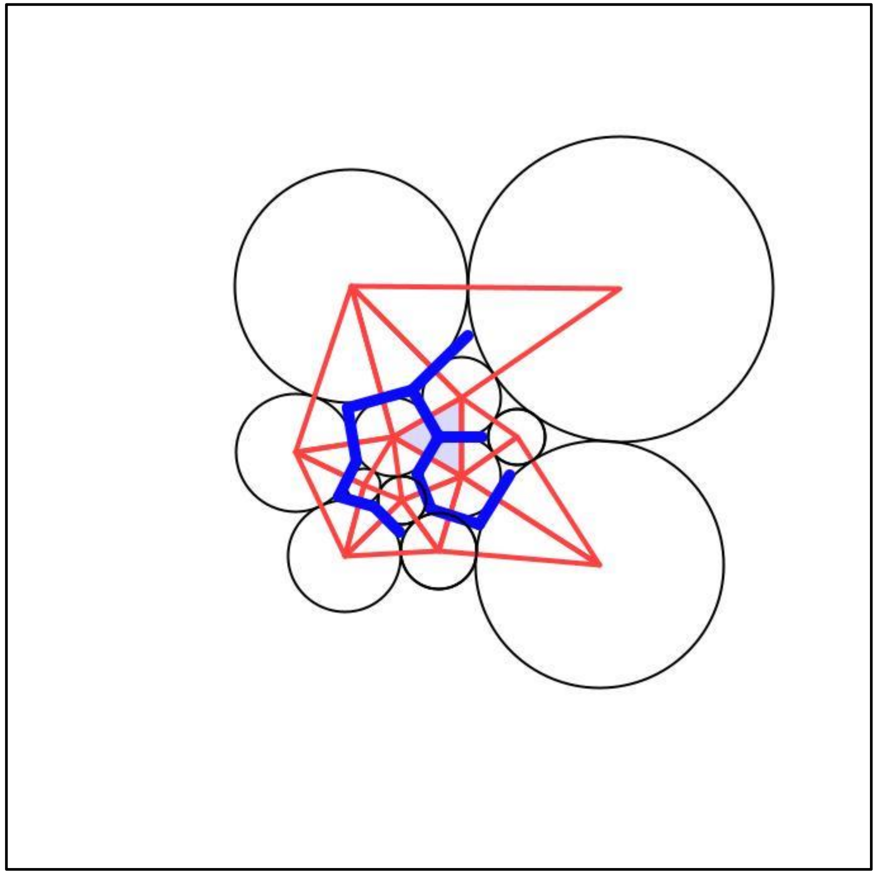

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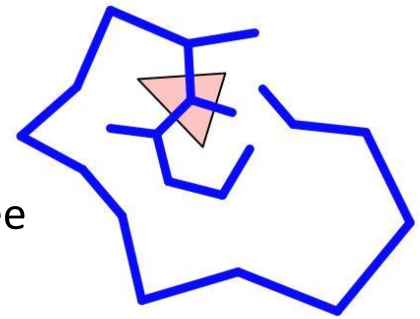

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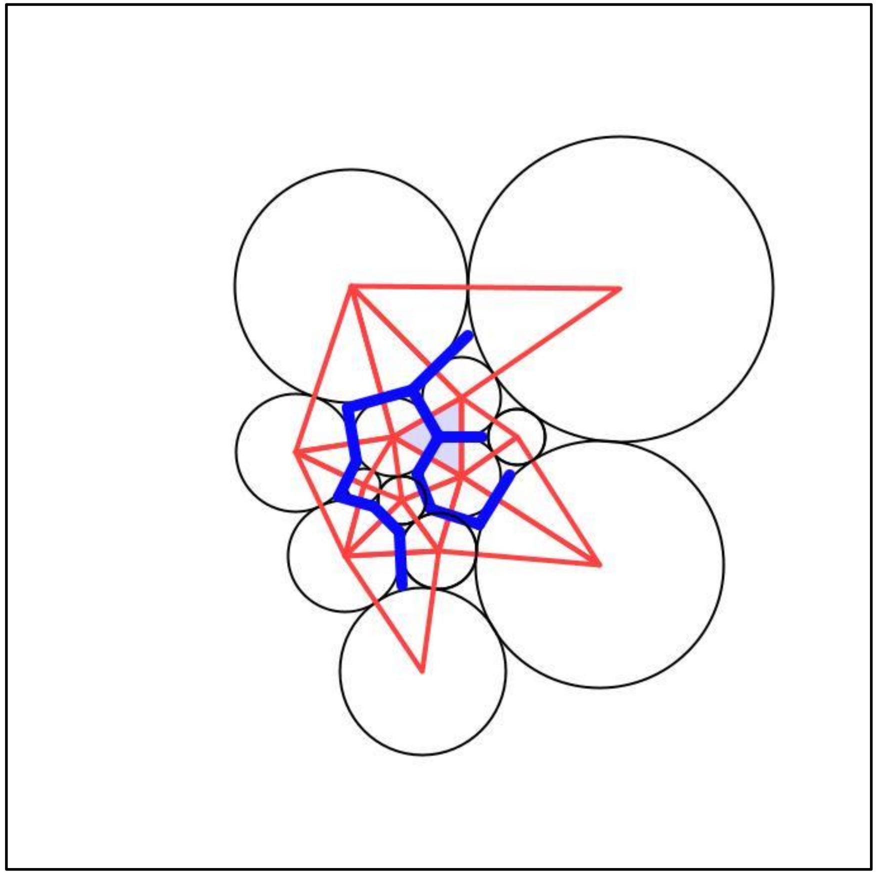

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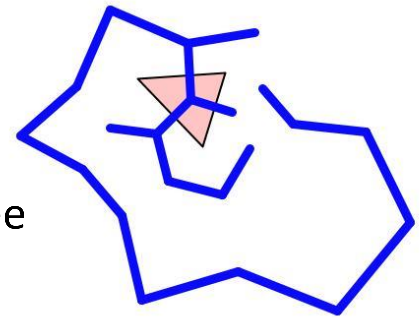

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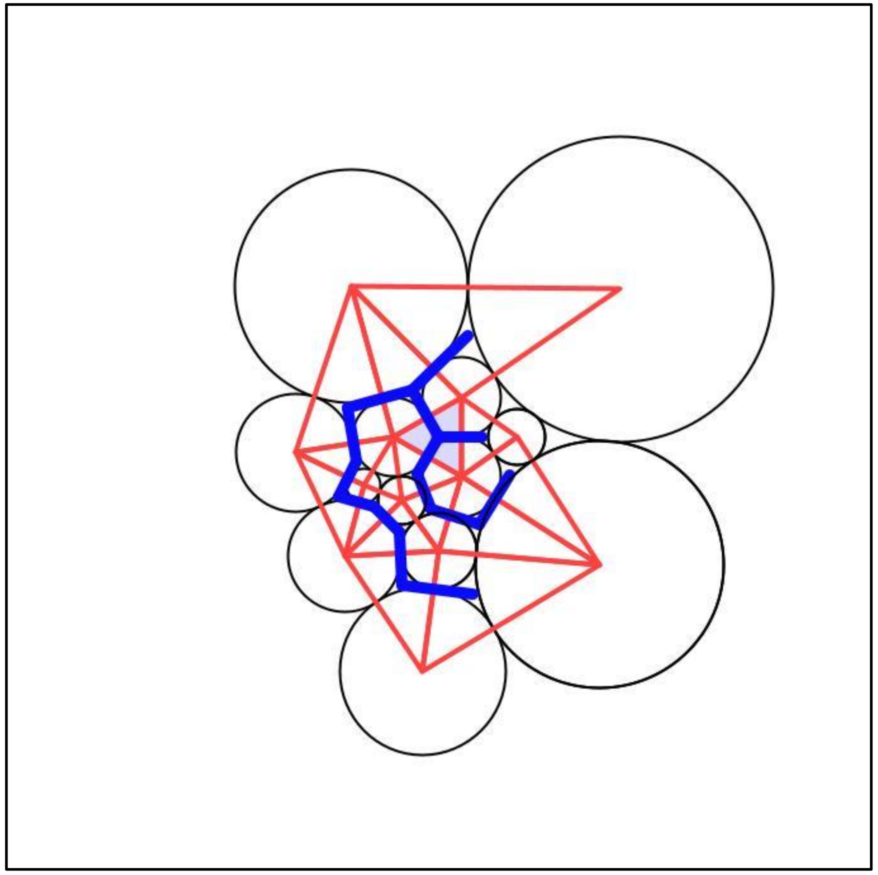
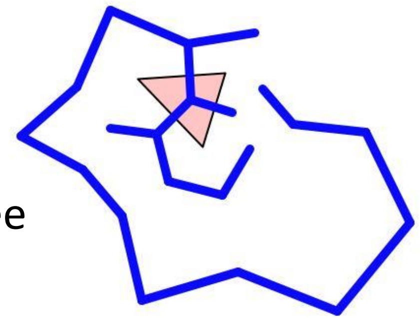
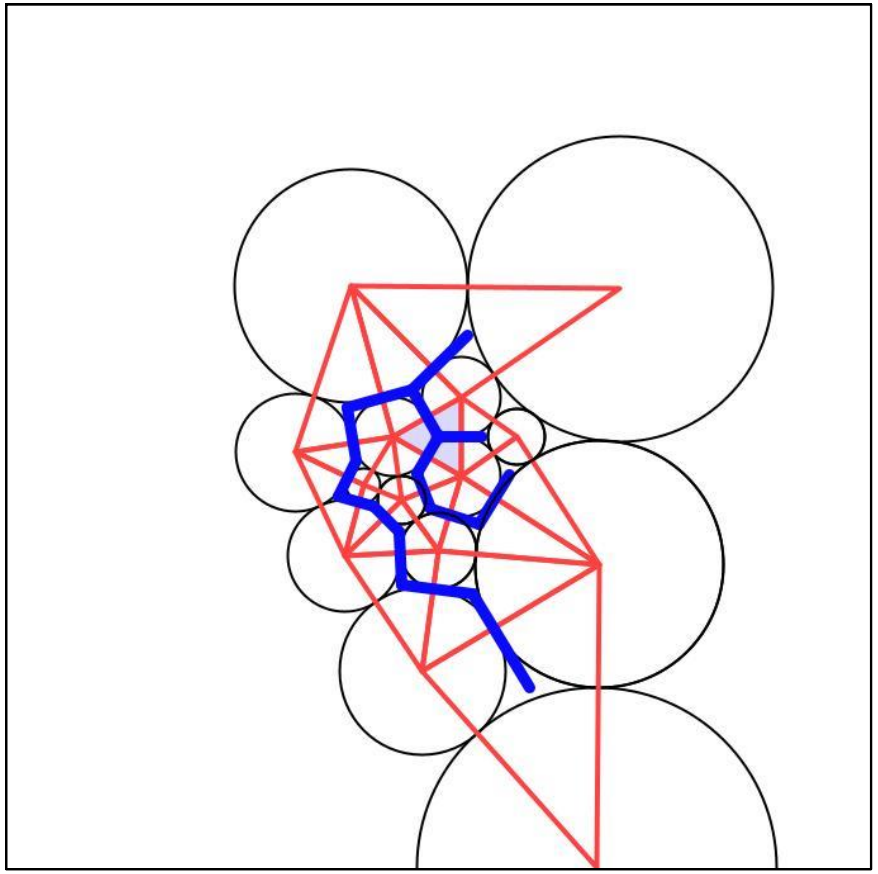


Using known discrete Schwarzians:

Dual spanning tree



On the plane



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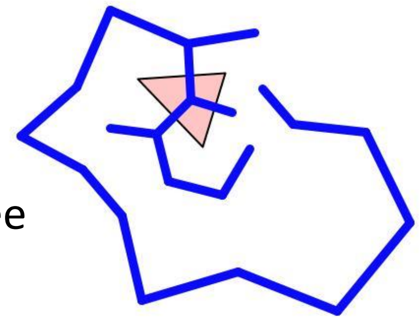

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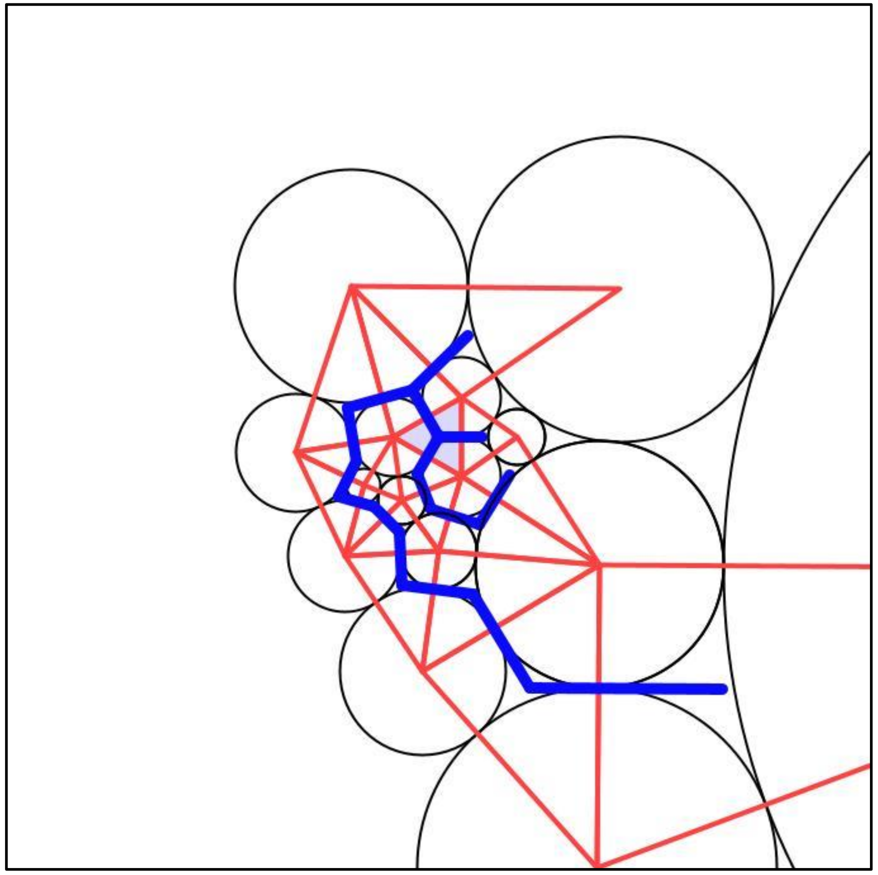

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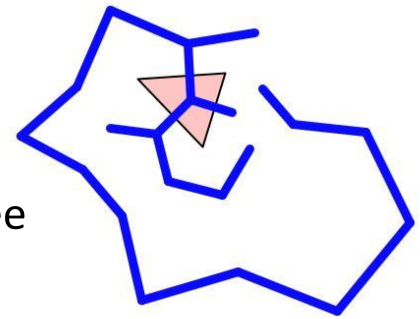

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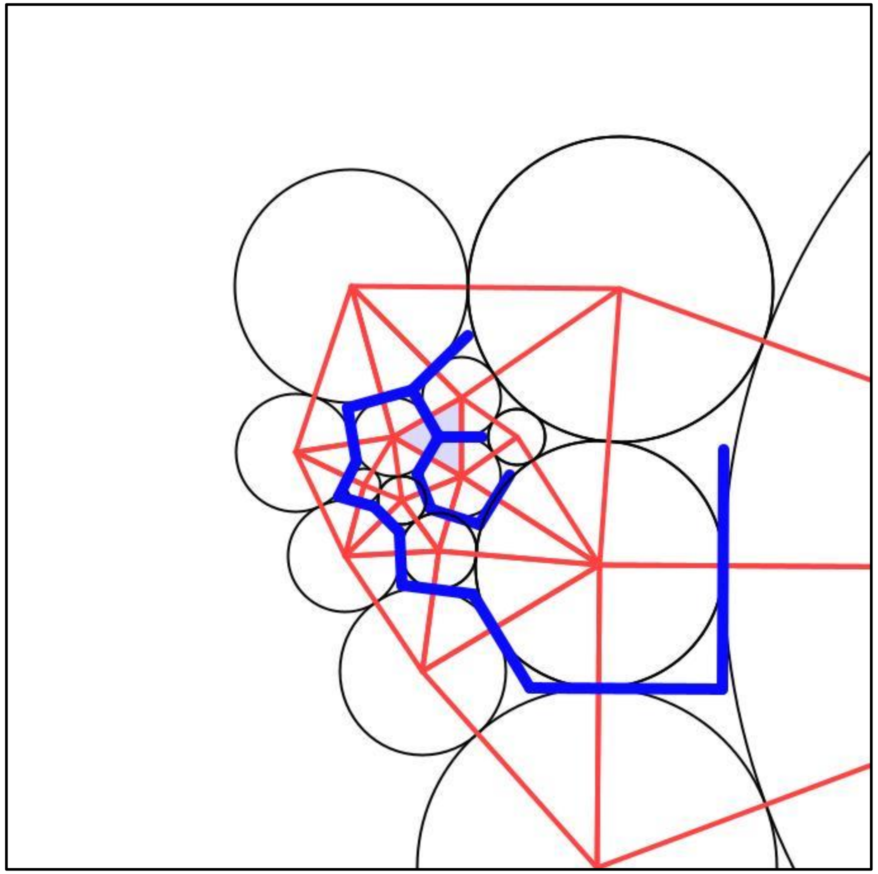
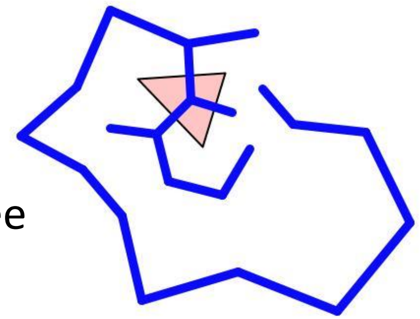


Using known discrete Schwarzians:

Dual spanning tree



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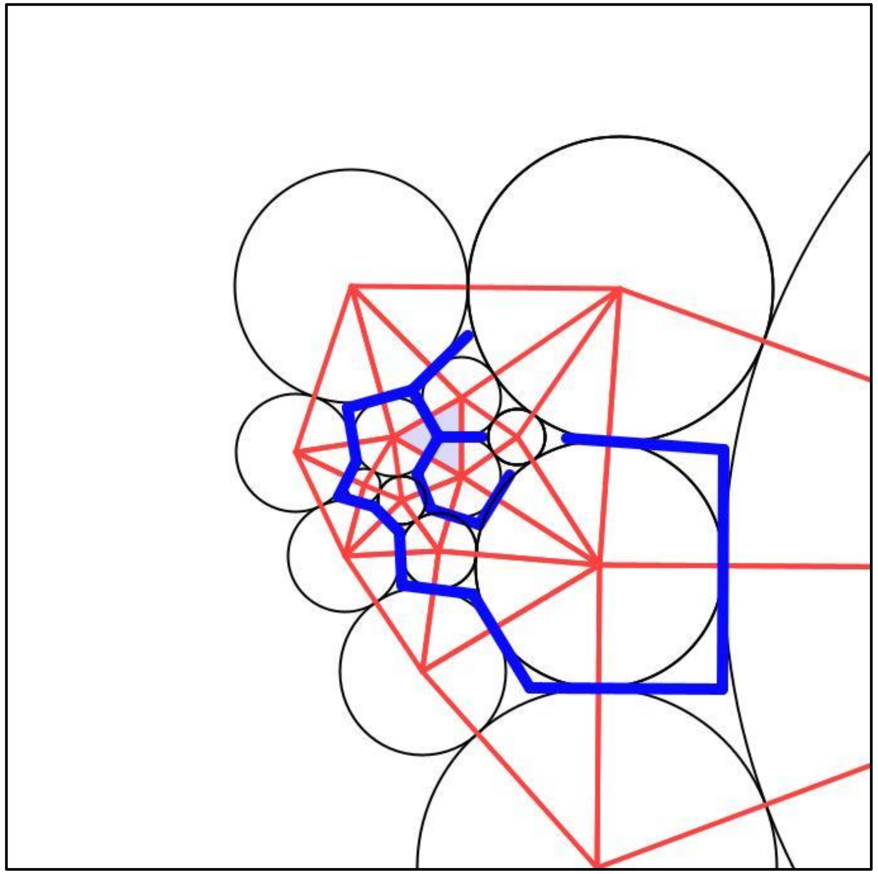
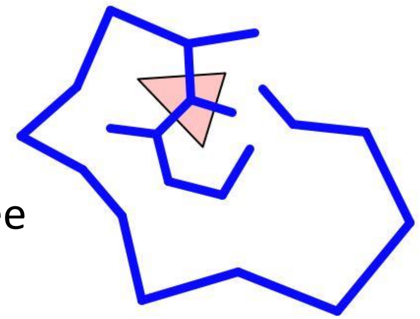


Using known discrete Schwarzians:

Dual spanning tree



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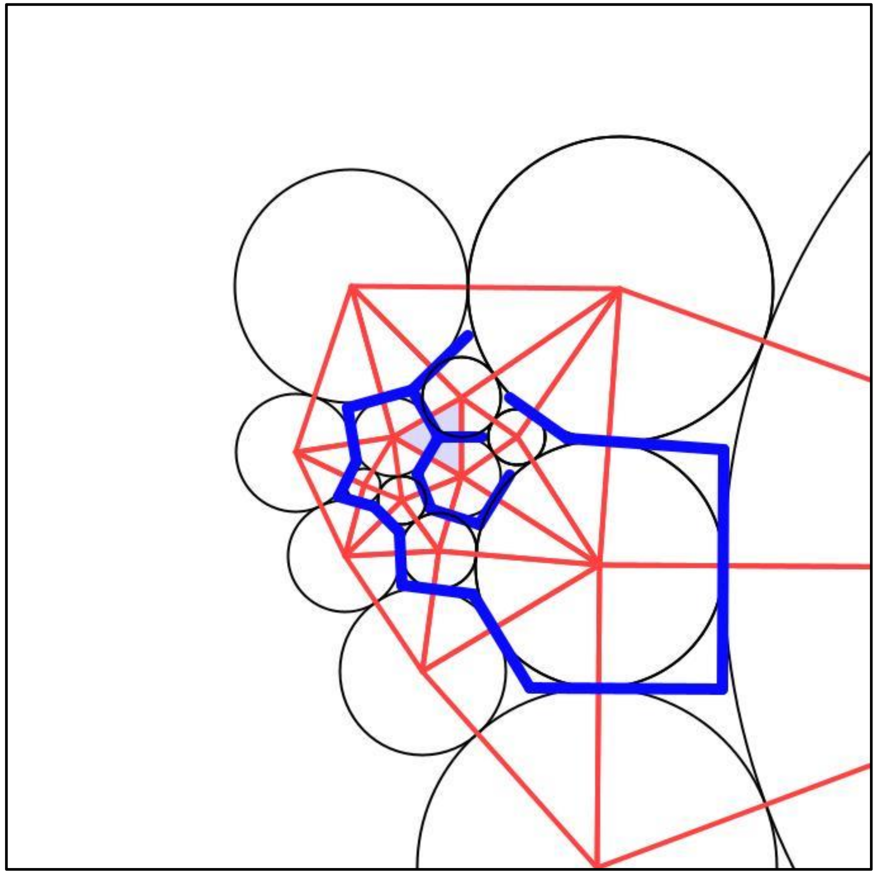

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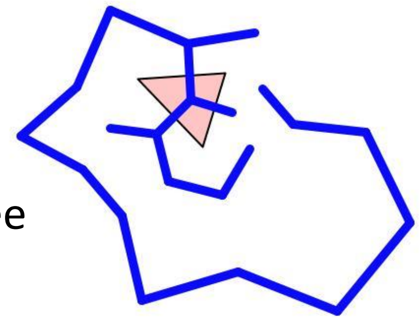
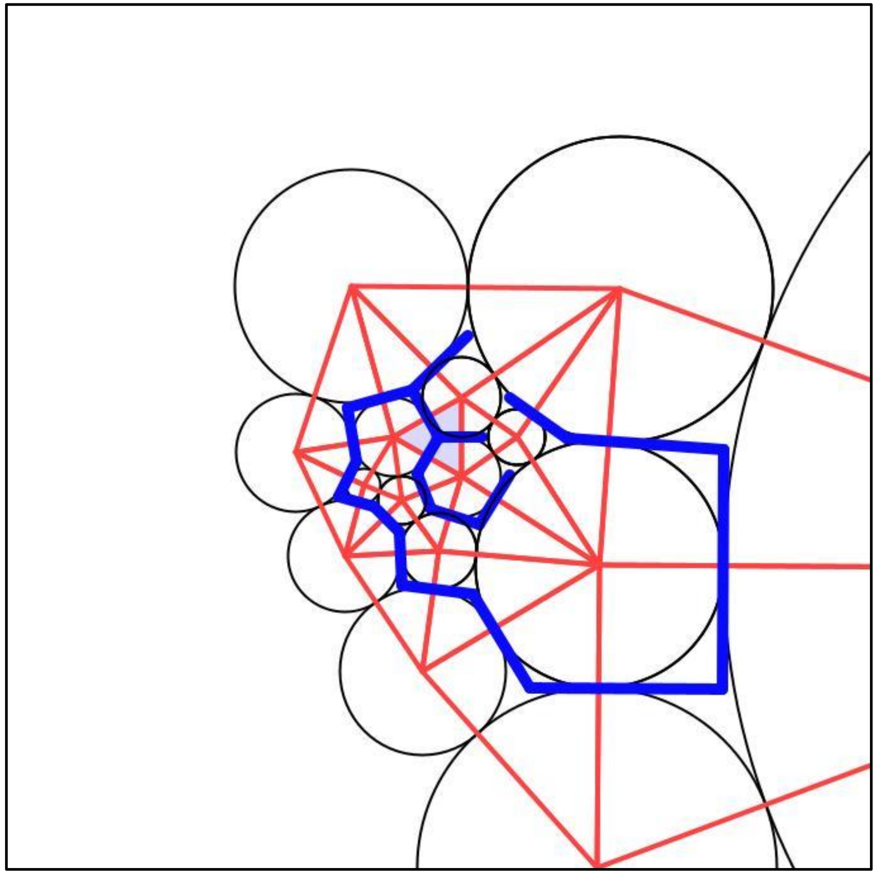


Dual spanning tree

On the plane



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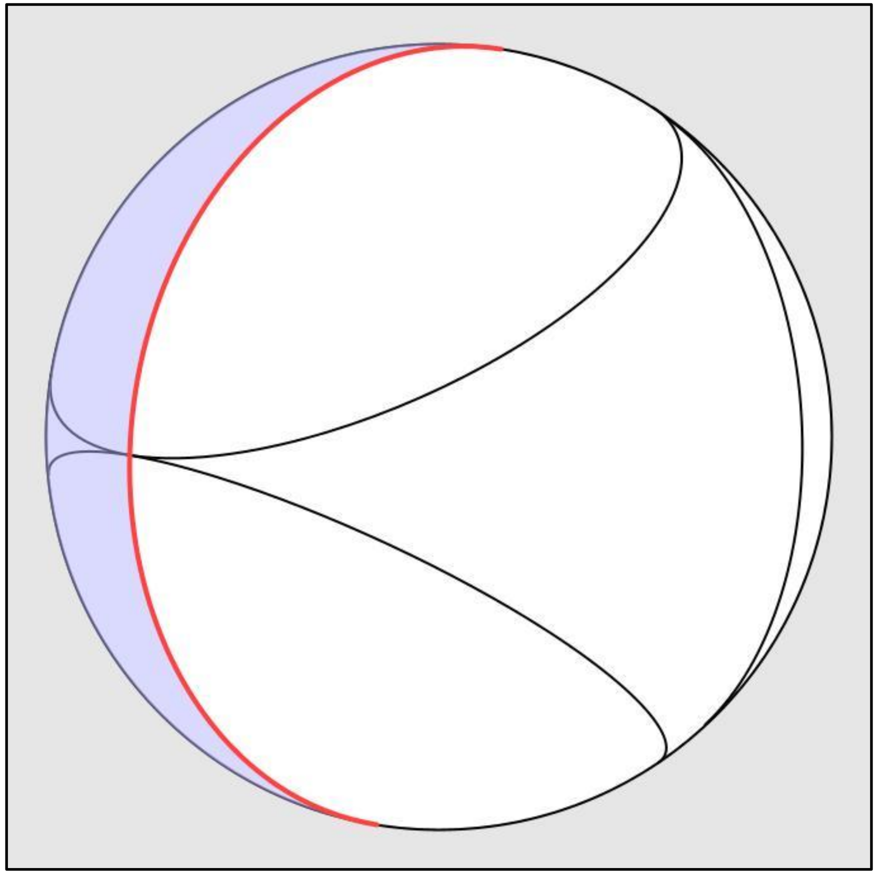
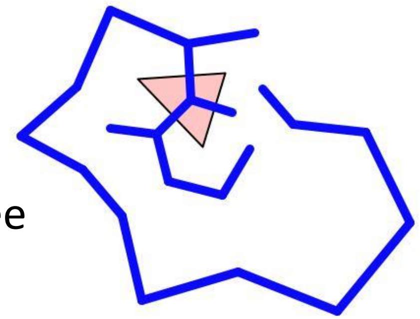


Using known discrete Schwarzians:

Dual spanning tree



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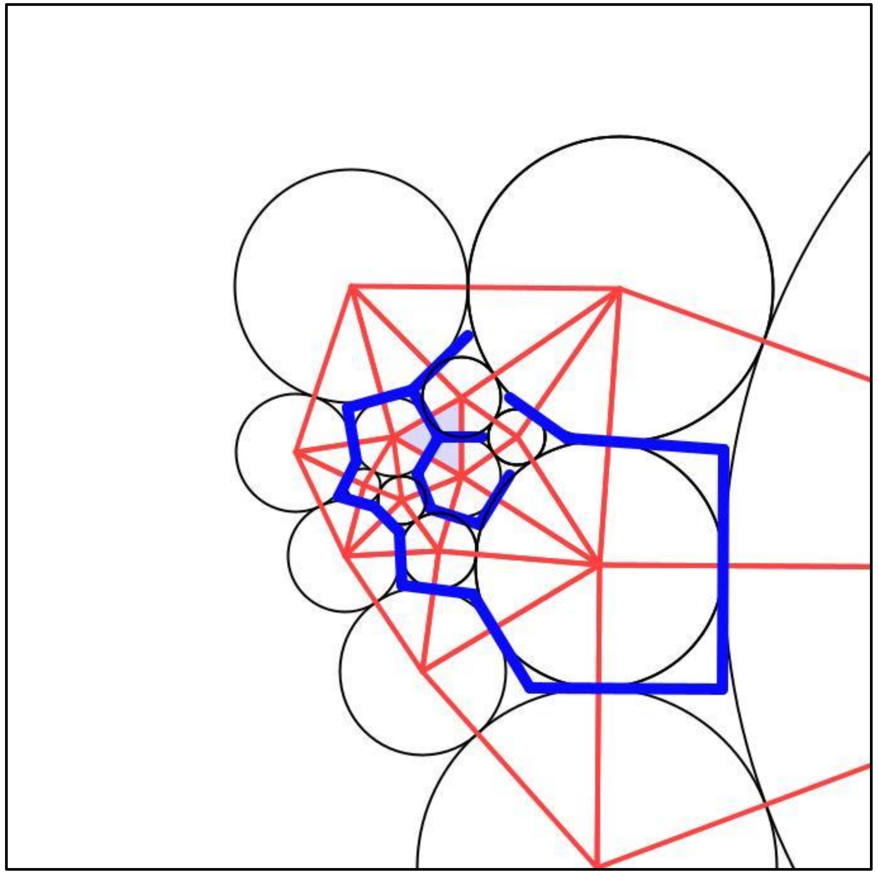

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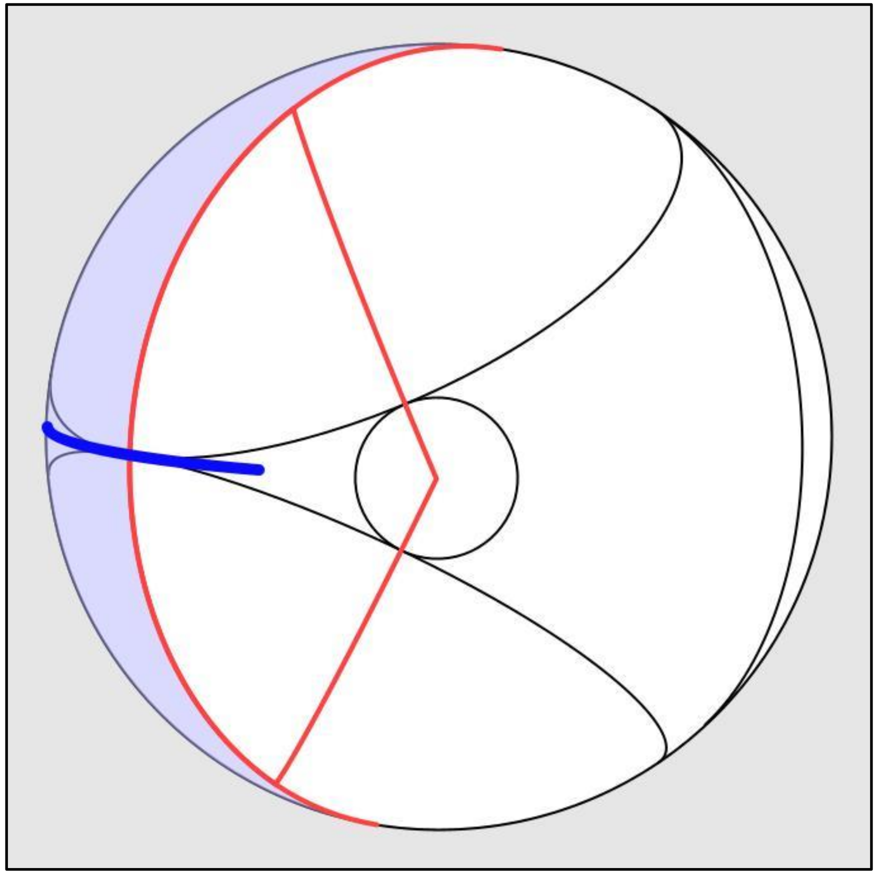

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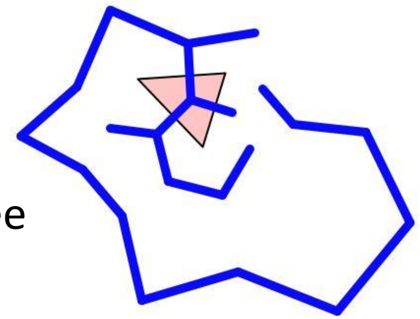
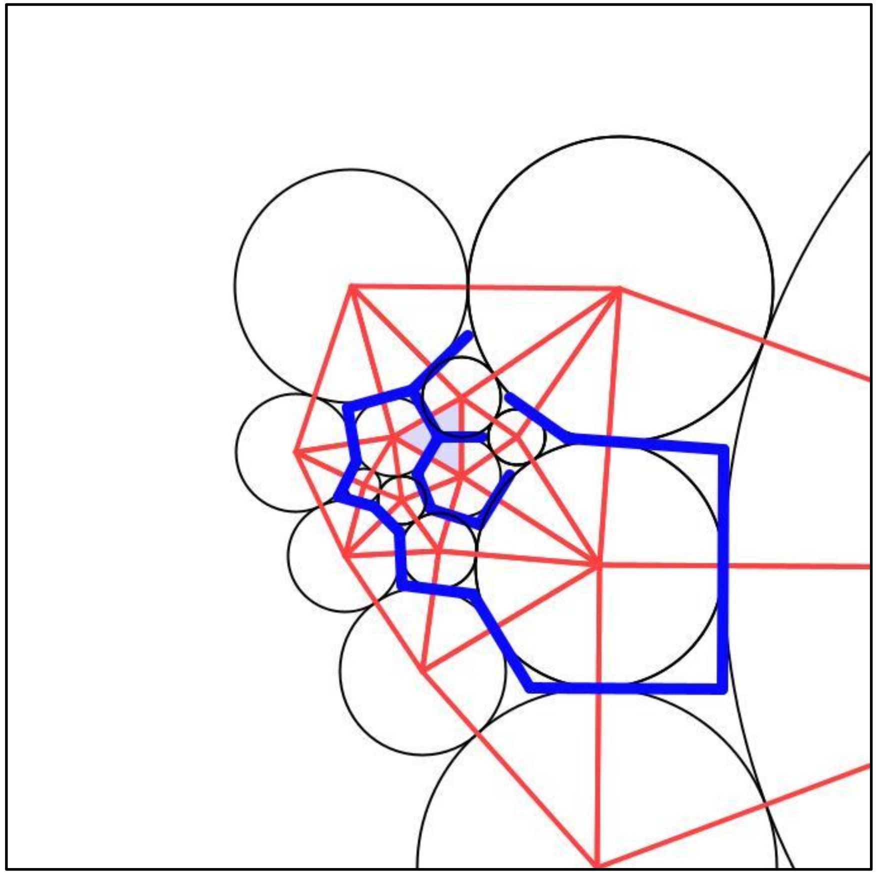


Dual spanning tree

On the plane



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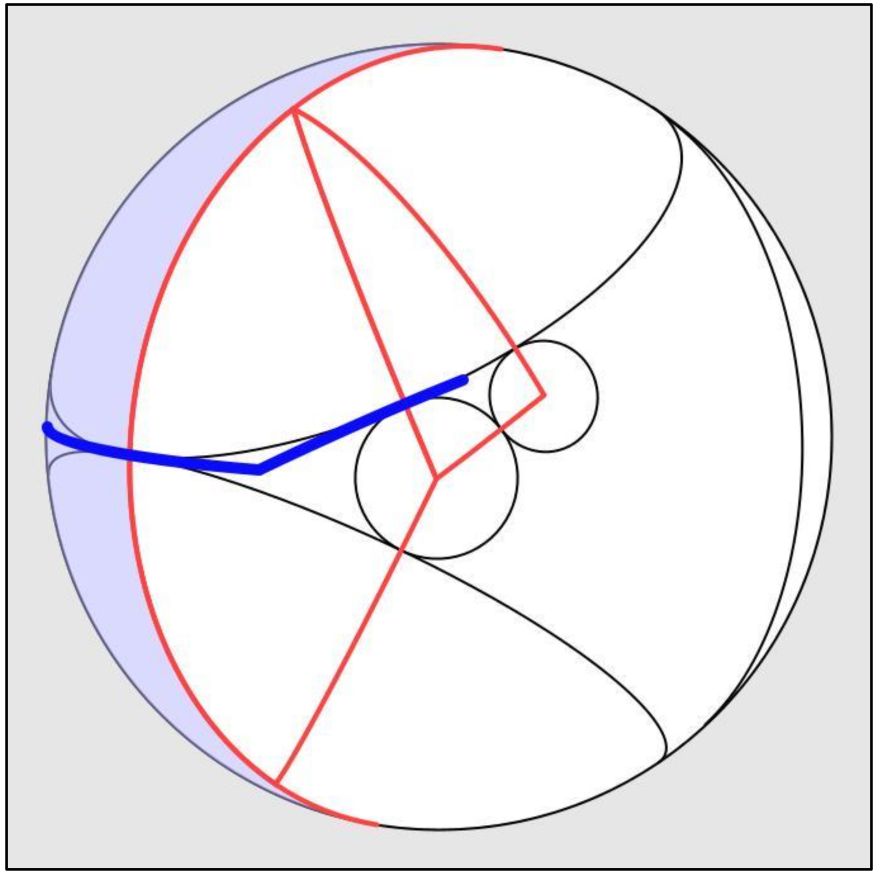

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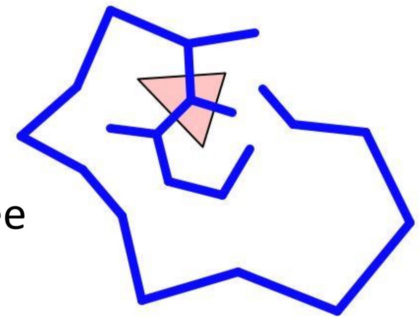
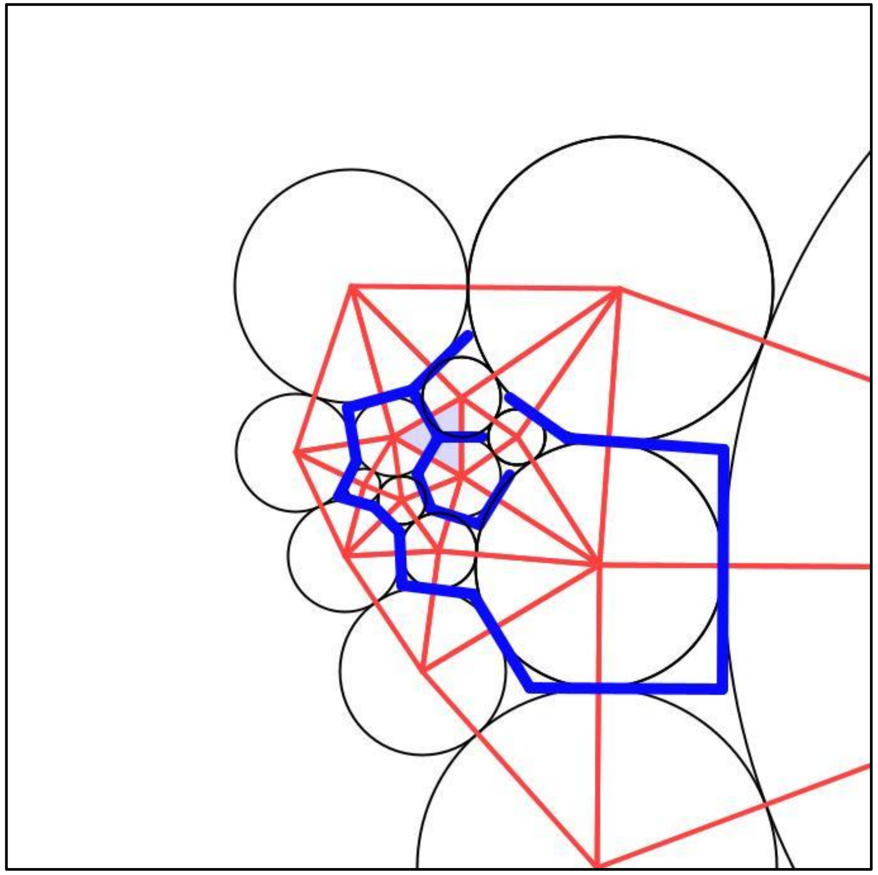
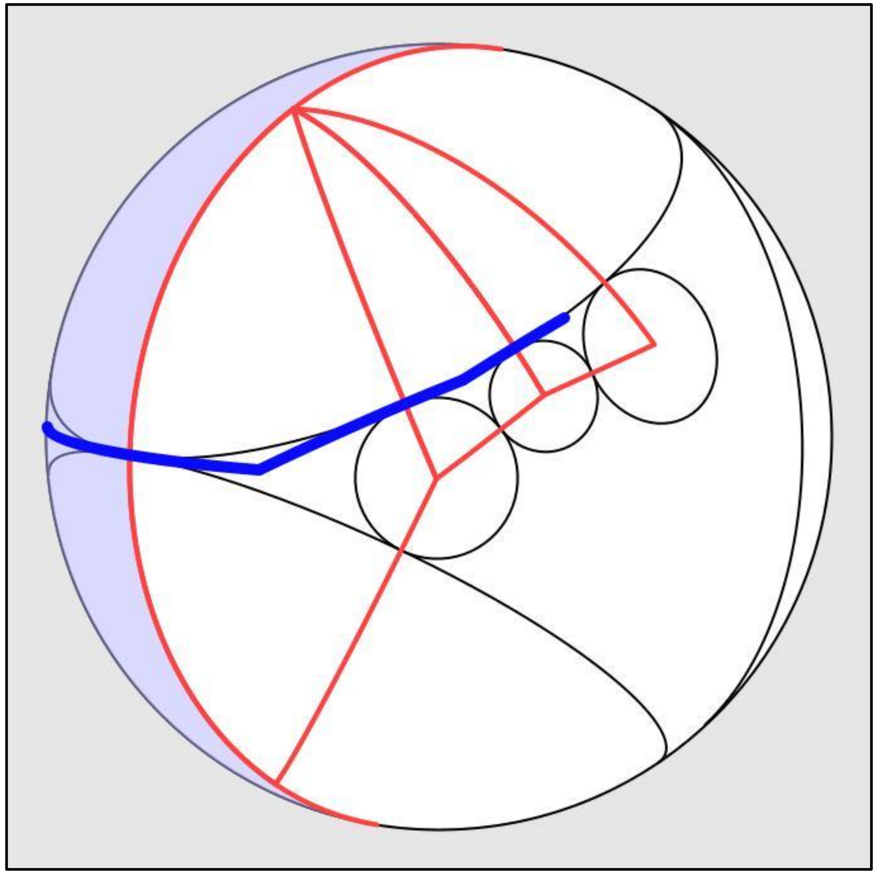


Dual spanning tree

On the plane

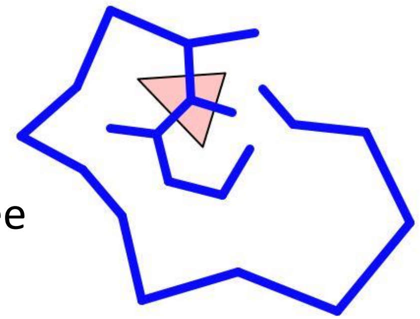


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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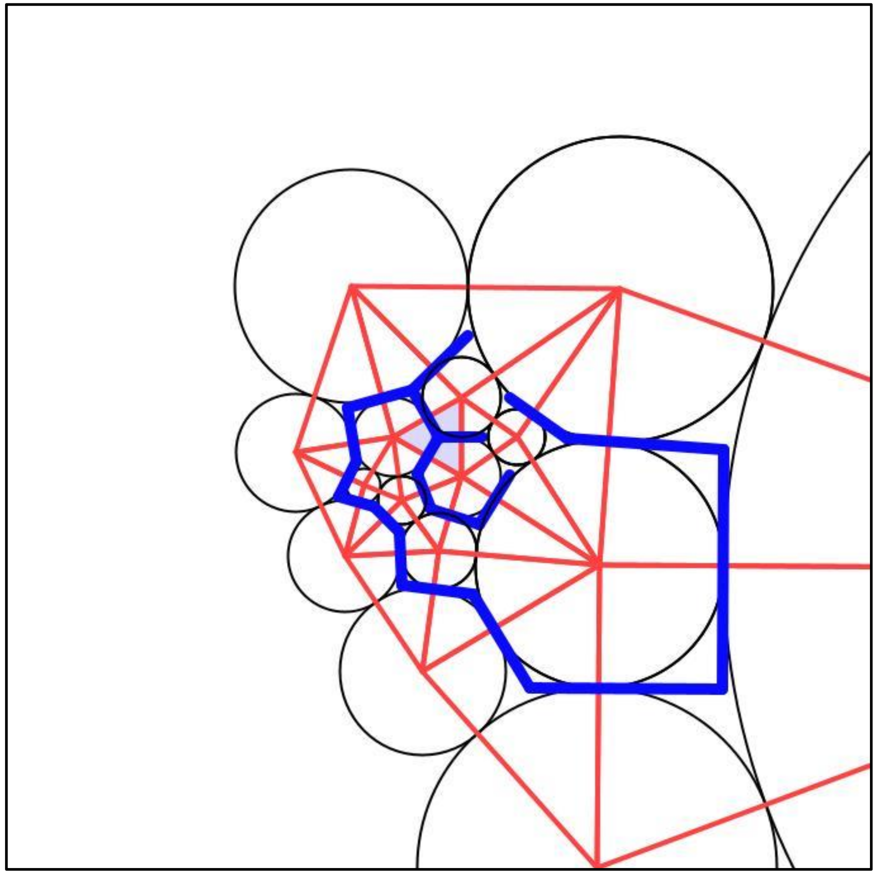
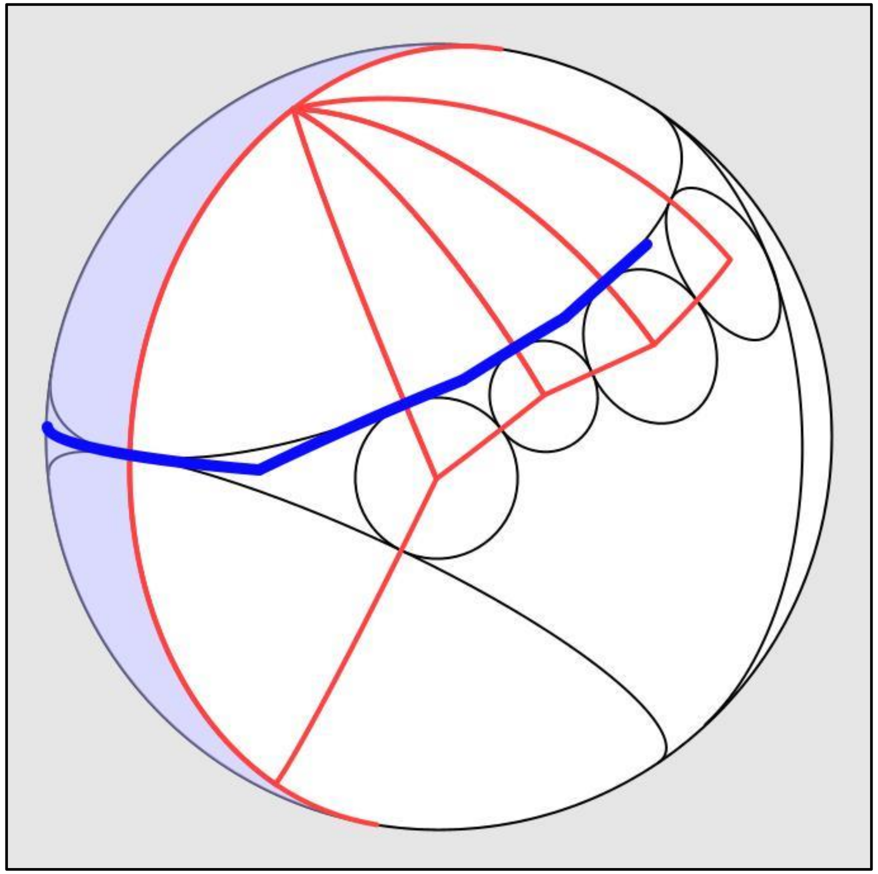
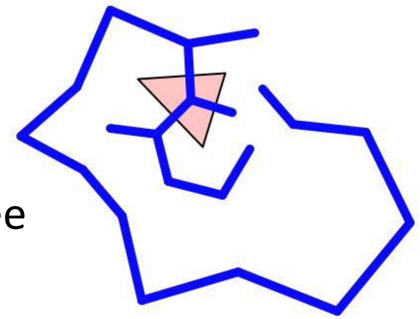


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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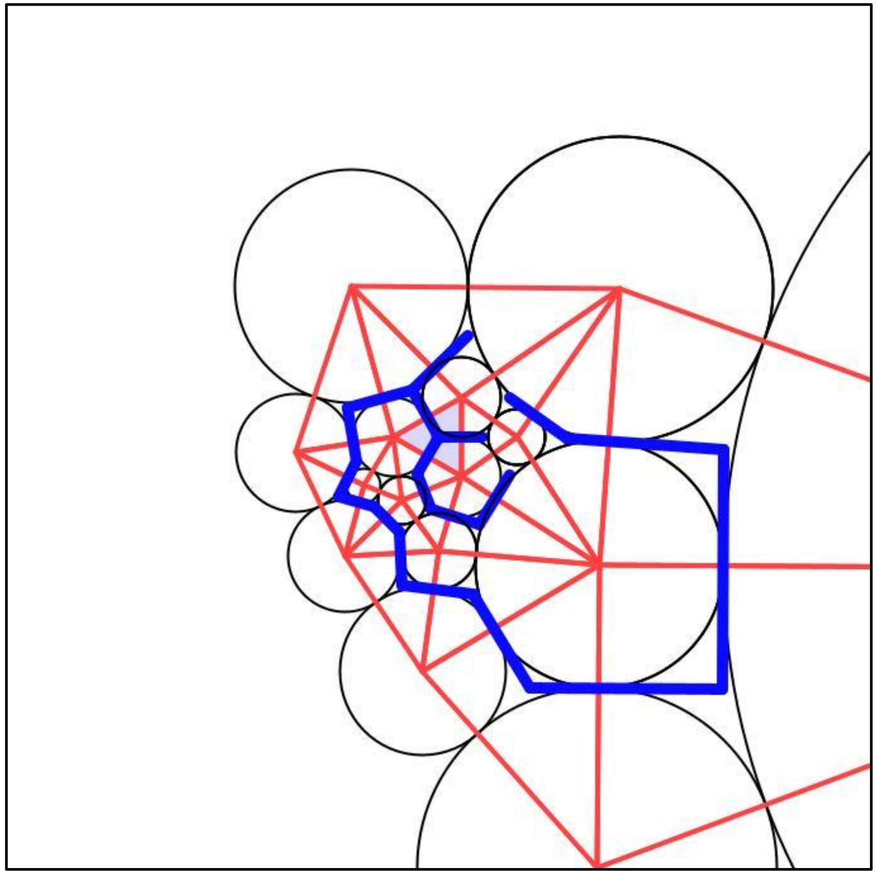
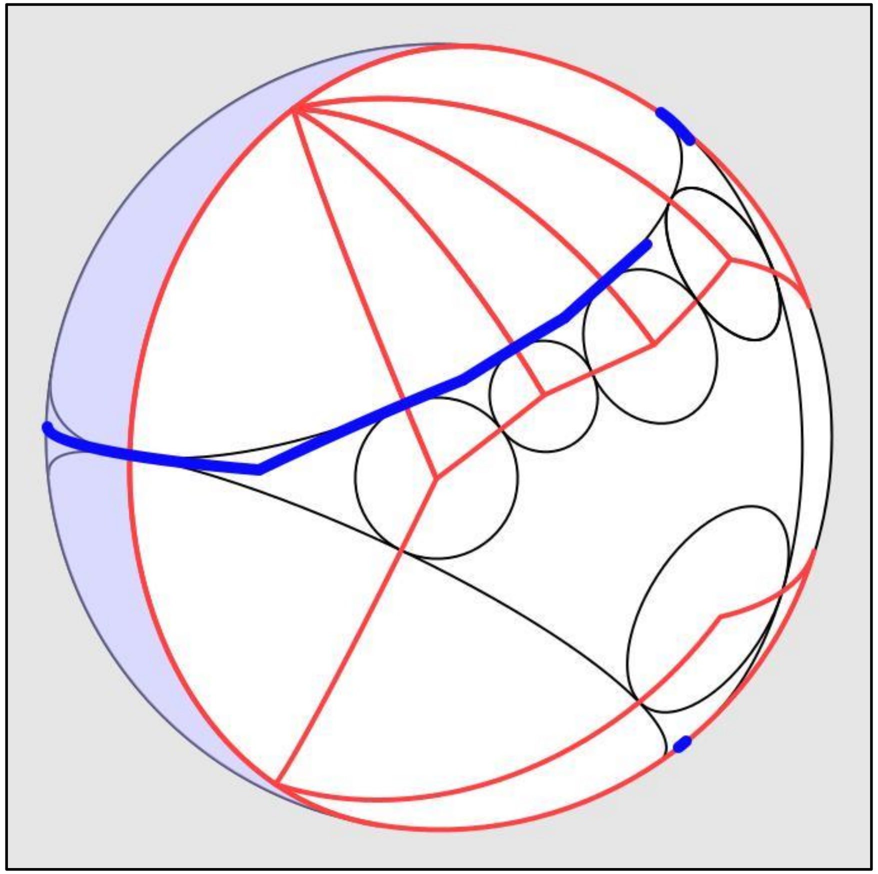
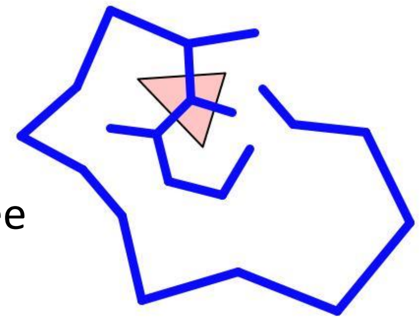


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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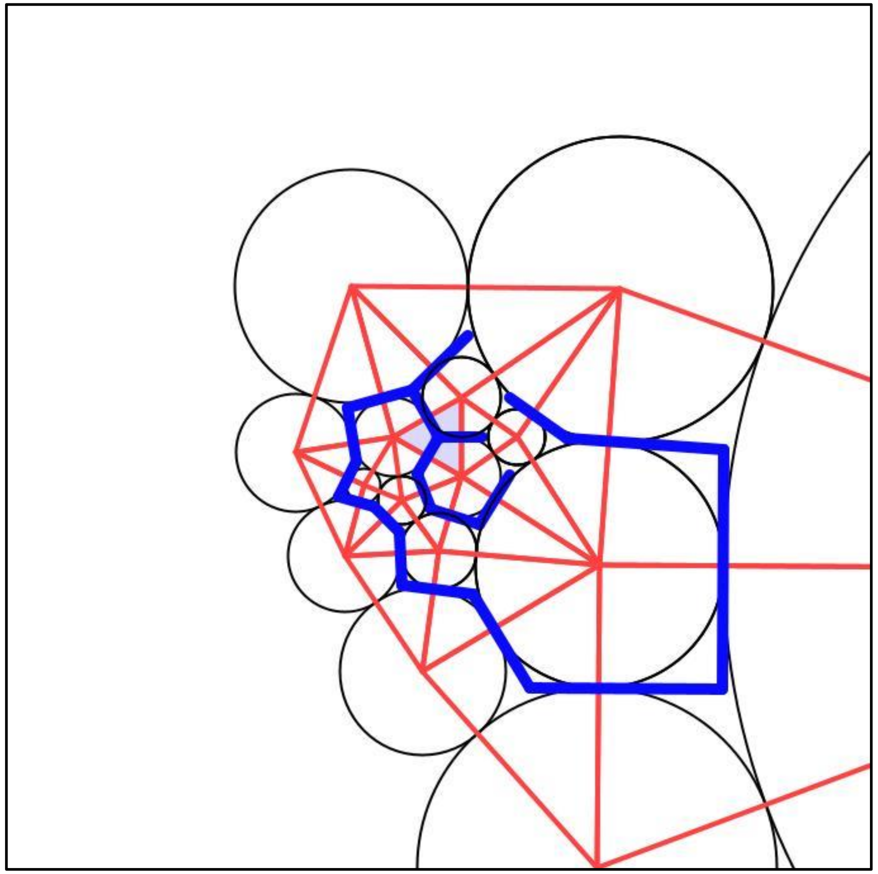

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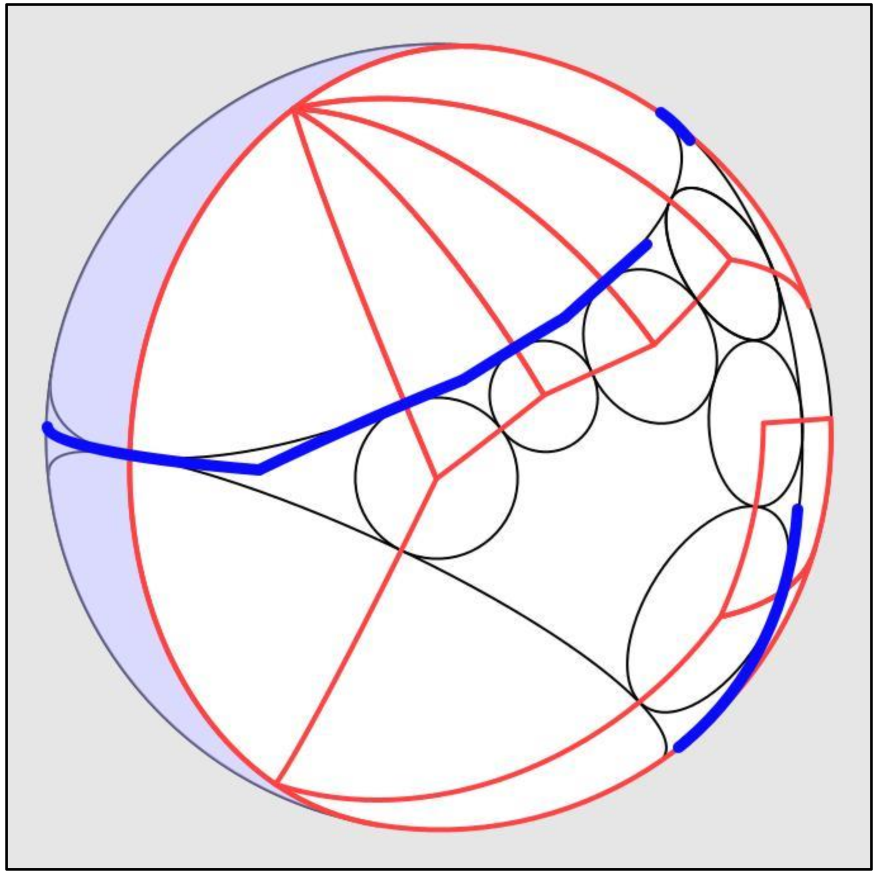

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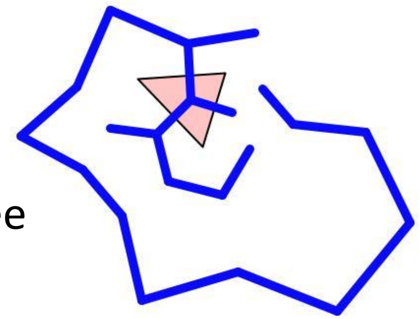
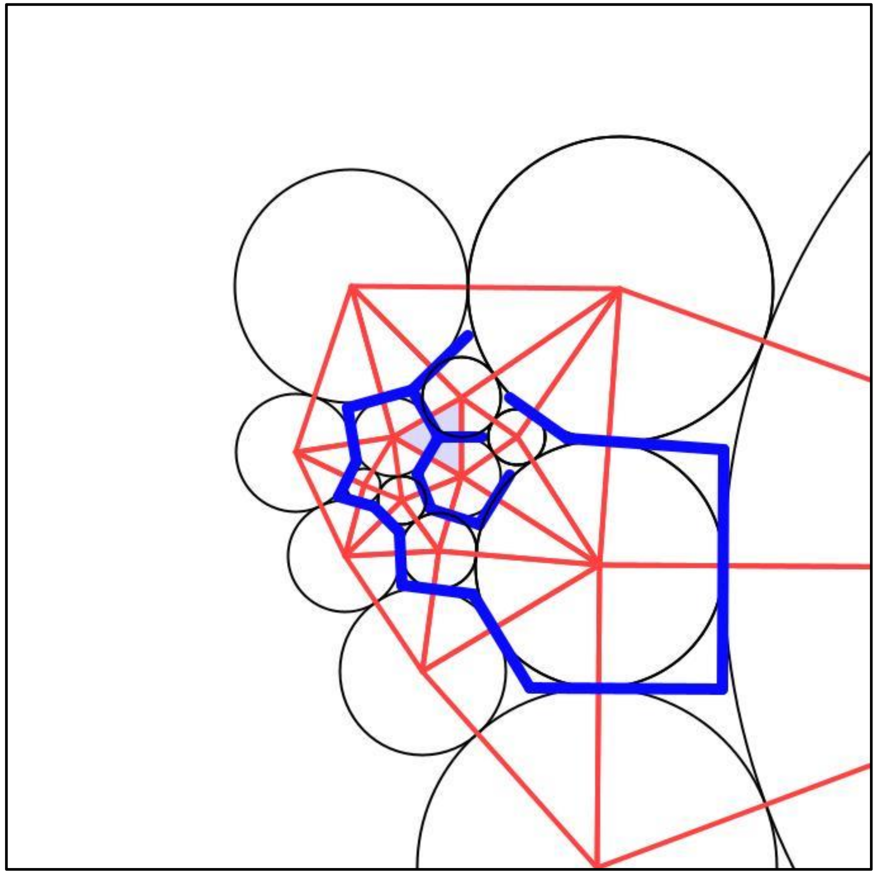


Dual spanning tree

On the plane



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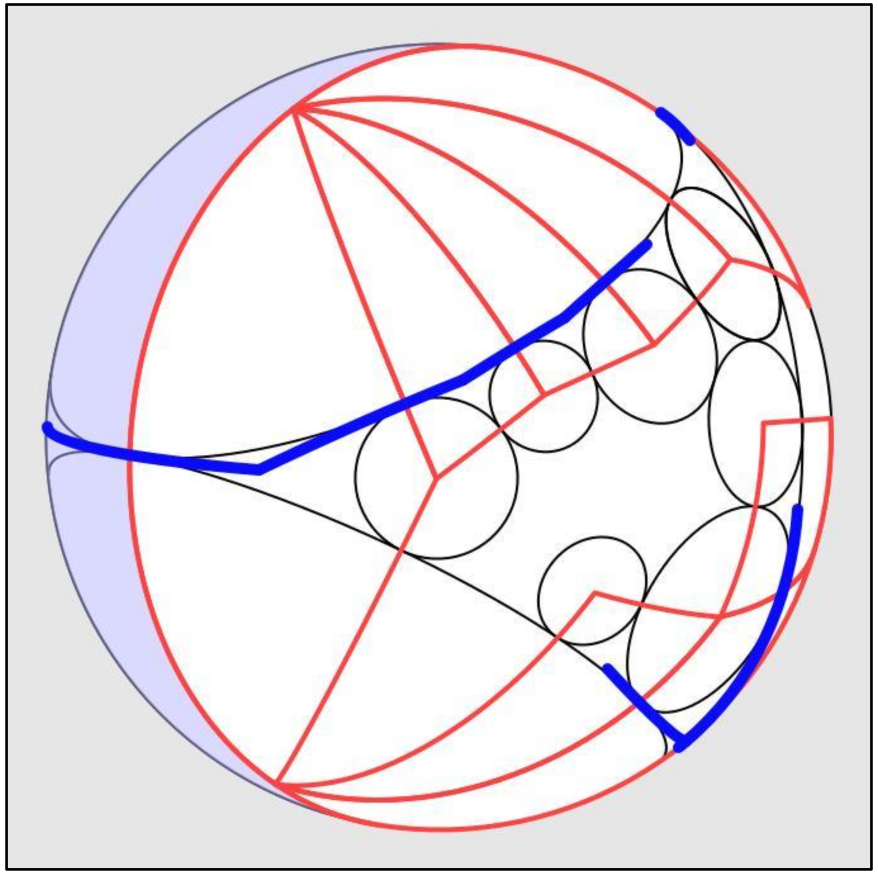
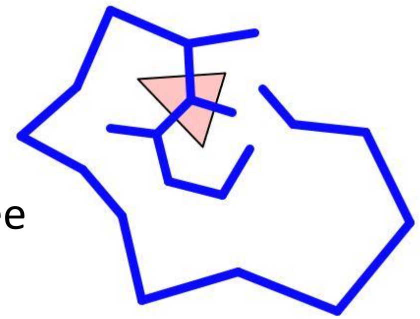


Using known discrete Schwarzians:

Dual spanning tree



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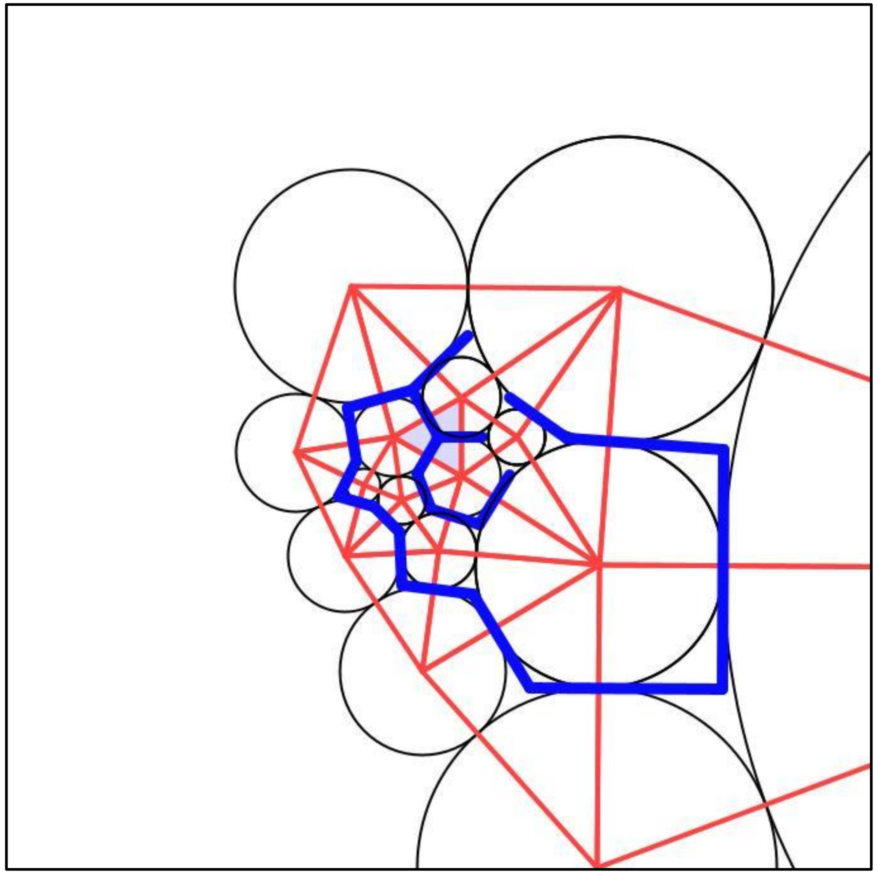
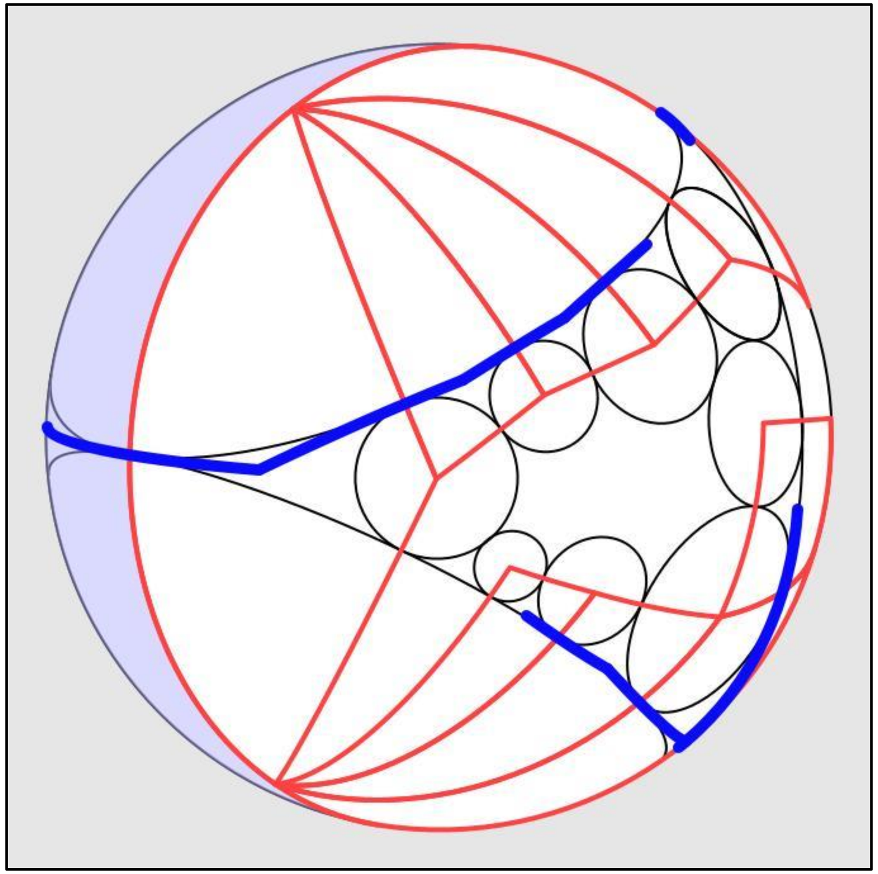


On the sphere



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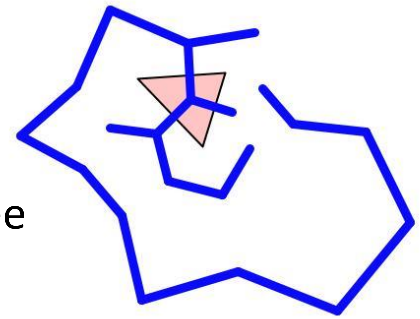
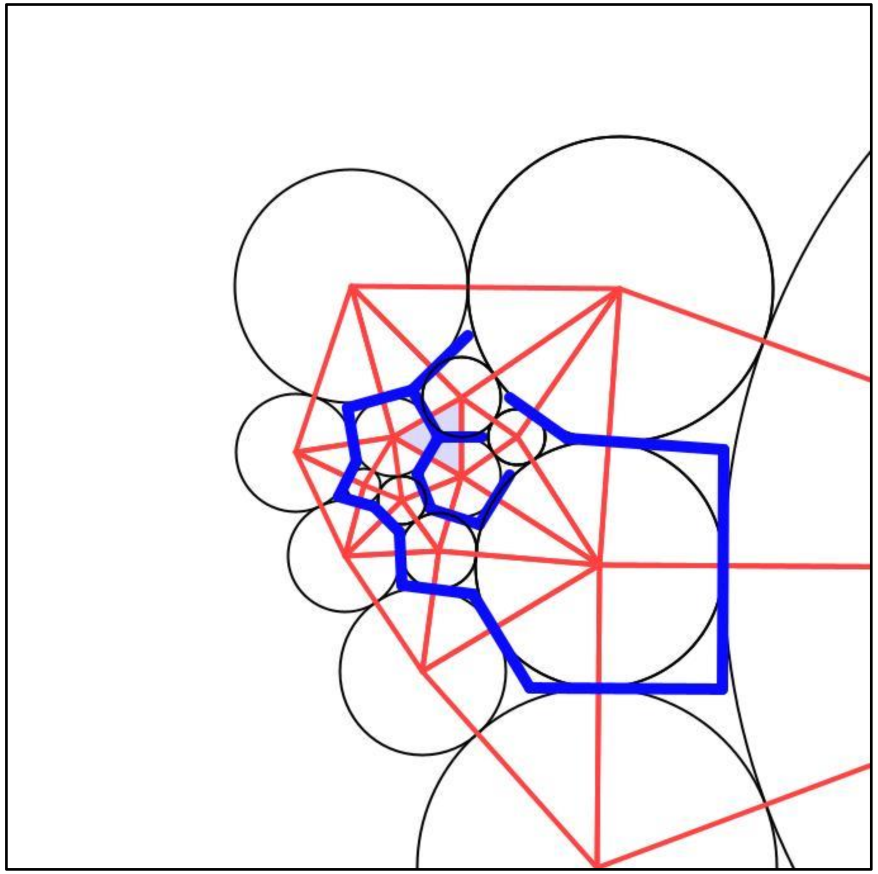
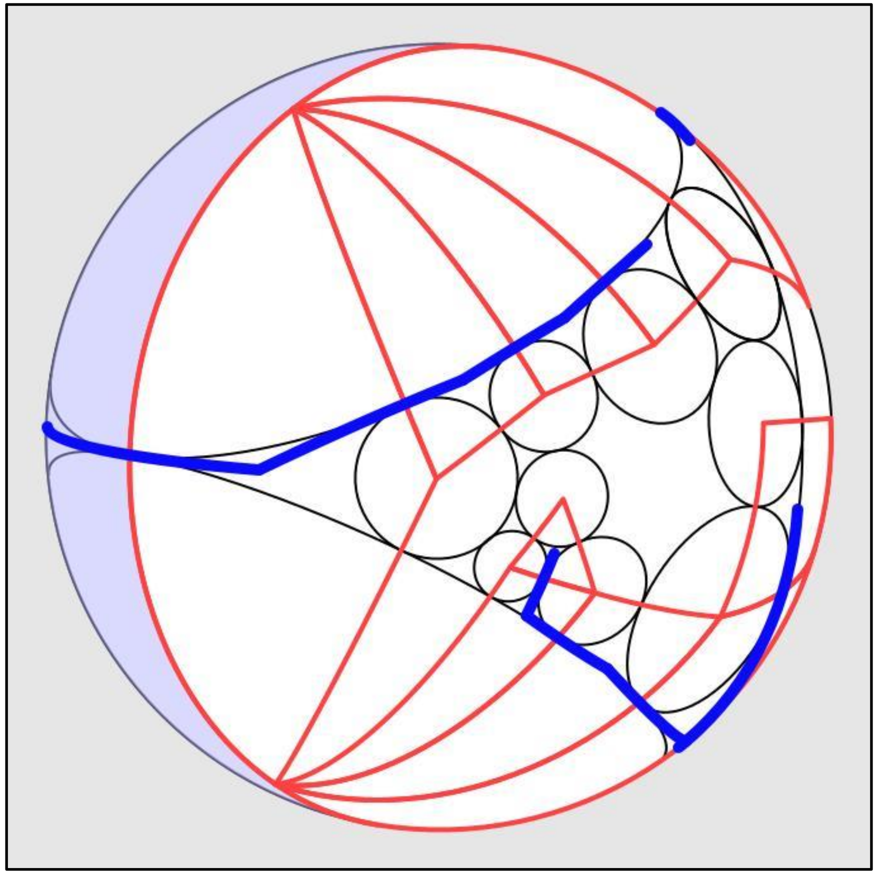


Dual spanning tree

On the plane

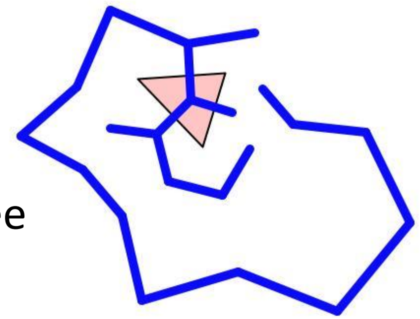


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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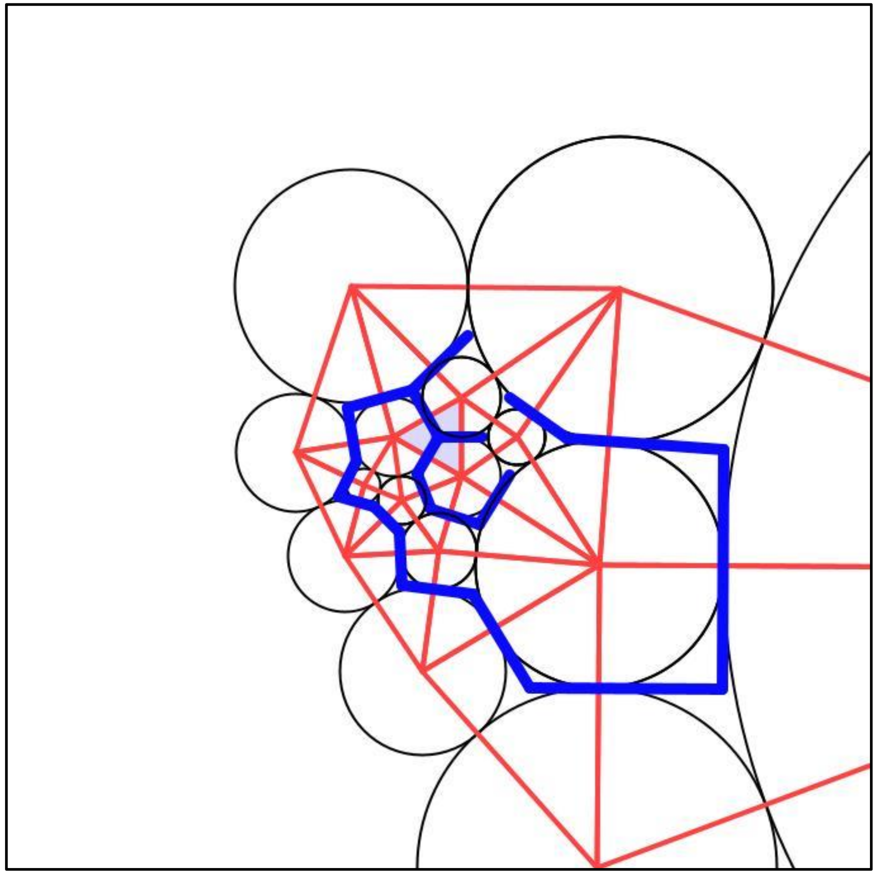

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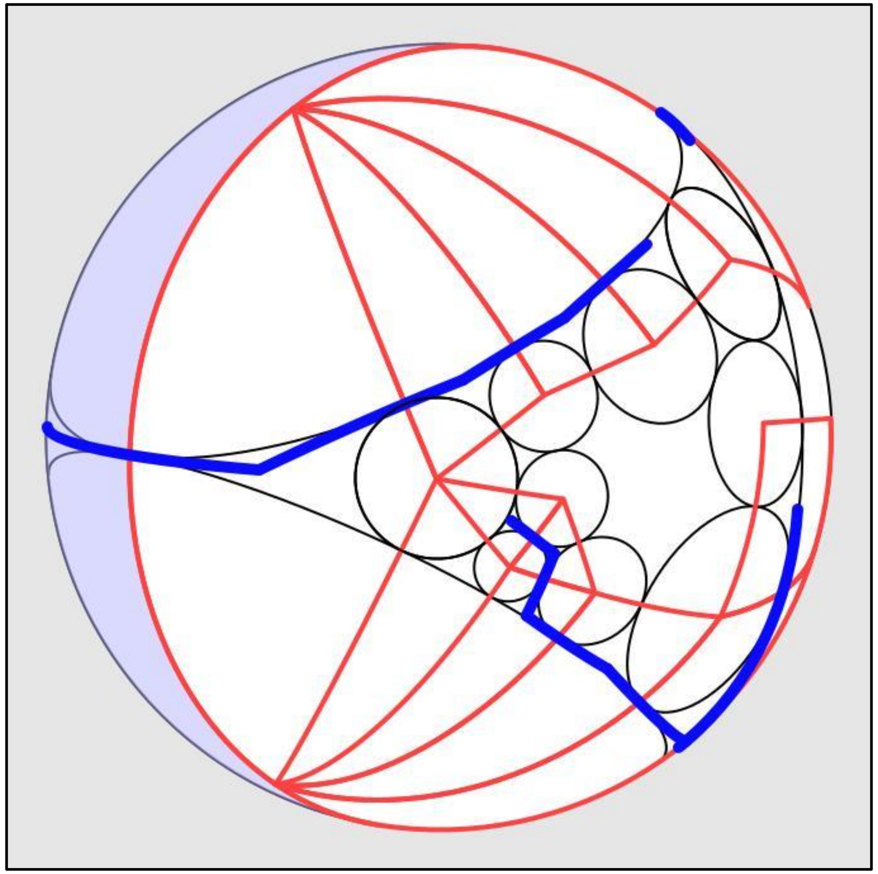

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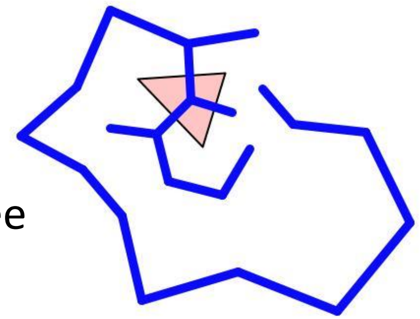

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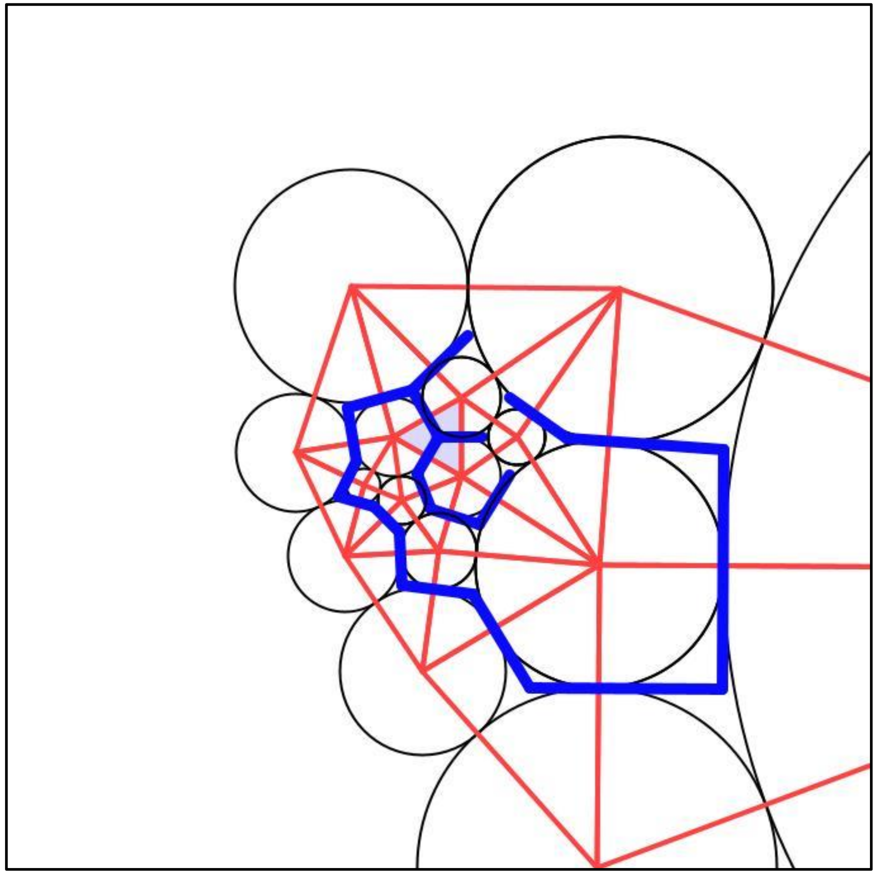

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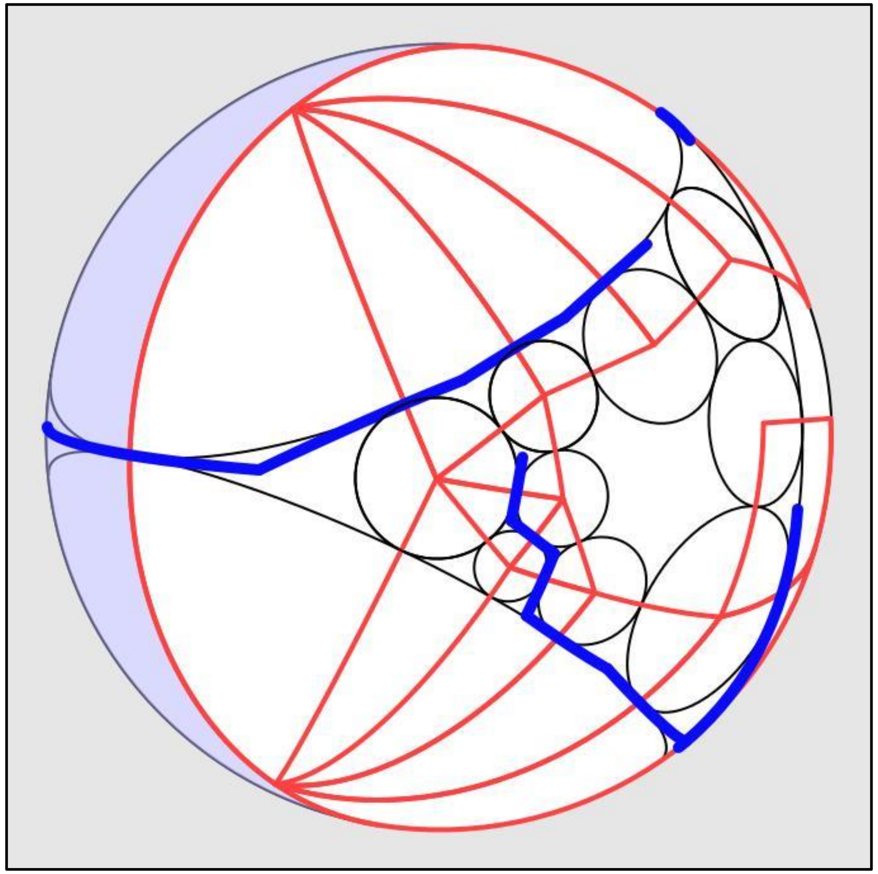
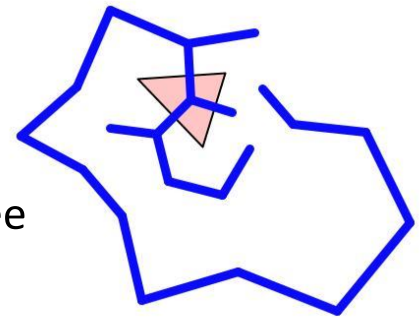


Using known discrete Schwarzians:

Dual spanning tree



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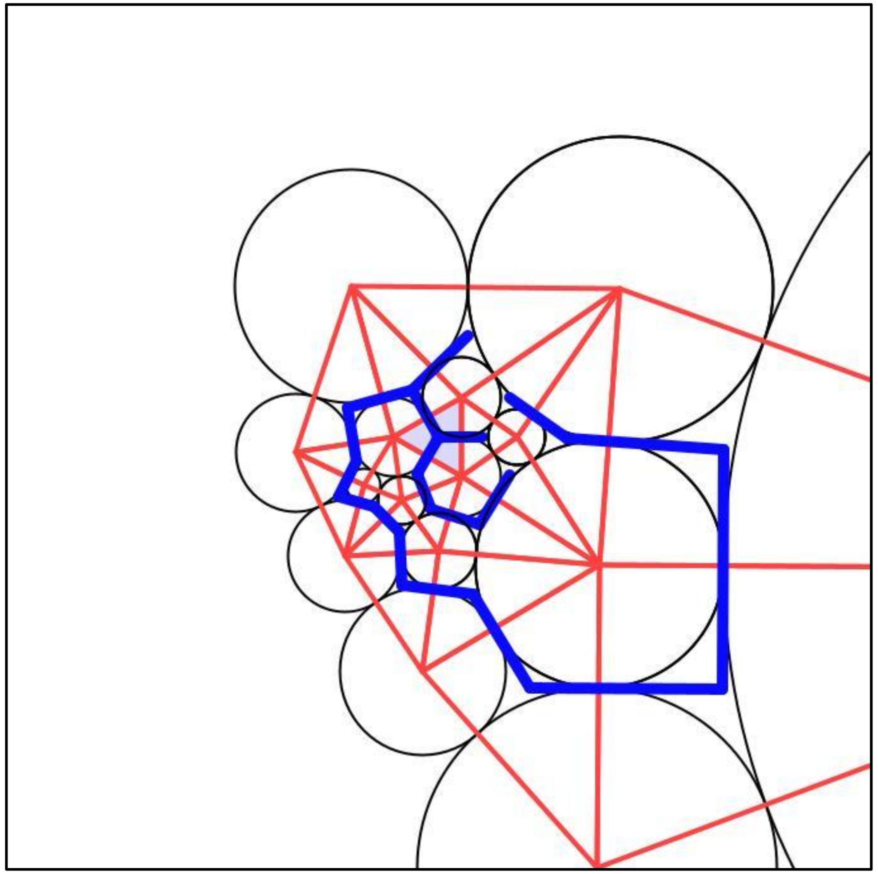
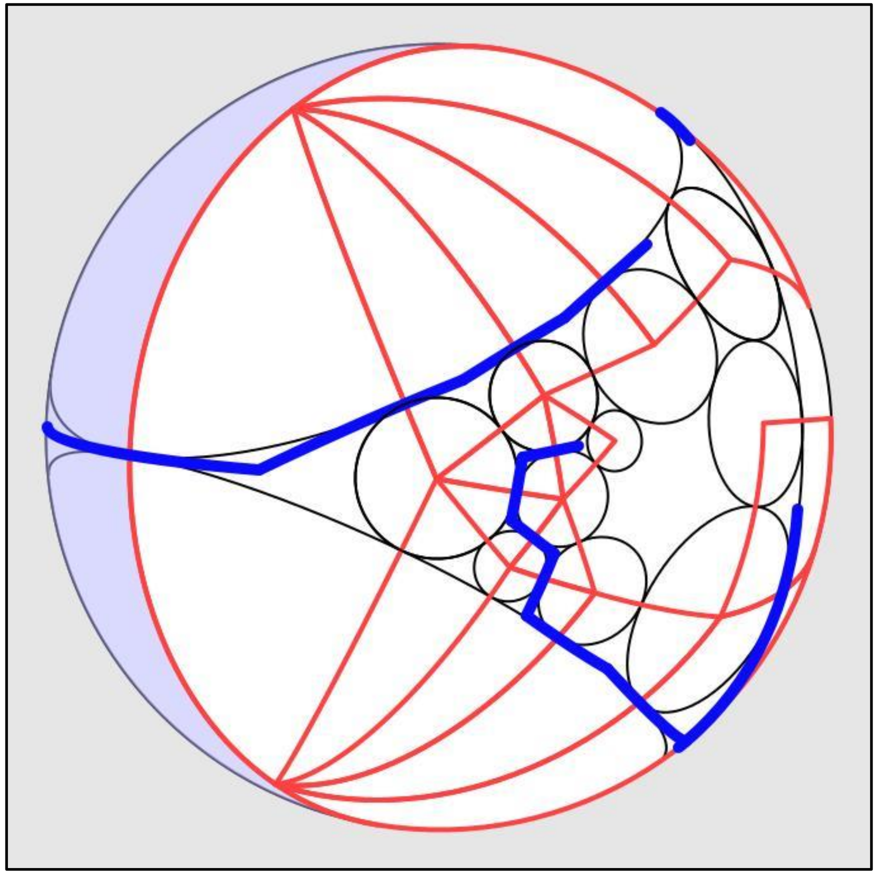
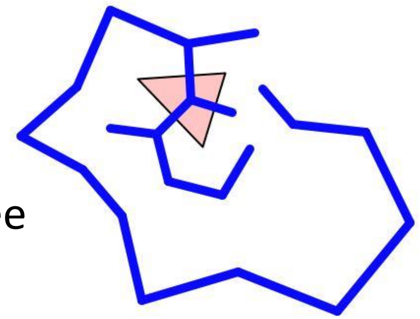


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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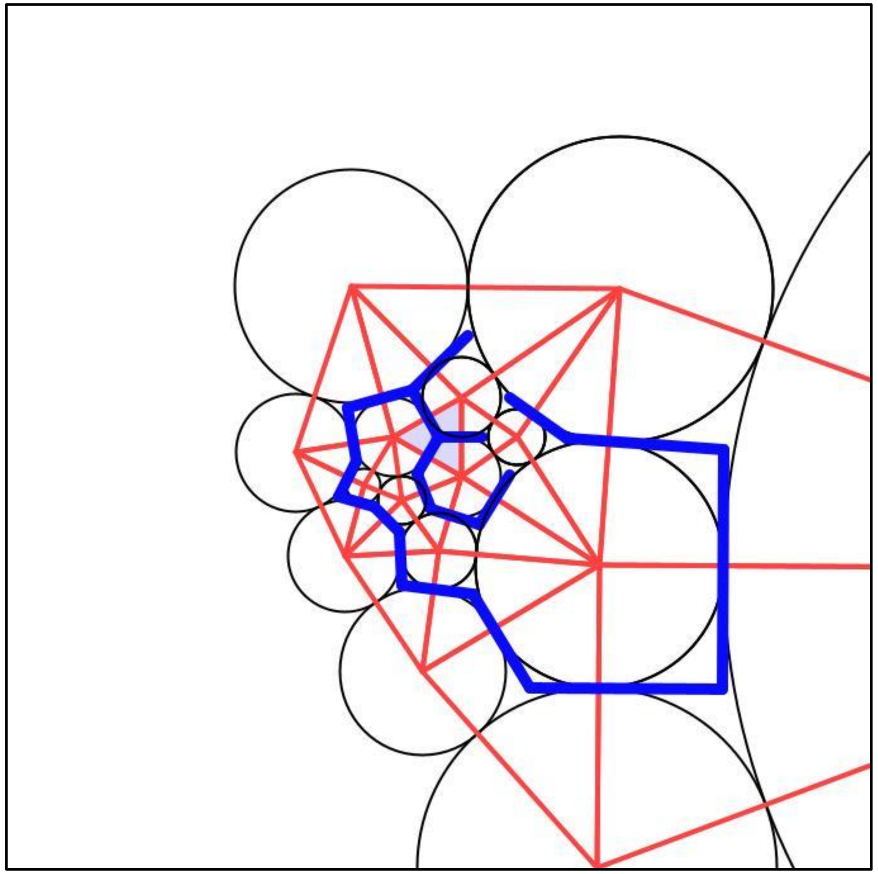

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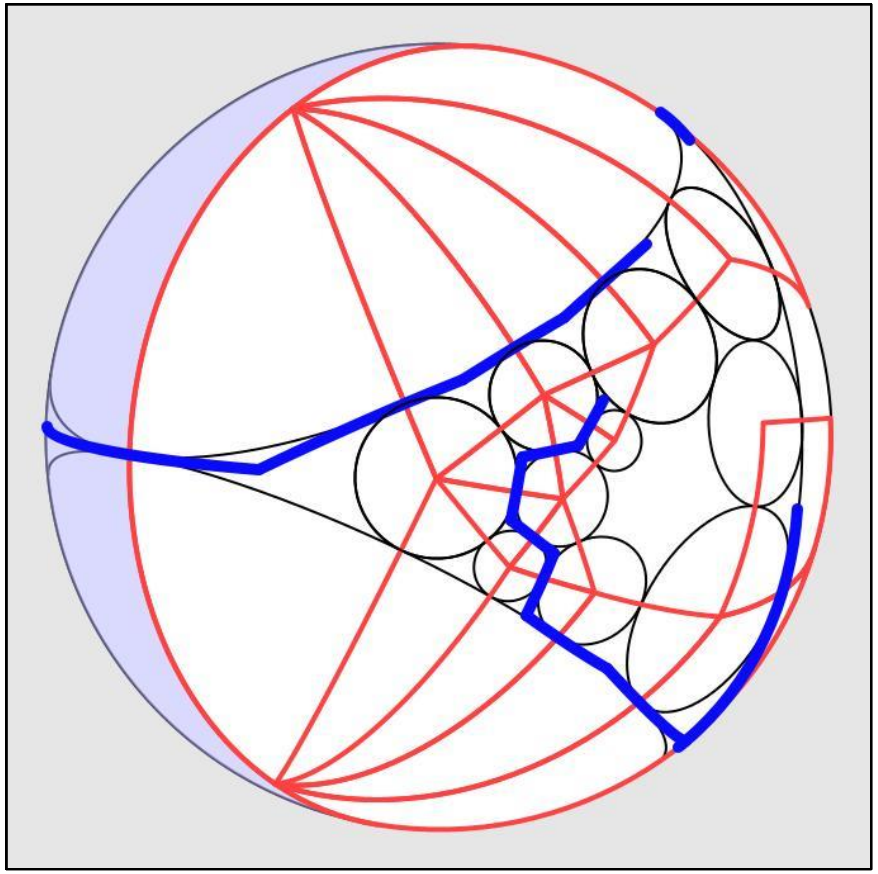

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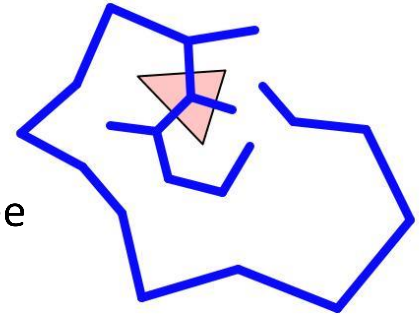

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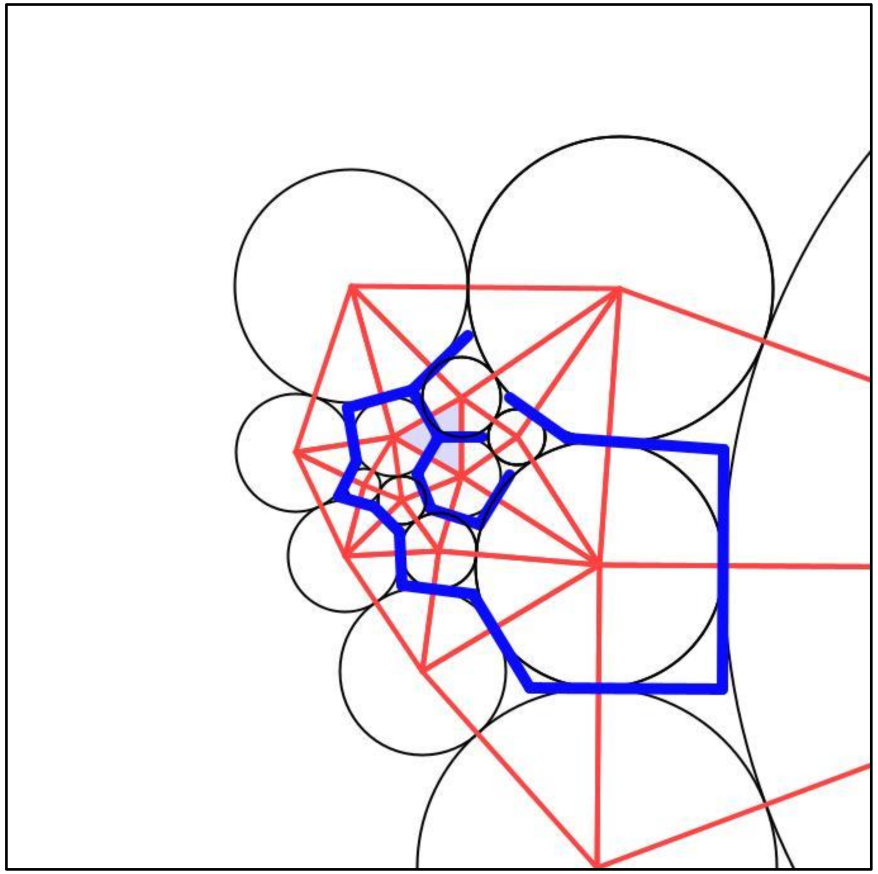

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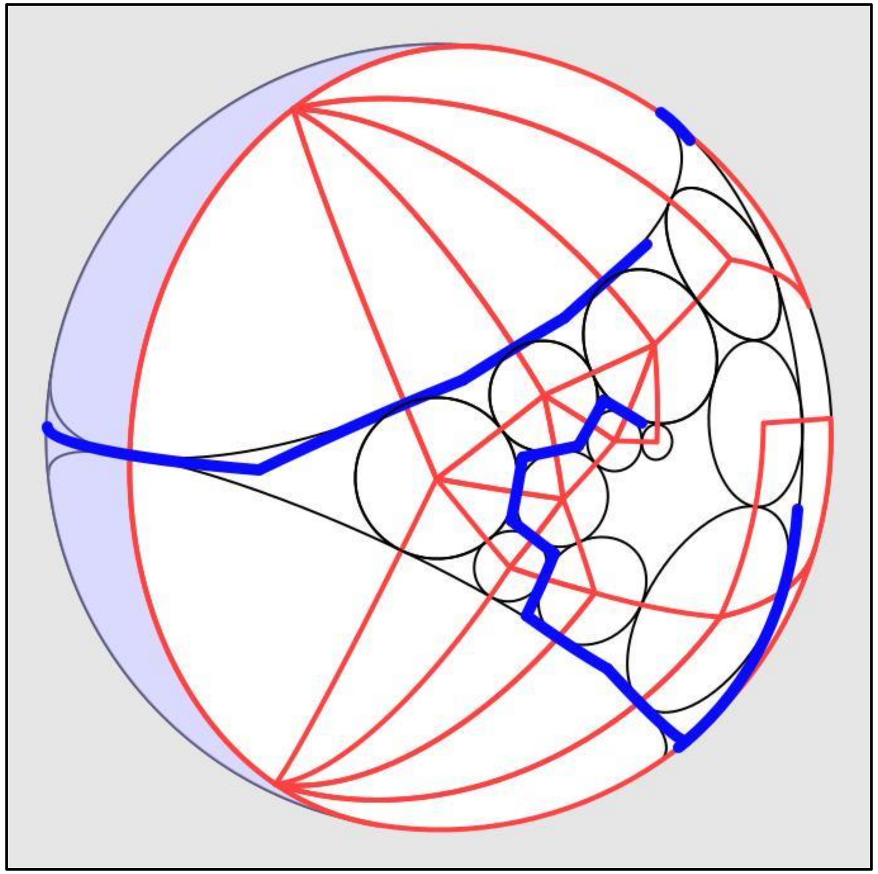
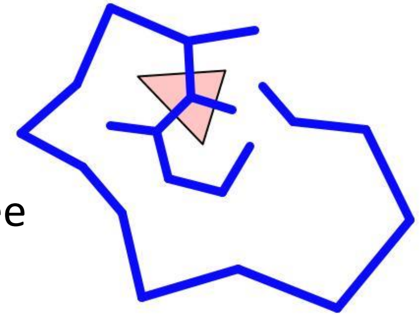
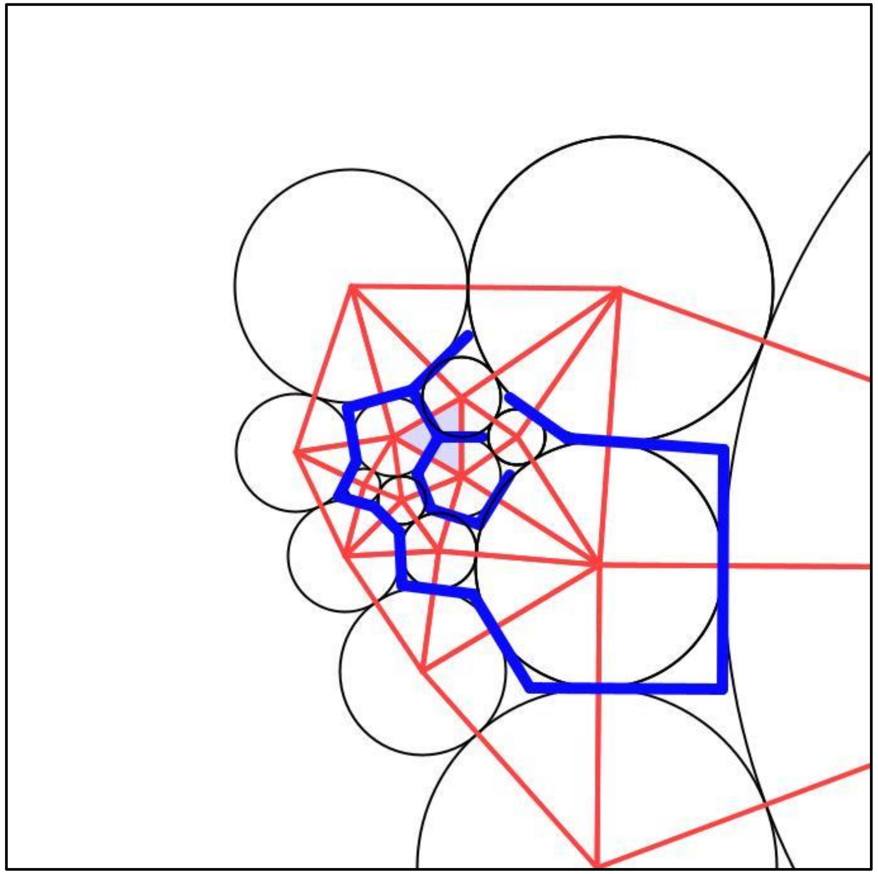


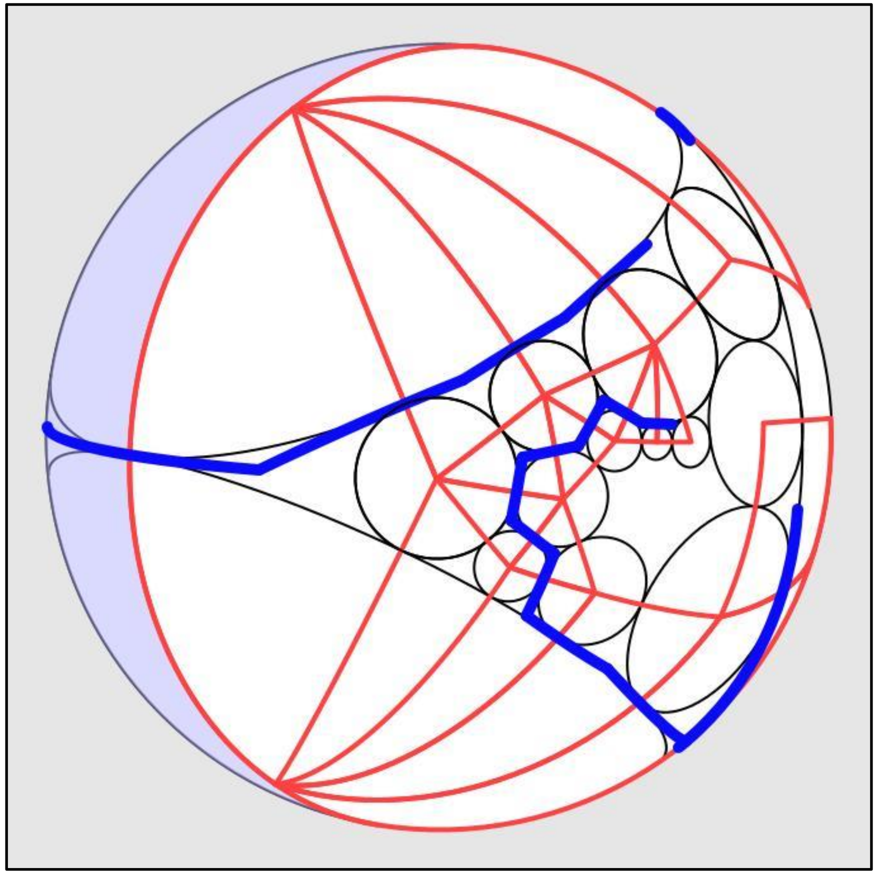
Using known discrete Schwarzians:



On the plane

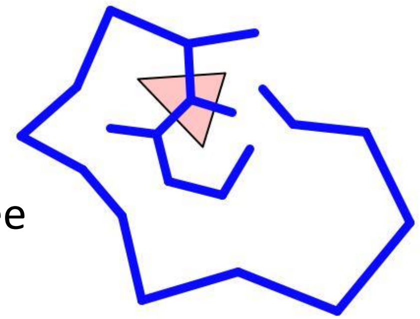


On the sphere



Using known discrete Schwarzians:

Dual spanning tree



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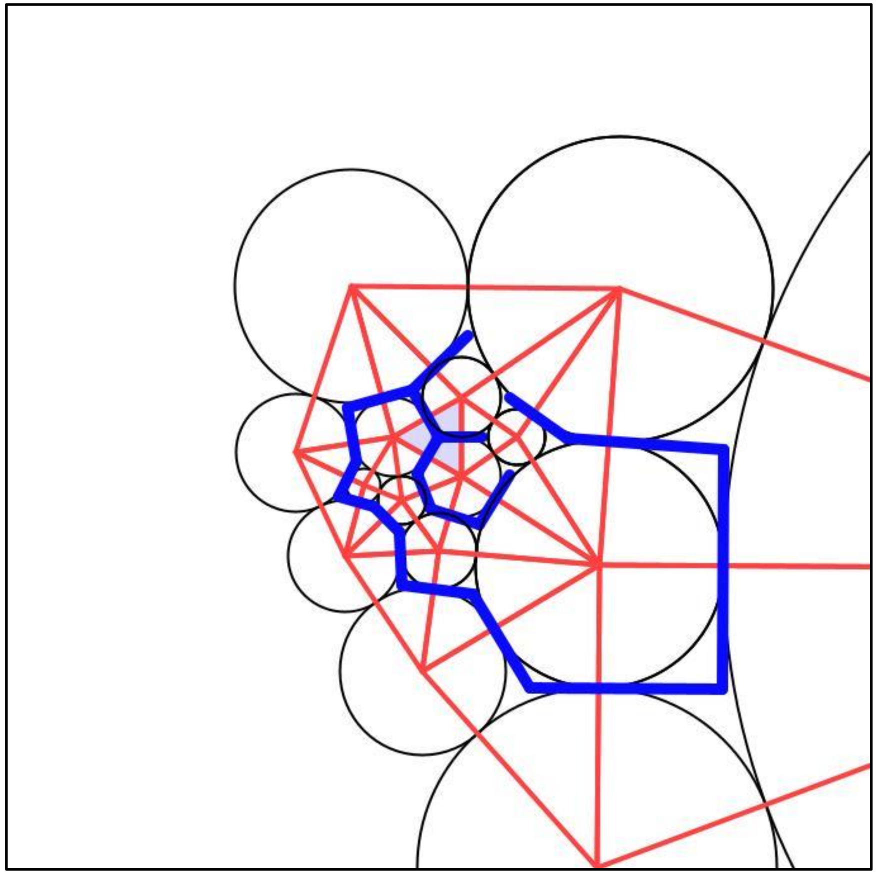

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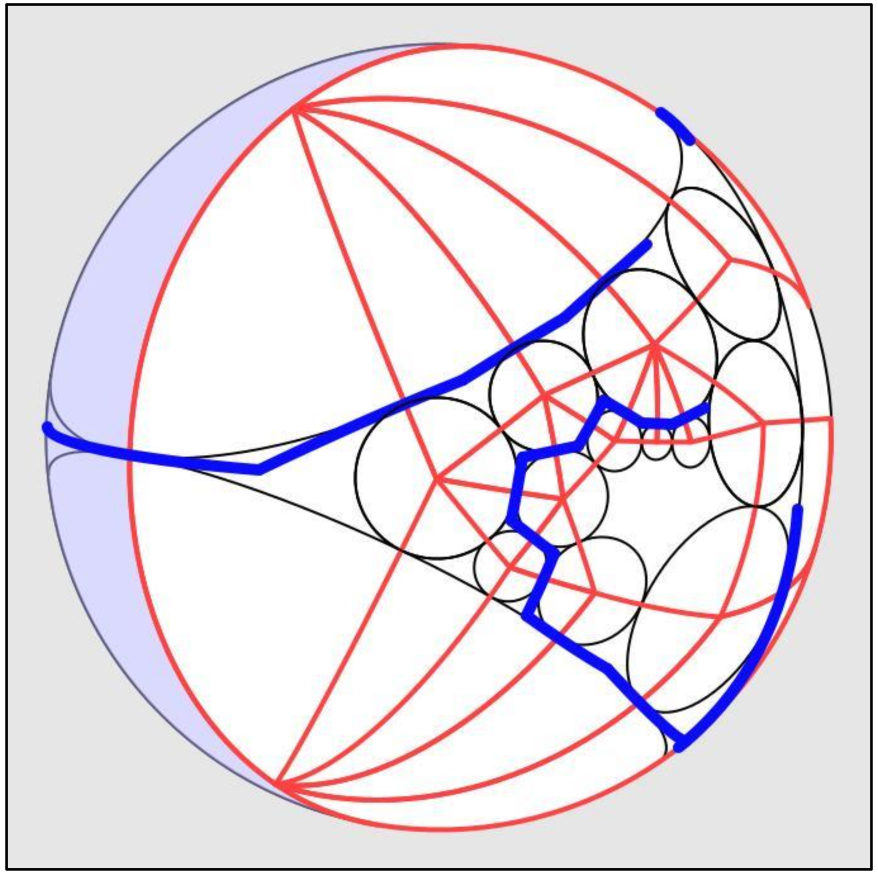


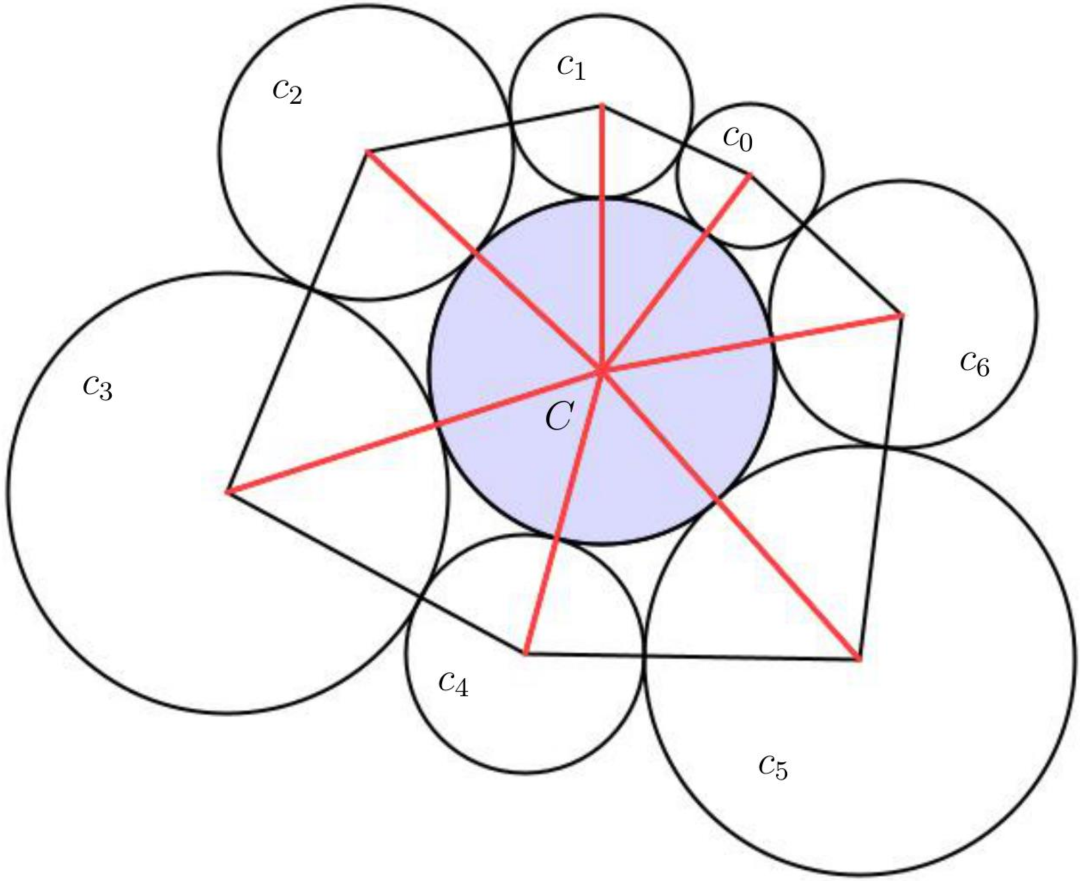
Goal (long term): Find packing (edge) labels

Definition. *Let K be a simply connected complex and let S be an edge label, that is, a set of real numbers, one for each interior edge of K. We call S a* ***packing (edge) label*** *if there exists a circle packing P on the Riemann sphere whose intrinsic schwarzians are given by S.*

Two traditional keys for working with radius labels:

- * **Criteria** for packing
- * **Monotonicities** in the data

If there exist monotonicities, they remain a mystery to me.



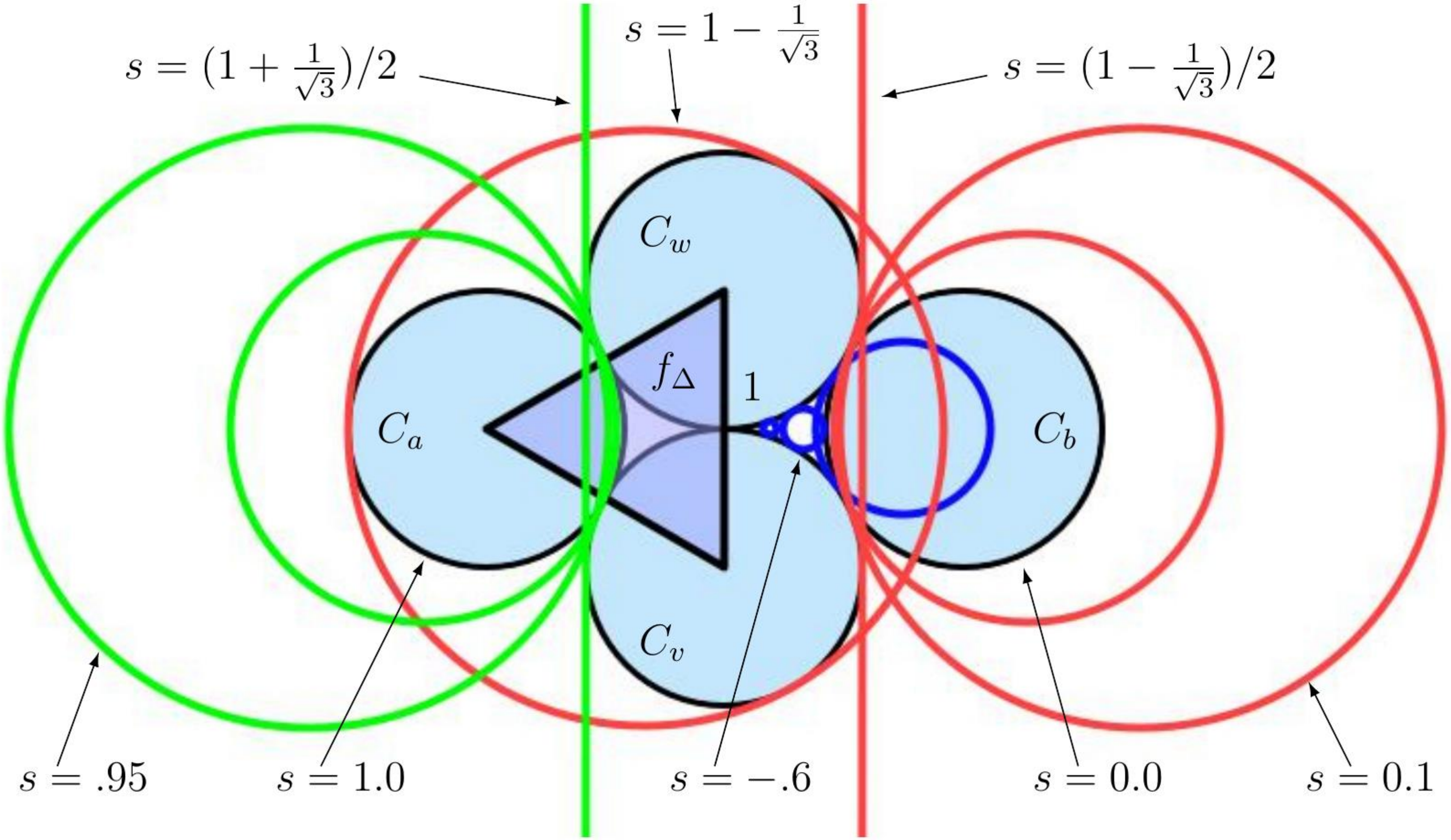
In this talk I concentrate on <u>criteria</u>: it suffices to have criteria for **packing labels for individual *n*-flowers**

Sample intrinsic schwarzians

$$c_b = M_s^{-1}(C_b)$$

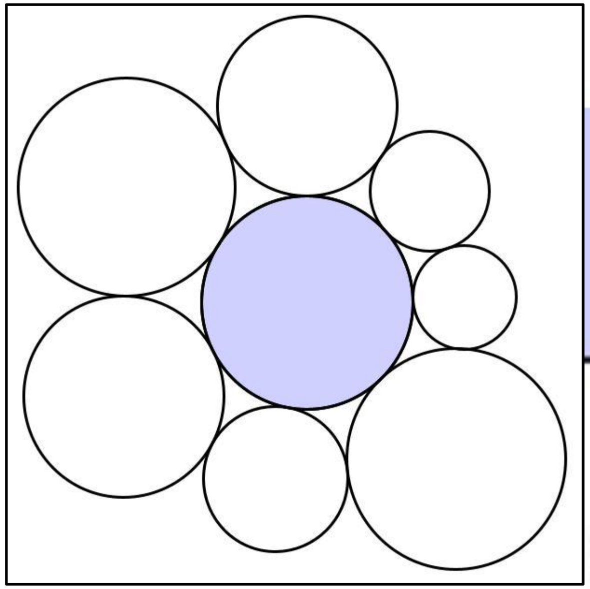
where

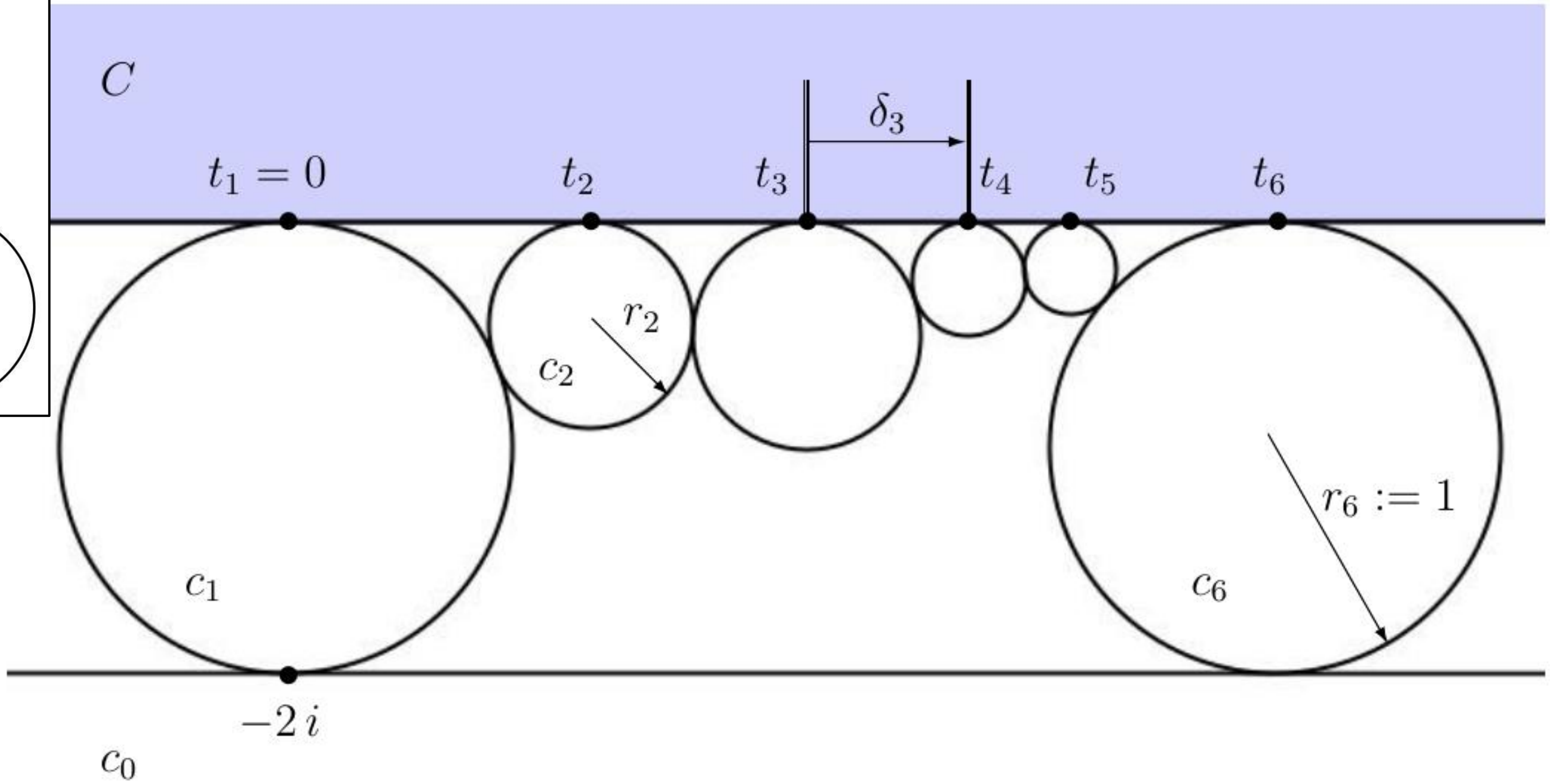
$$M_s^{-1} = \begin{bmatrix} 1-s & s \\ -s & 1+s \end{bmatrix}$$



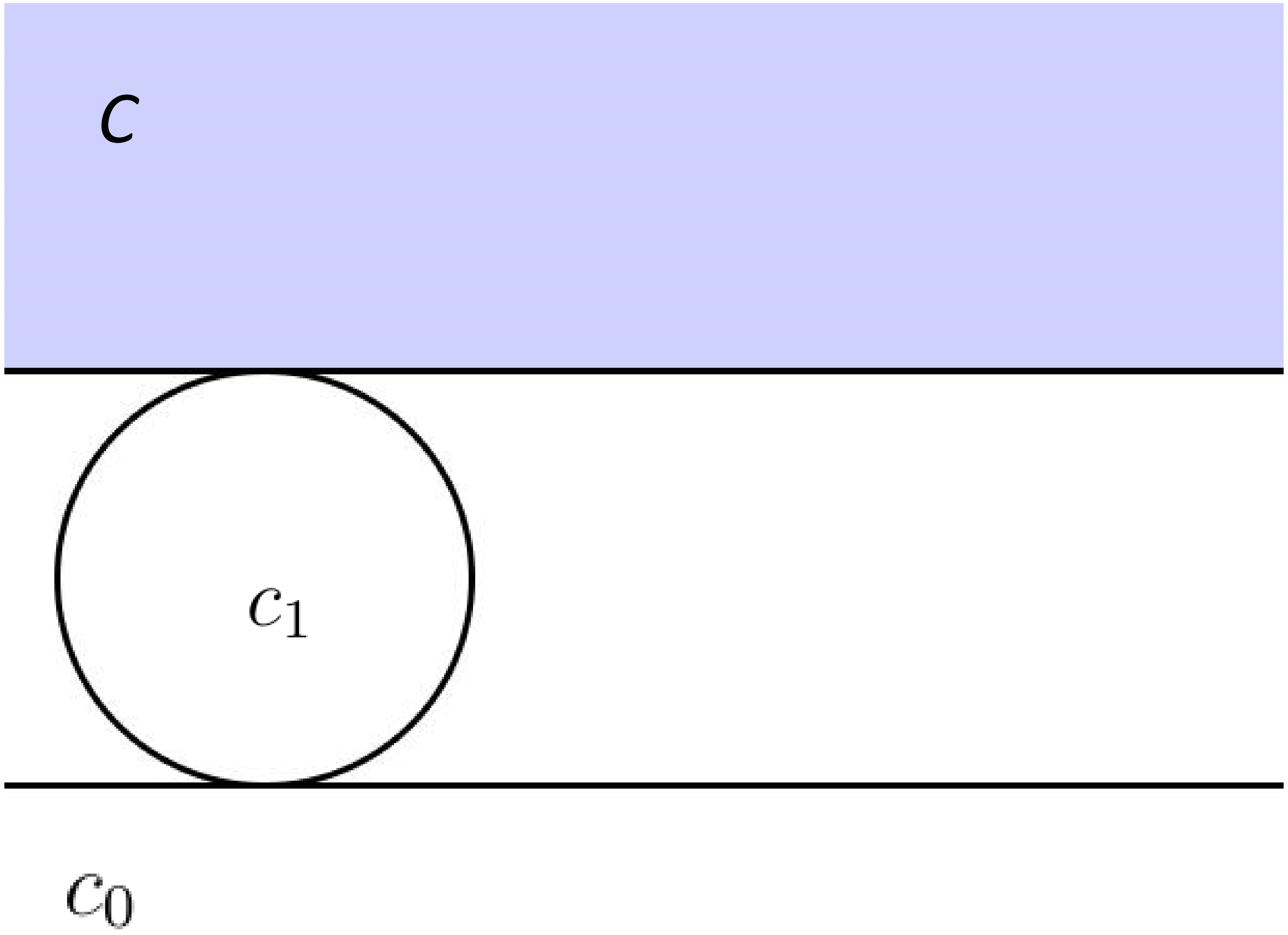
Packing Labels for Un-branched Interior Flowers

Flower Normalizations and Notations





Layout Process:



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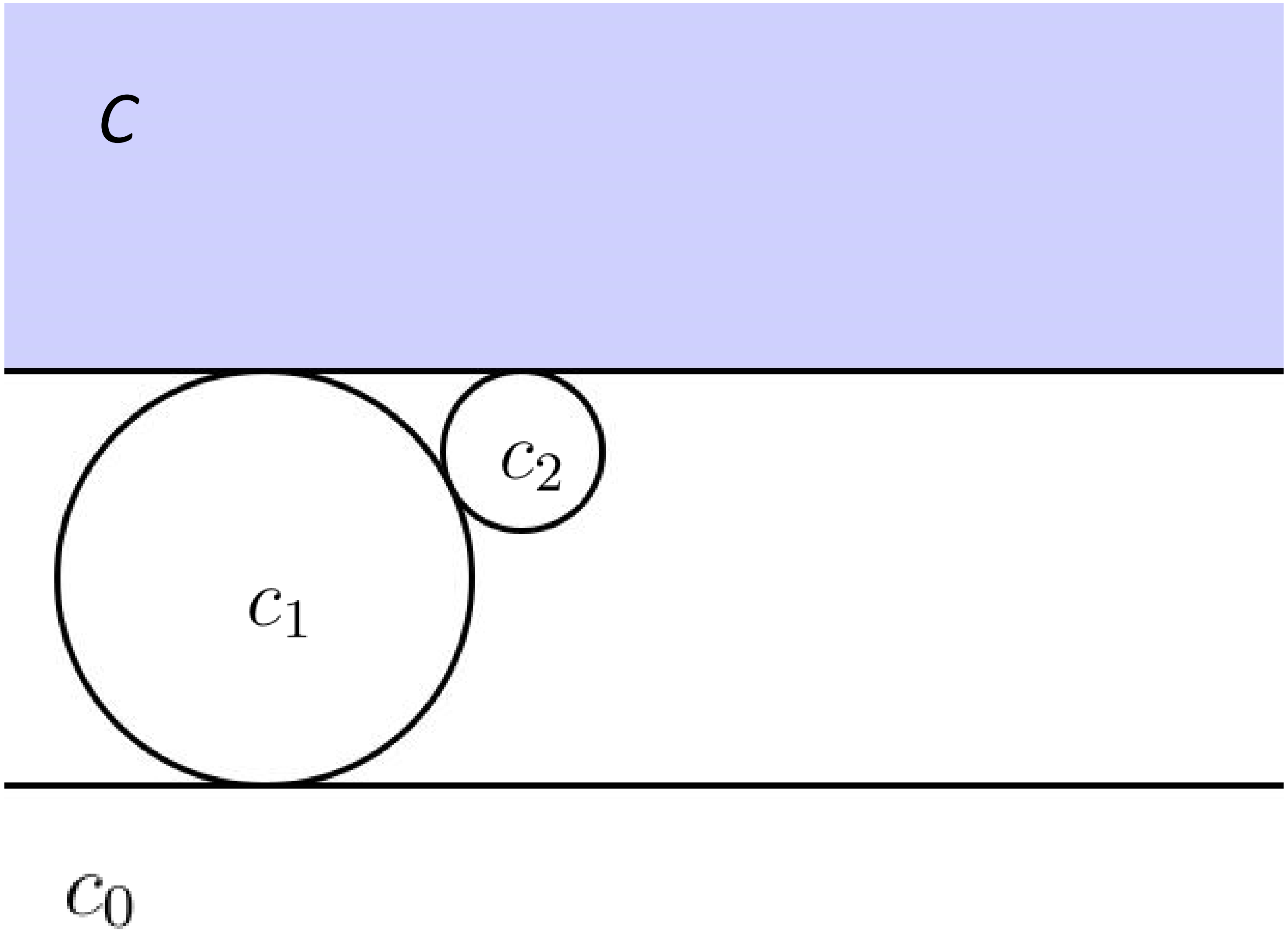
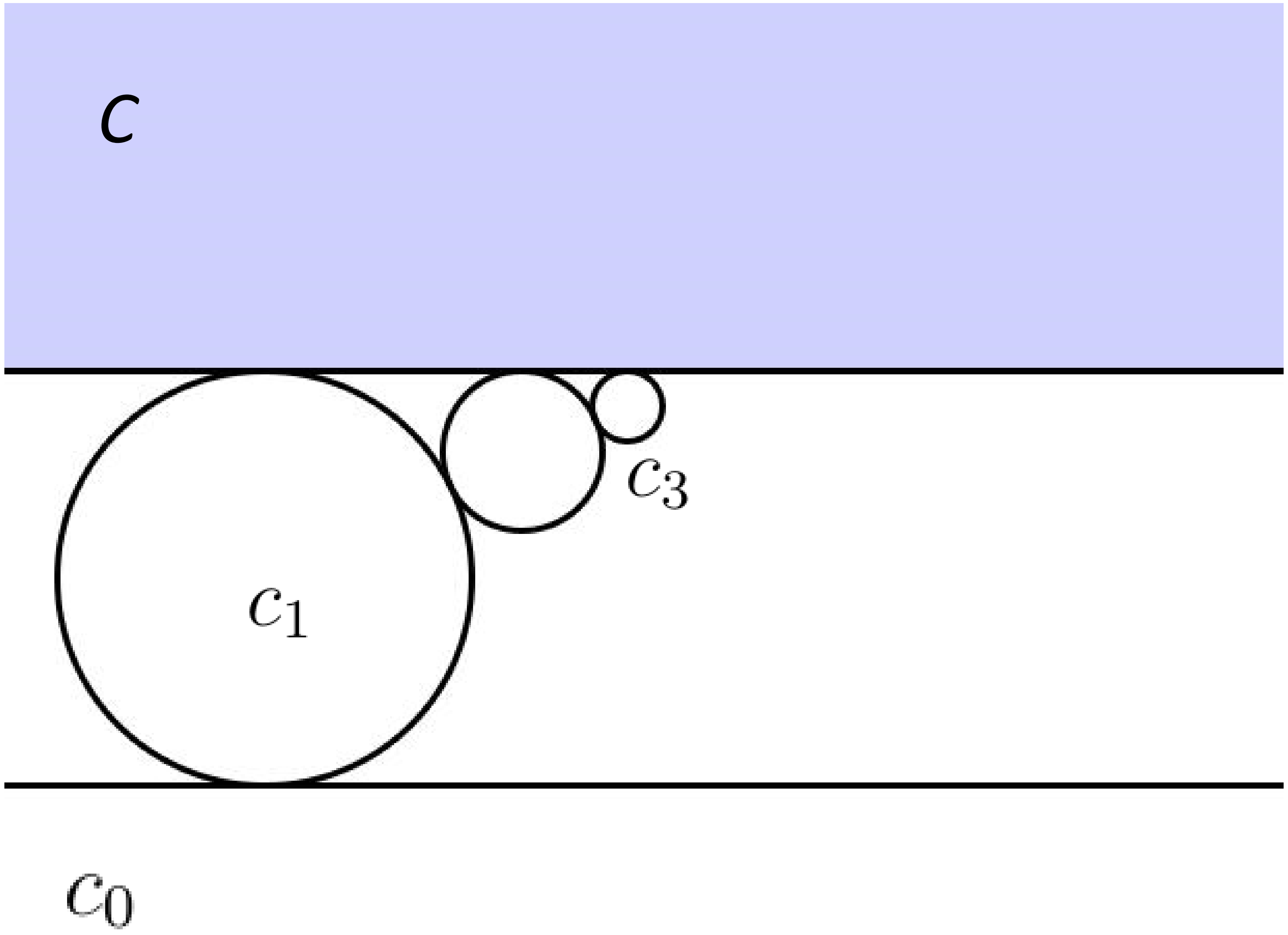
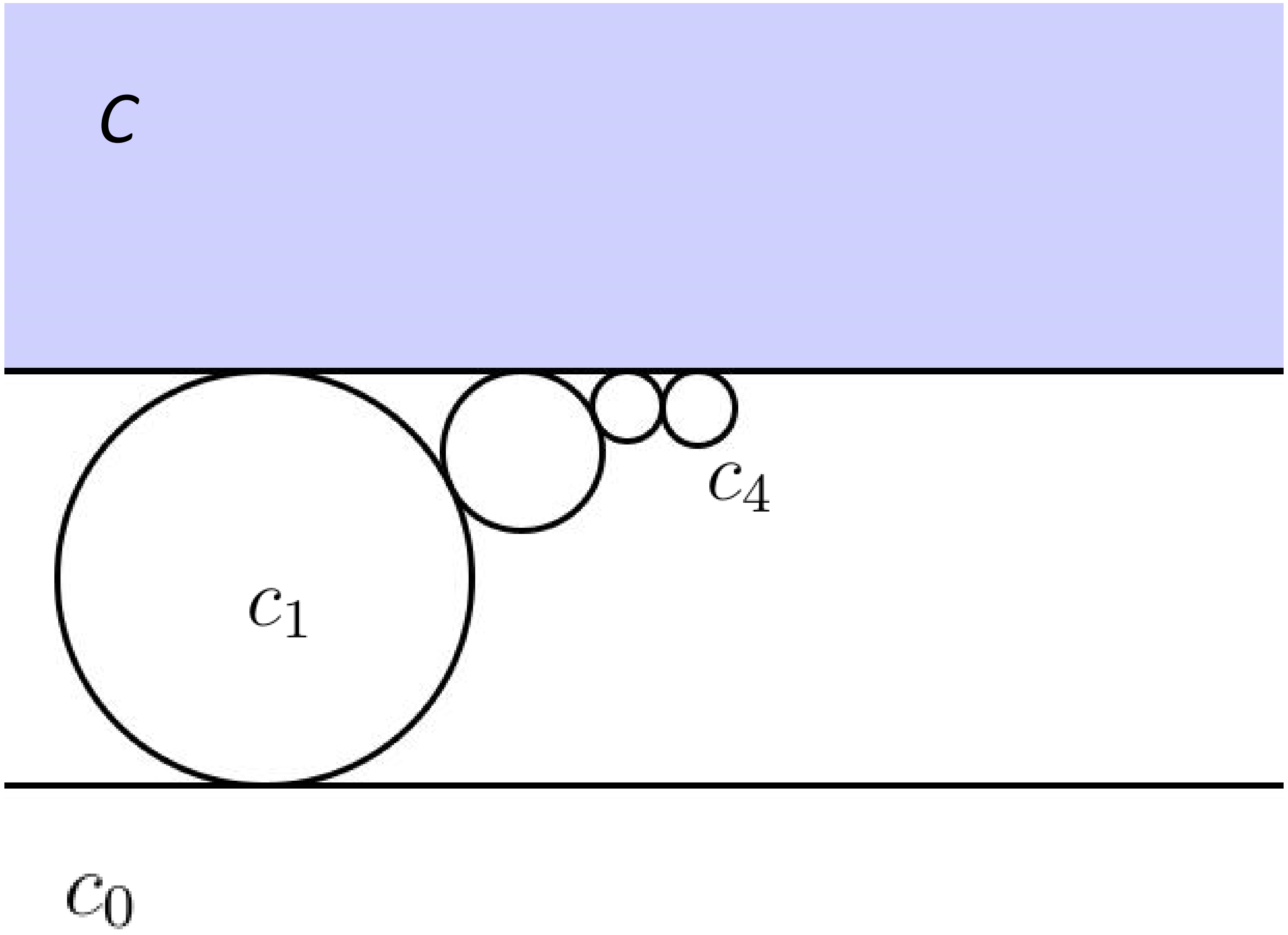


Layout Process:



Layout Process:



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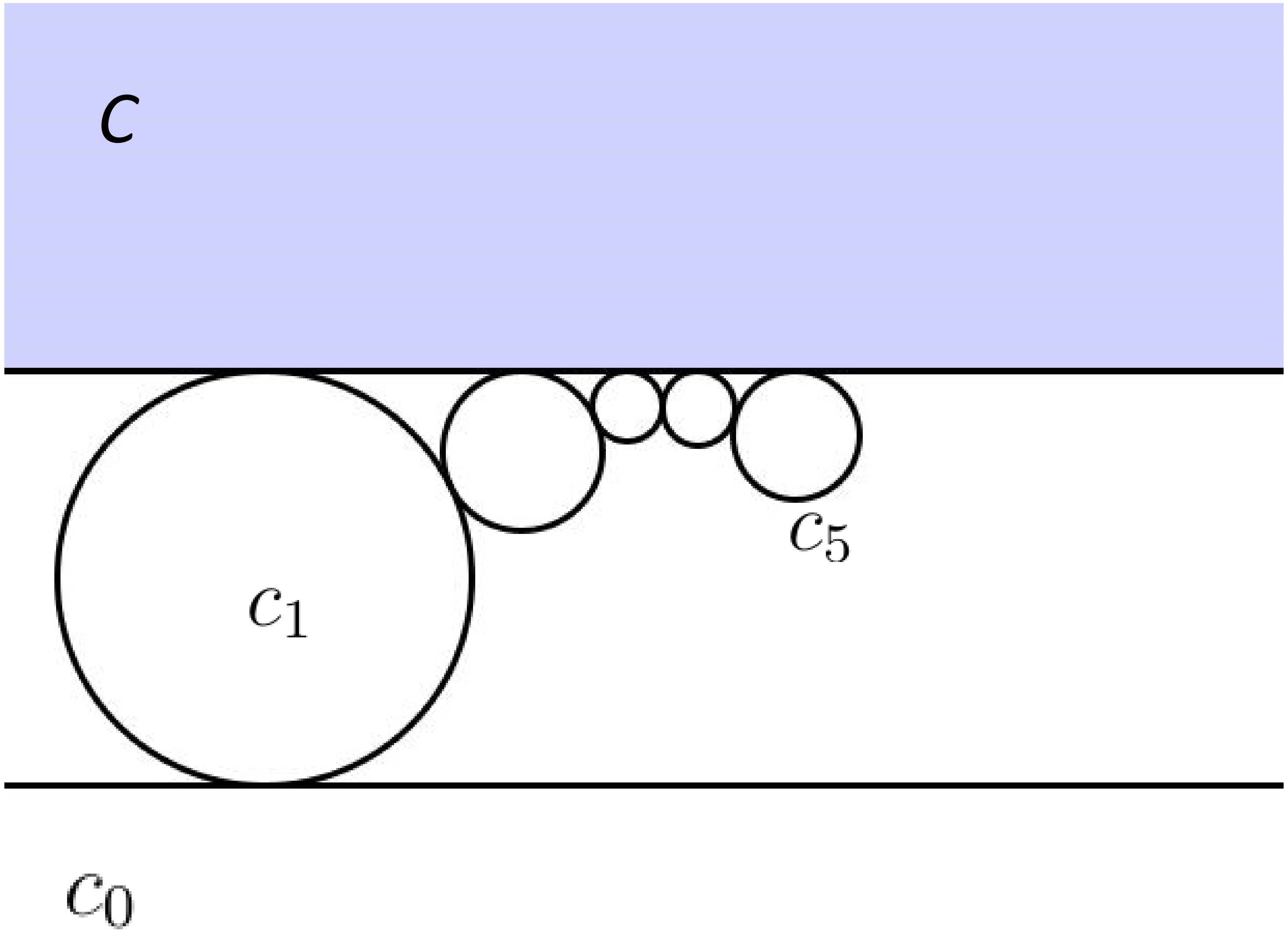

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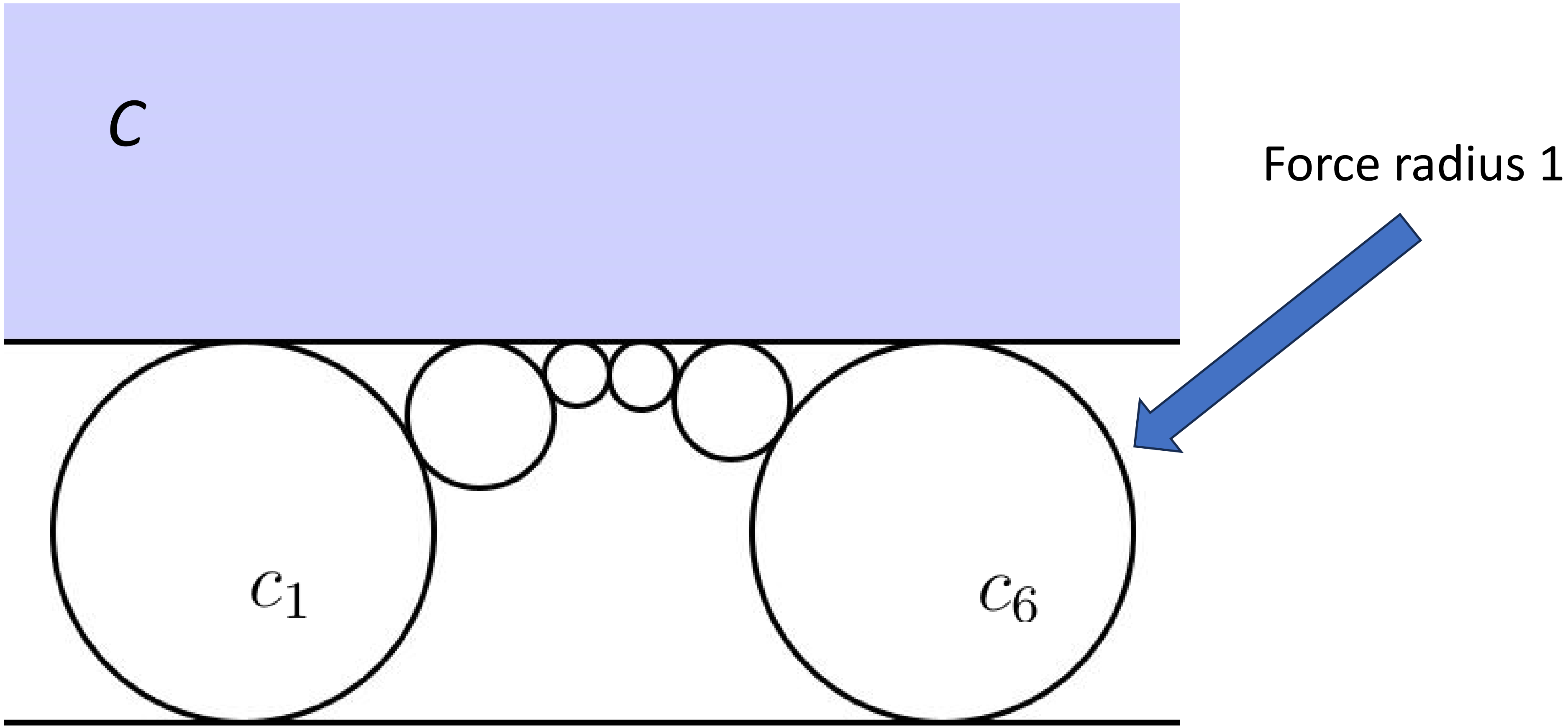
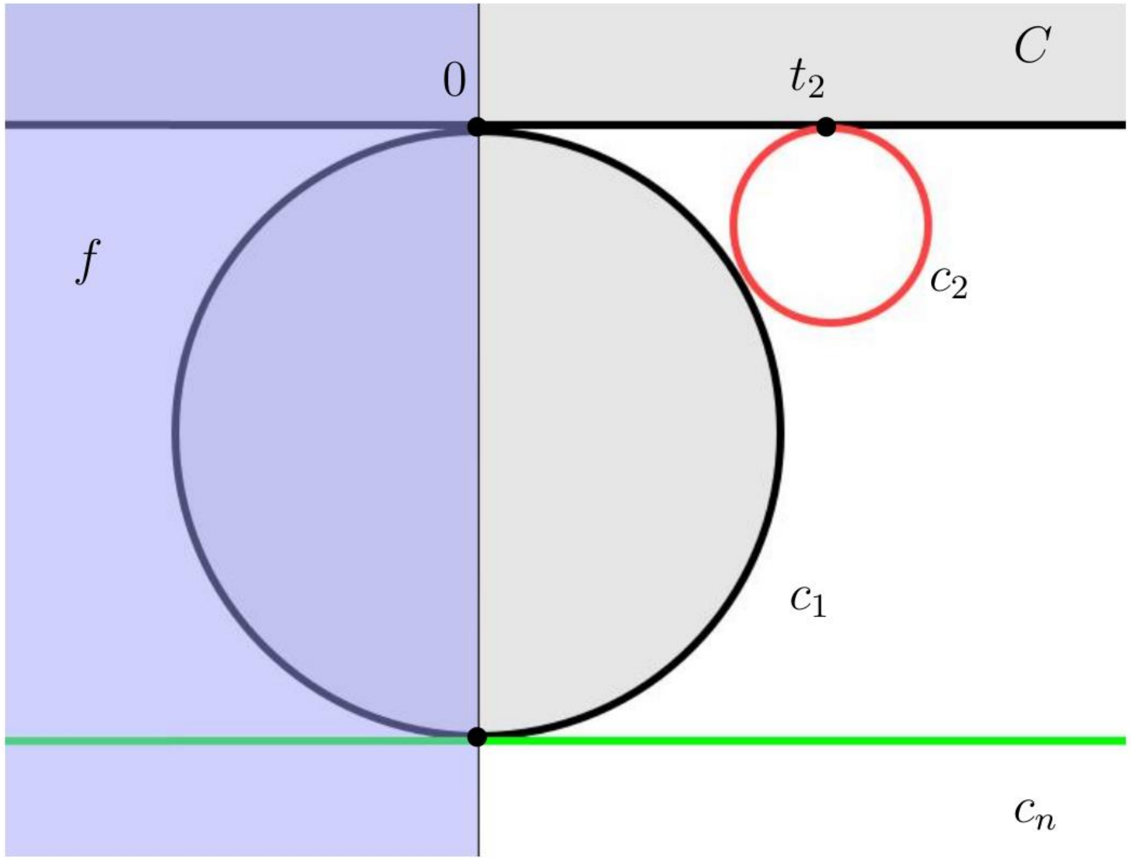


NOTE: This process used only *n*-3 of the *n* edge labels !!

Step-wise Computations:

This step-by-step construction encounters several computational situations:

NOTE: hand computations suggested introducing a new variable: $u = 1 - s$
so, our *n*-3 parameters are $\{u_1, \cdots, u_{n-3}\}$



Place the red petal based on the Previous edge label:

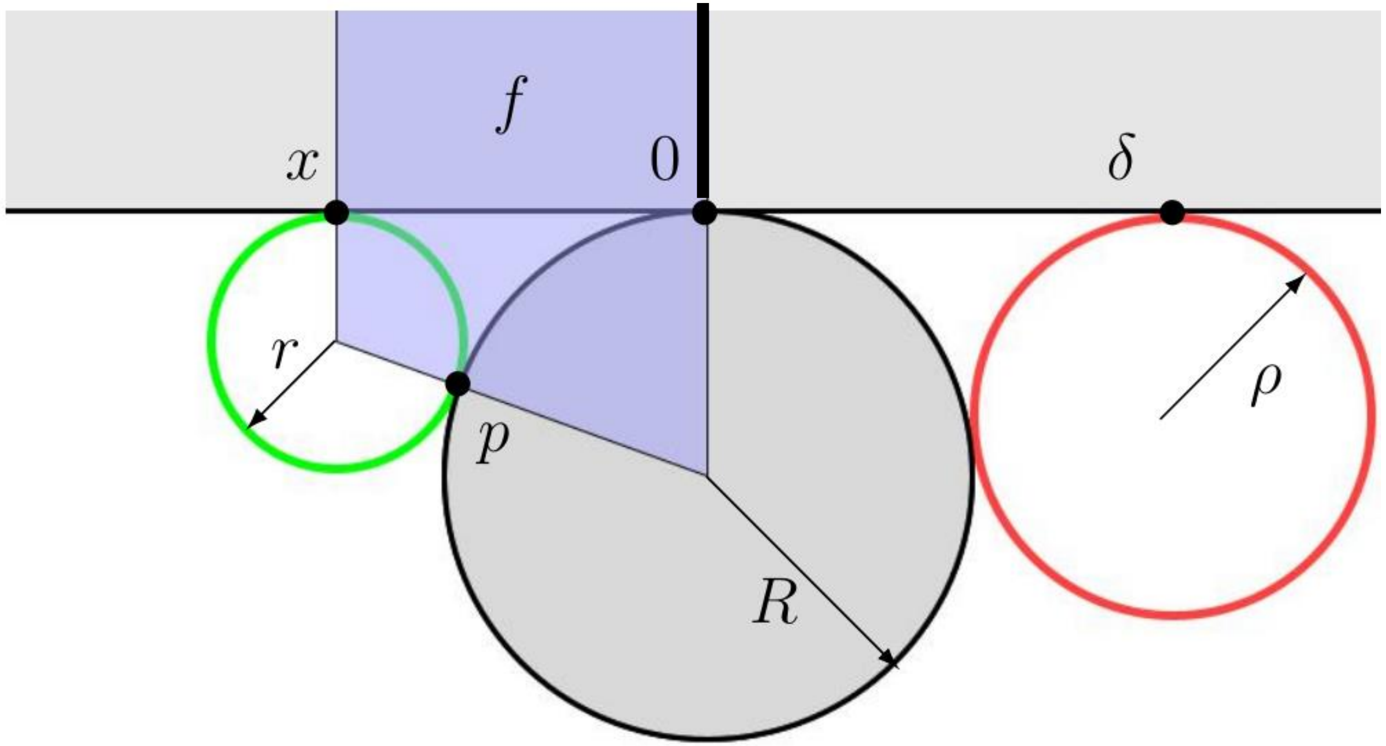
$$t_2 = 2/(\sqrt{3}\, u_1)$$

$$r_2 = 1/(\sqrt{3} u_1)^2$$

Step-wise Computations:

Next is the generic step which is repeated n-2 times:

Place the red petal based on the face *f* formed by the green and shaded petals and the upper halfplane *C*; use the schwarzian for the edge to the shaded petal:



$$\delta = \frac{2R}{(\sqrt{3}\,u - \sqrt{R/r}\,)}$$

$$\rho = \frac{1}{(\sqrt{3}\,u/\sqrt{R} - 1/\sqrt{r})^2}$$

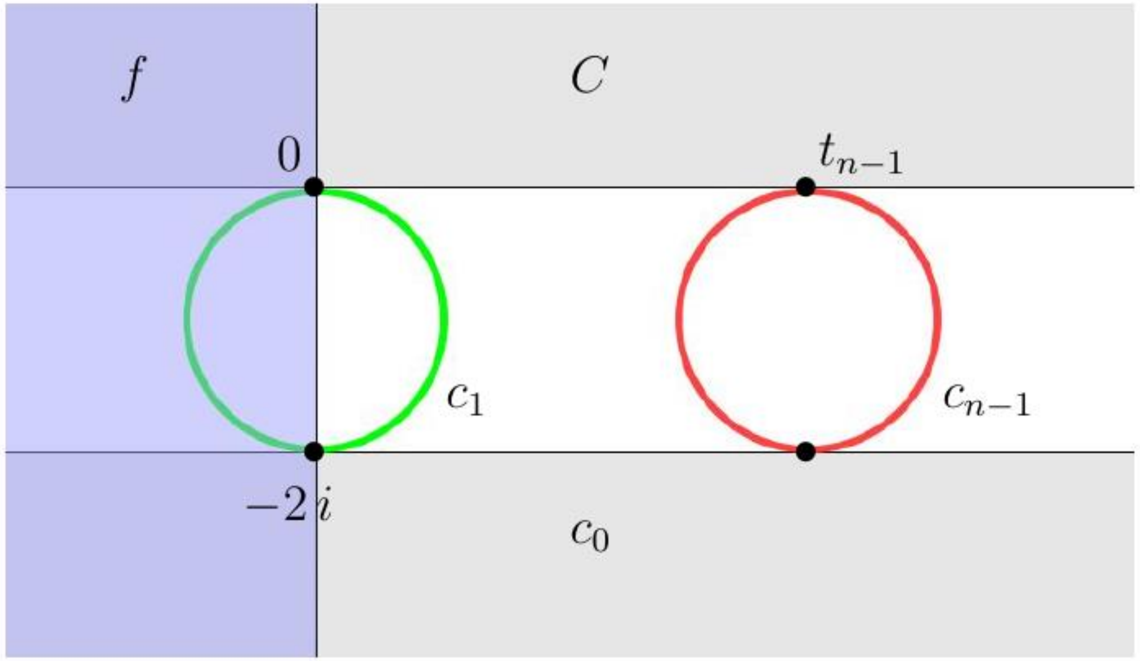
If $\rho = 1$, then

$$u = \frac{\sqrt{R} + \sqrt{R/r}}{\sqrt{3}}$$

*

Step-wise Computations:

The red and green petals are opposite across the edge connecting the half planes:



IF we where placing the last petal based on the first, we would get

$$t_{n-1} = 2\sqrt{3}\, u_0$$

But we do not know u_0 !!

Crucial Step: We MANDATE $r_{n-1} = 1$

With the last petal of radius 1 in place, we can directly compute u_{n-2}, u_{n-1}, u_0

Successive reciprocal roots:

A focus on reciprocal roots gives the valuable functions $\mathfrak{C}_j(u_1, \cdots, u_{j-1})$.
(Notation: $u_{i,j,k} = u_i u_j u_k$)

$$1/\sqrt{r_2} = \mathfrak{C}_2(u_1) = \sqrt{3}u_1,$$
$$1/\sqrt{r_3} = \mathfrak{C}_3(u_1, u_2) = 3u_{1,2} - 1,$$
$$1/\sqrt{r_4} = \mathfrak{C}_4(u_1, u_2, u_3) = \sqrt{3}(3u_{1,2,3} - u_1 - u_3),$$
$$1/\sqrt{r_5} = \mathfrak{C}_5(u_1, \cdots, u_4) = 9u_{1,2,3,4} - 3u_{1,4} - 3u_{3,4} - 3u_{1,2} + 1,$$
$$1/\sqrt{r_6} = \mathfrak{C}_6(u_1, \cdots, u_5) =$$
$$\sqrt{3}(9u_{1,2,3,4,5} - 3u_{1,4,5} - 3u_{3,4,5} - 3u_{1,2,5} - 3u_{1,2,3} + u_1 + u_3 + u_5).$$
$$\cdots\cdots$$

For $p = (u_0, u_1, \cdots, u_{n-1}) \in \mathbb{R}^n_+$ write $\mathfrak{C}_j(u_1, \cdots, u_{j-1}) = \mathfrak{C}_j(p)$.
Then computations show

$$\mathfrak{C}_{j+1}(p) = \sqrt{3}u_j\mathfrak{C}_j(p) - \mathfrak{C}_{j-1}(p)$$

Associated Functions $\mathfrak{U}_n(\cdot)$

Since the last petal has radius 1, (*) implies $u_{n-2} = \dfrac{1+\mathfrak{C}_{n-3}(p)}{\sqrt{3}\,\mathfrak{C}_{n-2}(p)}$.

Defining $\mathfrak{U}_n(u_1, \cdots, u_{n-3}) = \mathfrak{U}_n(p) = \dfrac{1+\mathfrak{C}_{n-3}(p)}{\sqrt{3}\,\mathfrak{C}_{n-2}(p)}$, we have:

$$u_2 = \mathfrak{U}_4(u_1) = \frac{2}{3u_1},$$

$$u_3 = \mathfrak{U}_5(u_, u_2) = \frac{u_1 + 1/\sqrt{3}}{3u_{1,2} - 1},$$

$$u_4 = \mathfrak{U}_6(u_1, u_2, u_3) = \frac{u_{1,2}}{3u_{1,2,3} - u_1 - u_3},$$

$$u_5 = \mathfrak{U}_7(u_1, u_2, u_3, u_4) = \frac{3(3u_{1,2,3} - u_1 - u_3) + 1/\sqrt{3}}{3(3u_{1,2,3,4} - u_{1,2} - u_{1,4} - u_{3,4}) + 1},$$

$$u_6 = \mathfrak{U}_8(u_1, u_2, u_3, u_4, u_5)$$
$$= \frac{3(3u_{1,2,3,4} - u_{1,2} - u_{1,4} - u_{3,4}) + 2}{3(9u_{1,2,3,4,5} - 3u_{1,2,3} - 3u_{1,2,5} - 3u_{1,4,5} - 3u_{3,4,5} + u_1 + u_3 + u_5)}.$$

$\cdots\cdots$

Un-branched Flowers (petals wrapping once around *C*)

Theorem: *Given $n > 3$, the parameters $\{u_1, \cdots, u_{n-3}\}$ are part of a packing label for an un-branched n-flower if and only if*

$$\mathfrak{C}_j(u_1, \cdots, u_{j-1}) > 0, \;\; j = 2, \cdots, (n-2).$$

In this case, these expressions

$$\begin{aligned} u_{n-2} &= \mathfrak{U}_n(u_1, \cdots, u_{n-3}), \\ u_{n-1} &= \mathfrak{U}_n(u_2, \cdots, u_{n-2}), \\ u_0 &= \mathfrak{U}_n(u_3, \cdots, u_{n-1}), \end{aligned} \tag{1}$$

allow computation of the three remaining labels.

Simultaneously a **characterization**, **parameterization**, and **computational tool**

The Intriguing Functions $\mathfrak{U}_n(\cdot)$ $\mathfrak{C}_n(\cdot)$

Work with vectors $p = (u_0, u_1, \cdots, u_{n-1}) \in \mathbb{R}^n_+$. For **un-branched** n-flowers:

$$(\star) \qquad u_{n-2} = \mathfrak{U}_n(p), \text{ with } \mathfrak{C}_j(p) > 0, j = 2, \cdots, (n-2)$$

- $\mathfrak{U}_n$ is **rational**, $\mathfrak{C}_j$ are **polynomial** with coefficients in $\mathbb{Q}[\sqrt{3}]$. The zeros of $\mathfrak{C}_{n-2}$ are poles of $\mathfrak{U}_n$.
- $(\star)$ is invariant under cyclic rotation (or reversal) of the coords of p.
  If $\vec{p}^{\,k}$ is the cyclic permutation moving u_k to u_0, then

$$u_{n-1} = \mathfrak{U}_n(\vec{p}^{\,1}) \text{ and } u_0 = \mathfrak{U}_n(\vec{p}^{\,n-3}).$$

- Self-referential:

$$\begin{aligned} u_{n-2} &= \mathfrak{U}_n(\vec{u}_{1,n-3}) \\ u_{n-1} &= \mathfrak{U}_n(\vec{u}_{2,n-3}, \mathfrak{U}_n(\vec{u}_{1,n-3})) \\ u_0 &= \mathfrak{U}_n(\vec{u}_{3,n-3}, \mathfrak{U}_n(\vec{u}_{1,n-3}), \mathfrak{U}_n(\vec{u}_{2,n-3}, \mathfrak{U}_n(\vec{u}_{1,n-3}))). \end{aligned}$$

- Define $\mathcal{V}_n \in \mathbb{R}^n$ as the $(n-3)$-dim alg variety defined by

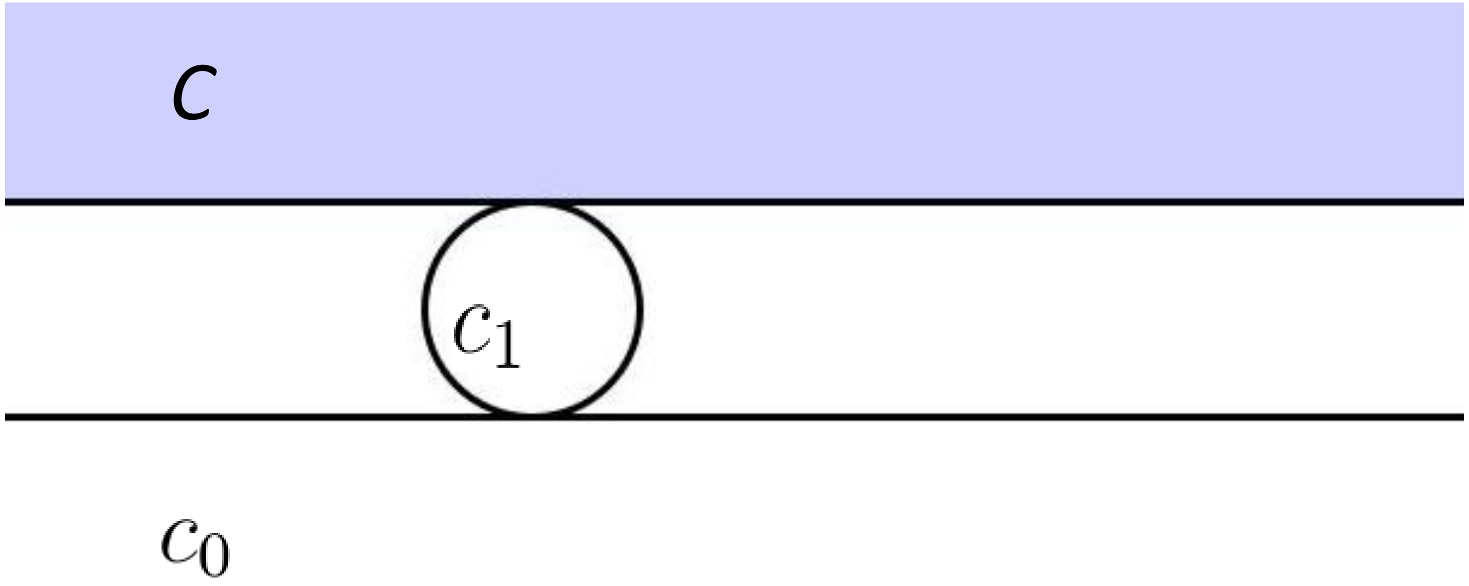
$$u_{n-2} = \mathfrak{U}_n(p) \qquad u_{n-1} = \mathfrak{U}_n(\vec{p}^{\,1}) \qquad u_0 = \mathfrak{U}_n(\vec{p}^{\,2})$$

  and $\mathcal{C}_n$ as cone $\{\mathfrak{C}_j(p) > 0, j = 2, \cdots, (n-2)\}$. Then:

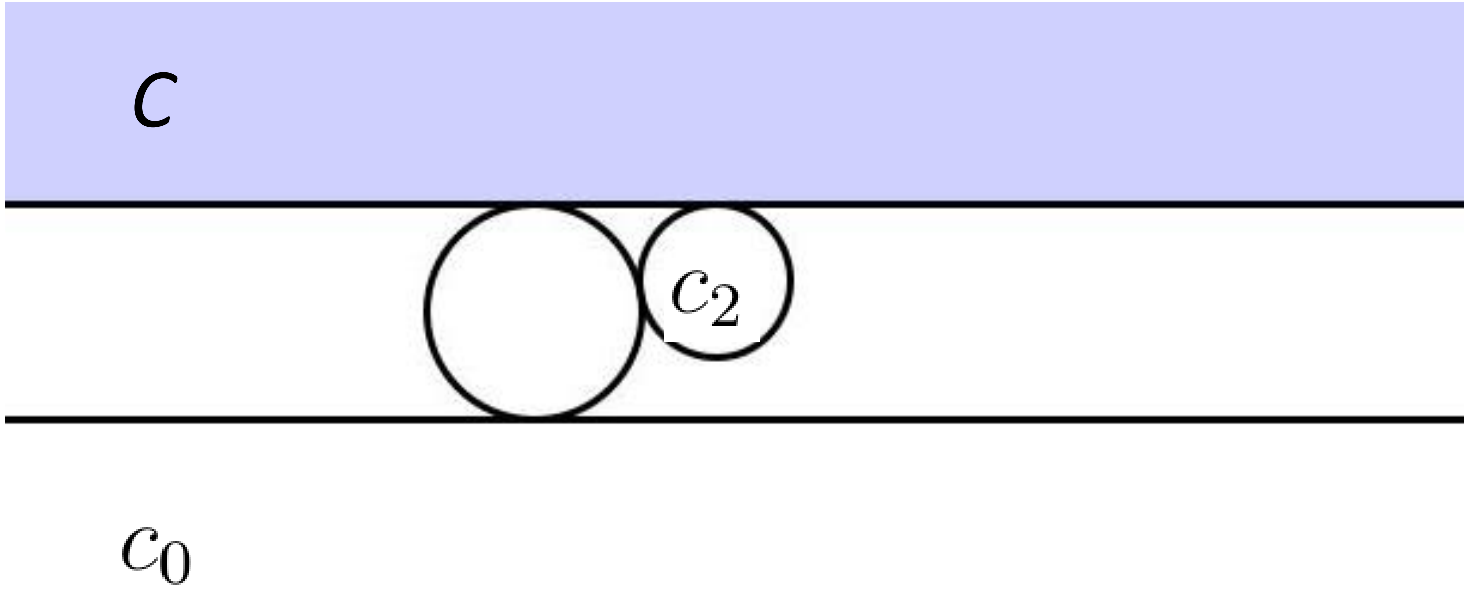
  Parameter Space for un-branched n-flowers is $\boxed{\mathcal{F}_n = \mathcal{V}_n \cap \mathcal{C}_n}$

Packing Labels for Branched Interior Flowers

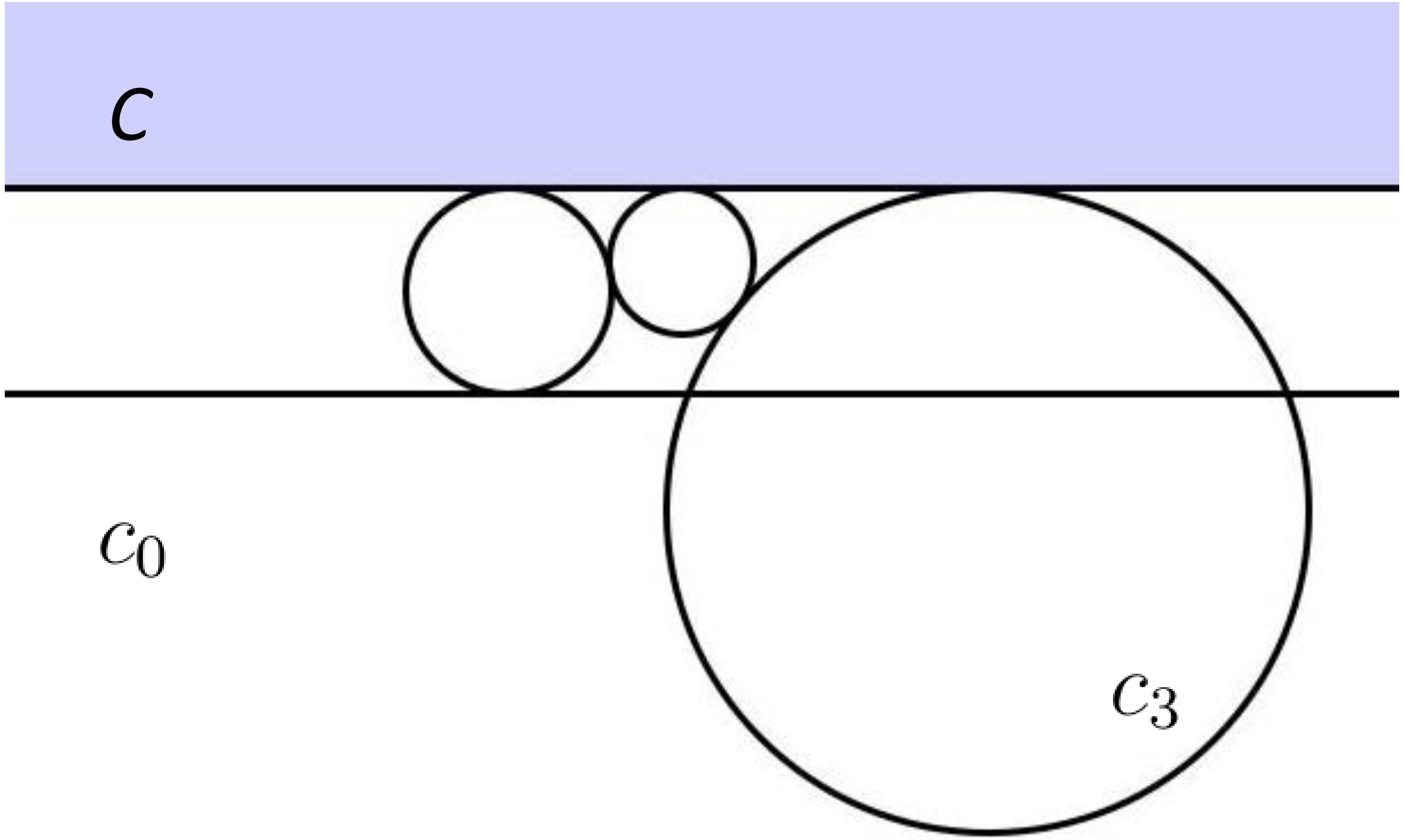
Branching appears:



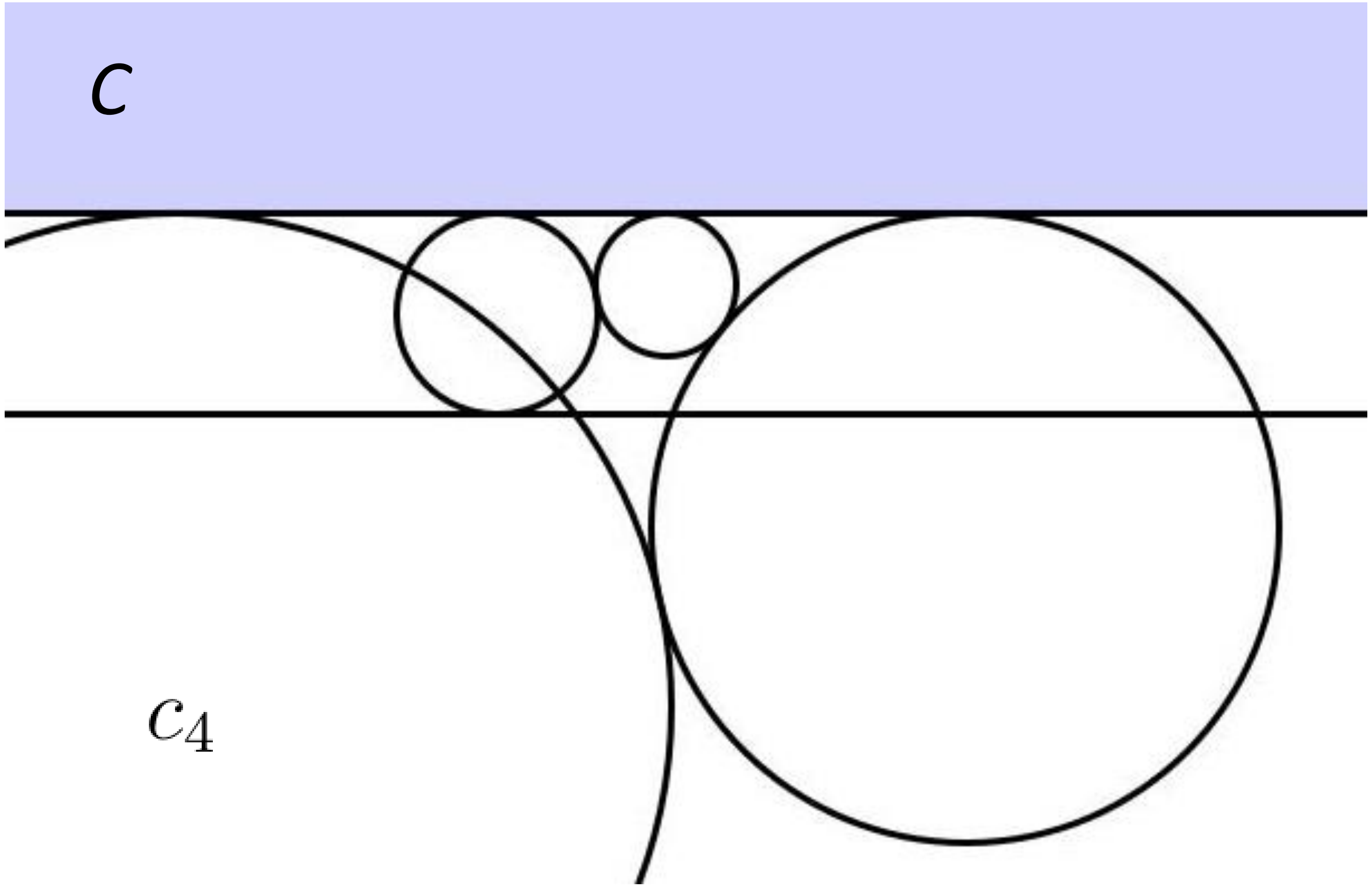
Branching appears:



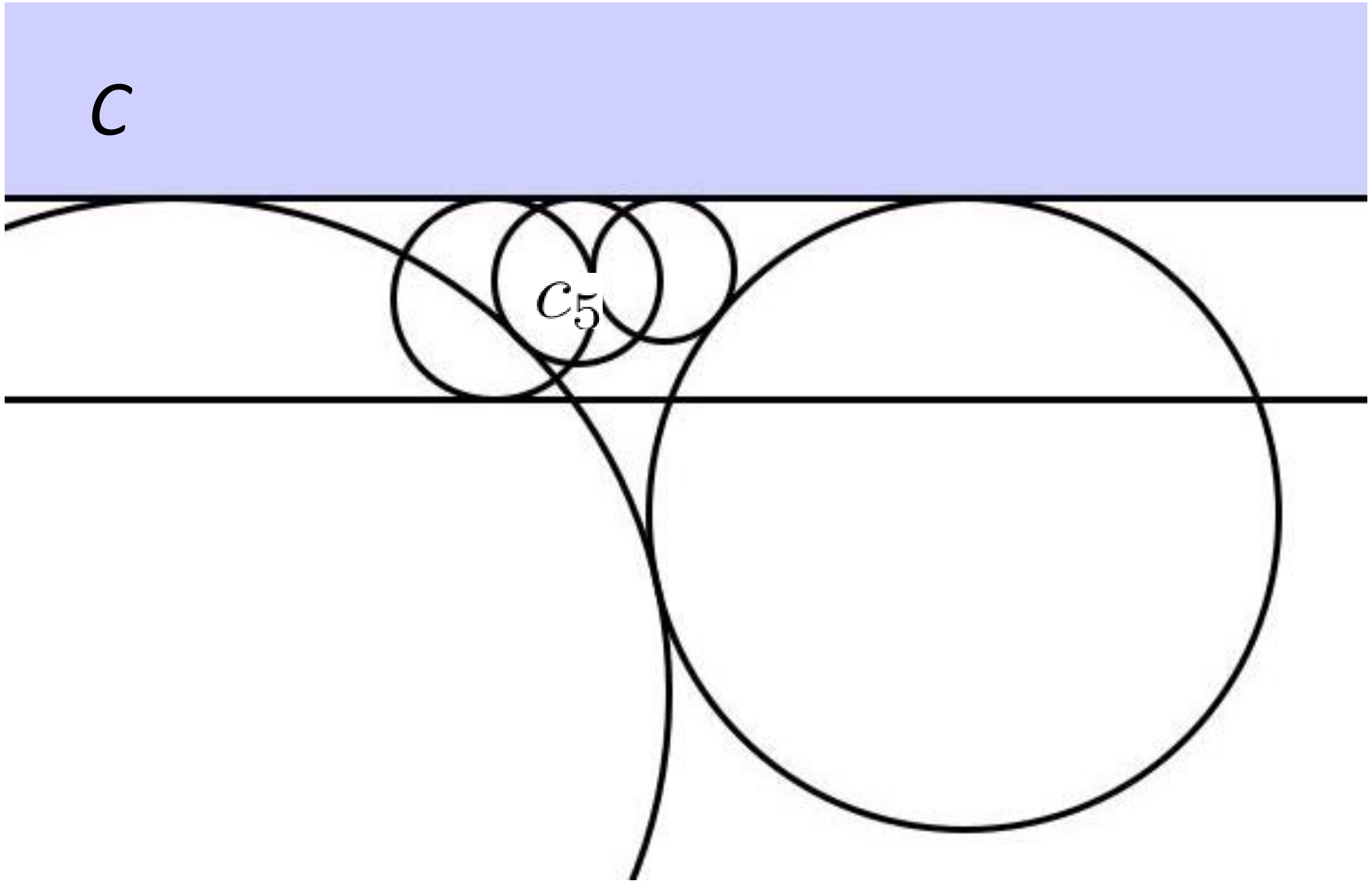
Branching appears:



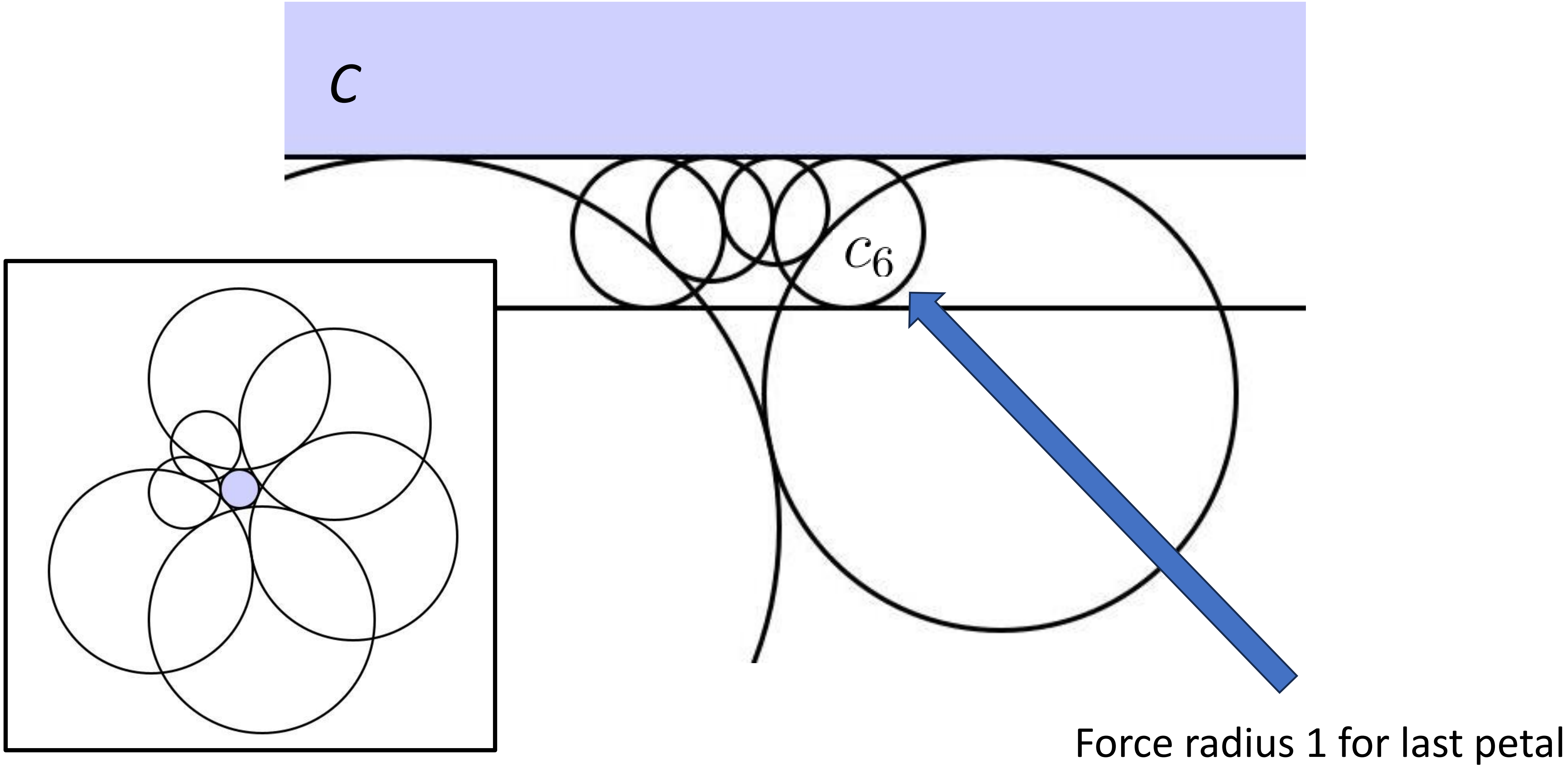
Branching appears:



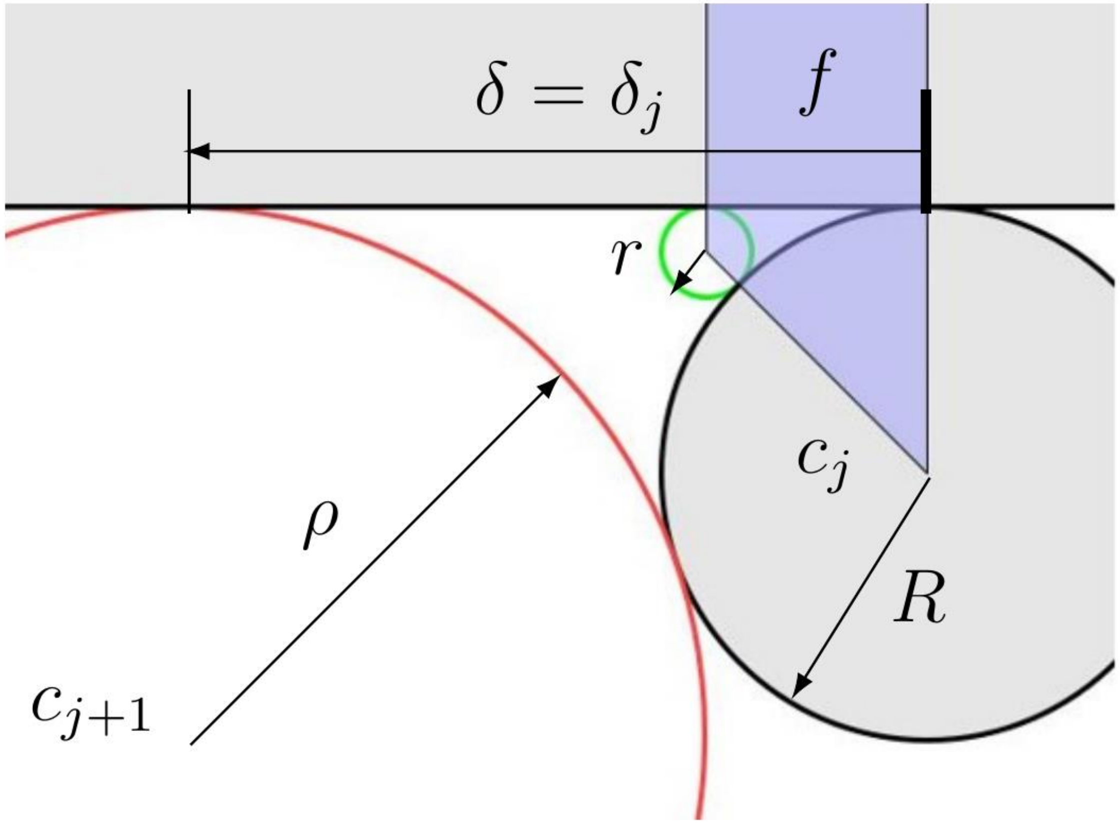
Branching appears:

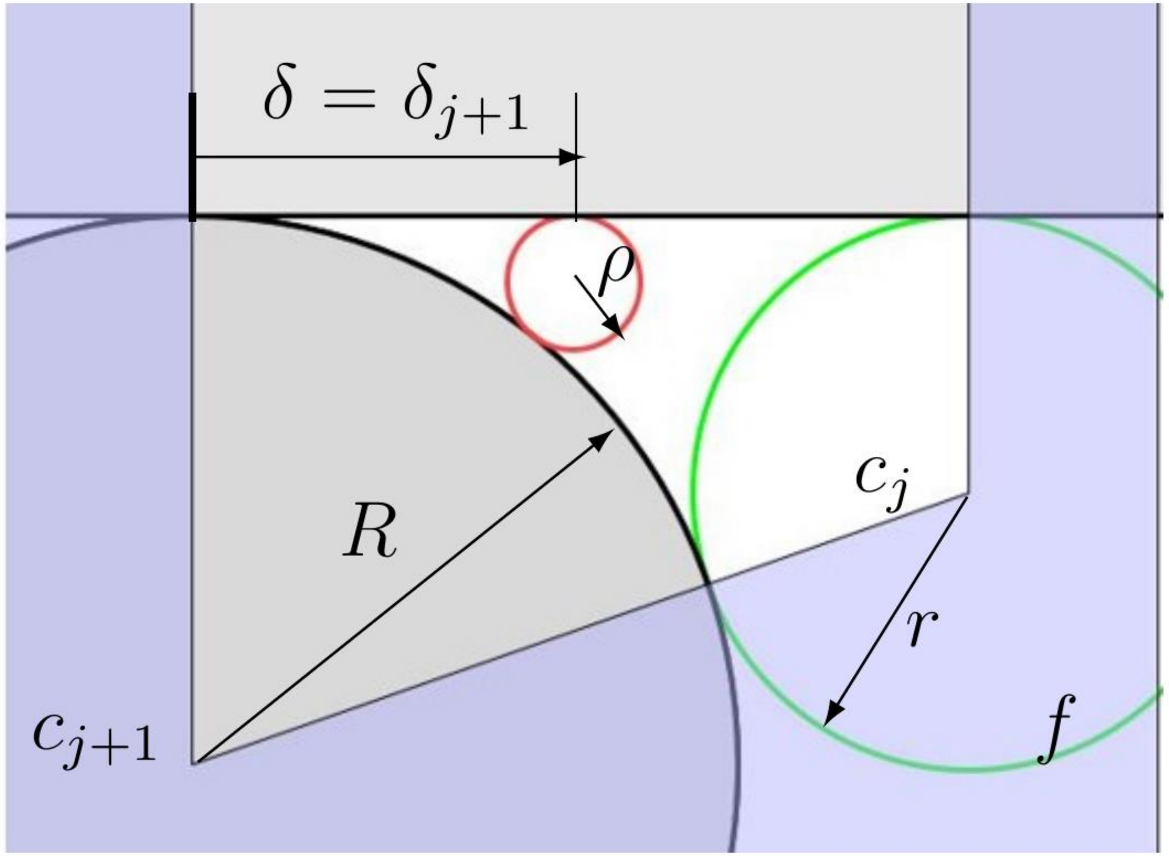


Branching appears:



Branching Starts:



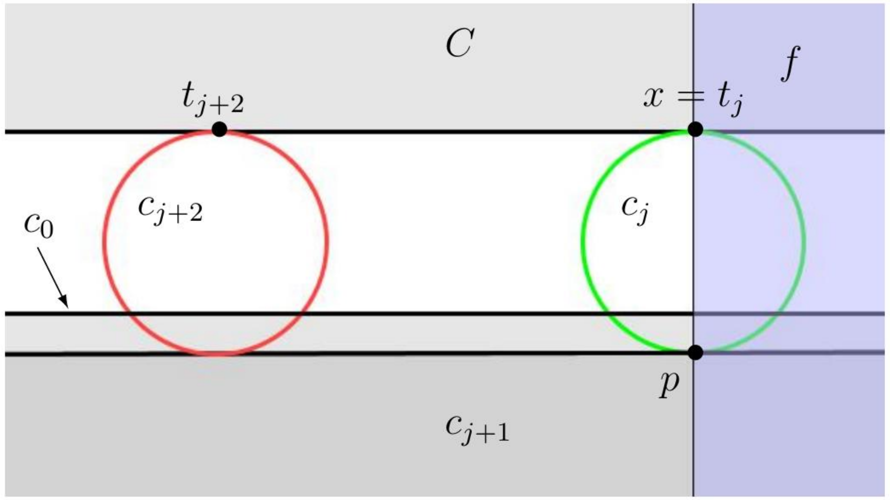


A negative displacement

$$\delta(u, r, R) = \frac{2R}{(\sqrt{3}\,u + \sqrt{R/r}\,)} \qquad \rho = \frac{1}{(\sqrt{3}\,u/\sqrt{R} + 1/\sqrt{r})^2}$$

Theorem: *Given schwarzians $\{s_1, \cdots, s_{n-3}\}$, the* ***Layout Process*** *results in a legitimate n-flower except in the following two situations:*

(a) *when c_{n-2} is tangent to C at infinity or*
(b) *when the computed s_0 exceeds 1.*

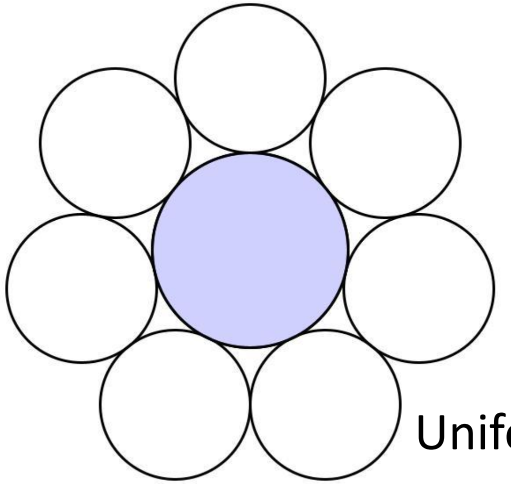


Upshot in the case of branching?

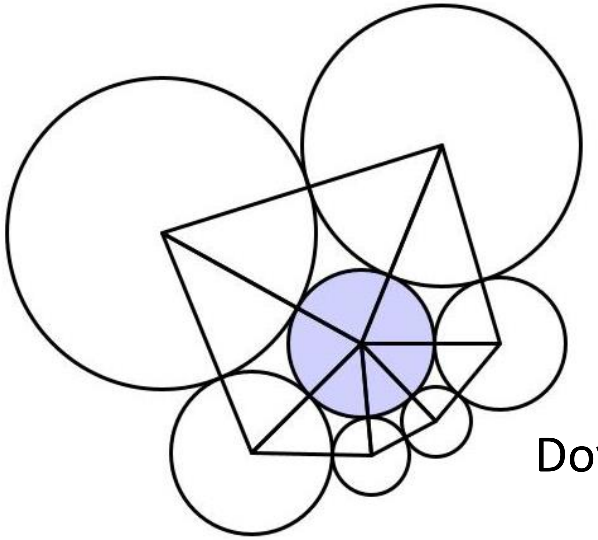
Computations/formulas for normalized flowers continue to hold.
The $\mathfrak{U}_n$ are no longer explicit and must be interpreted as algorithms.

Schwarzians for Special Classes of Flowers

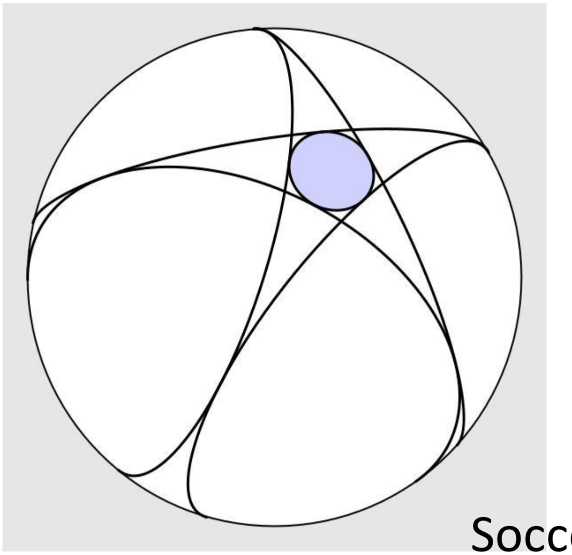
Special flowers



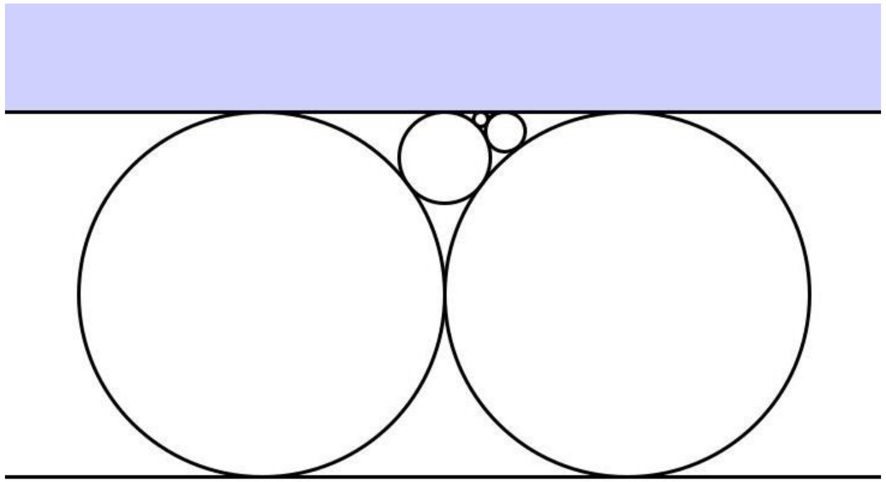
Uniform flower



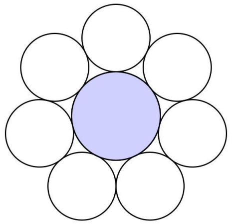
Doyle flower



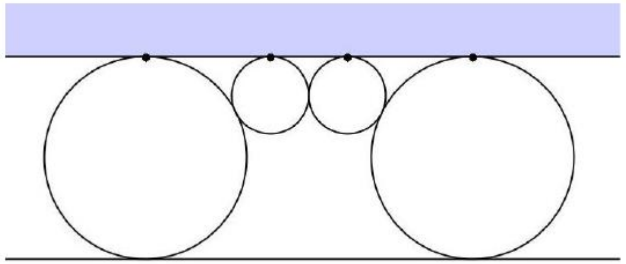
Soccerball flower



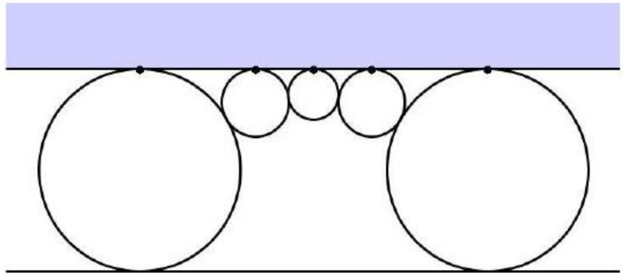
Ring Lemma flower



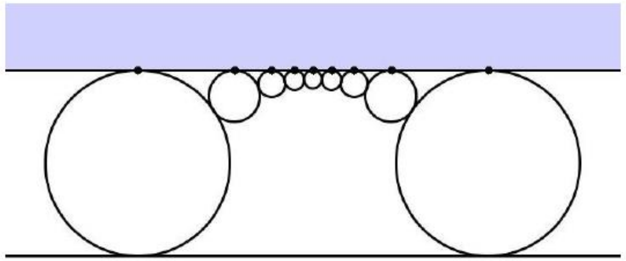
Uniform Flowers:



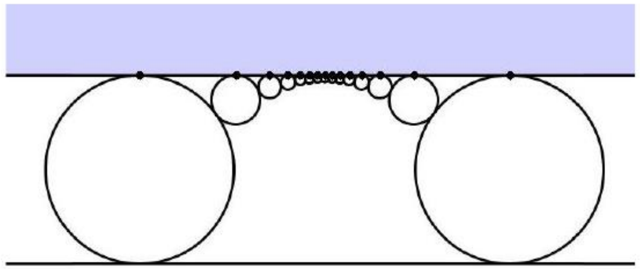
5-flower



6-flower



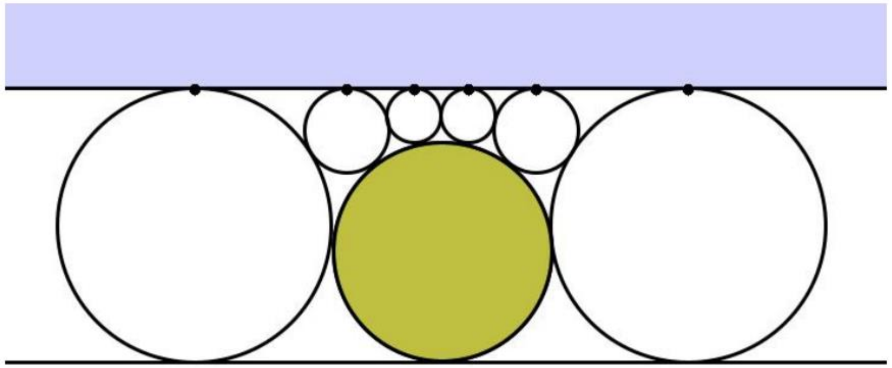
10-flower

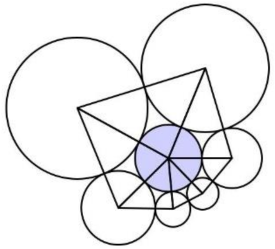


16-flower

For an *n*-flower: $$\mathfrak{s}_n = 1 - \frac{2\cos(\pi/n)}{\sqrt{3}}$$

More general: $$s = 1 - \frac{2\cos(\alpha)}{\sqrt{3}}$$





Doyle Flowers:

Two-parameter family: For $a > 0, b > 0$ the pattern of petal radii is:

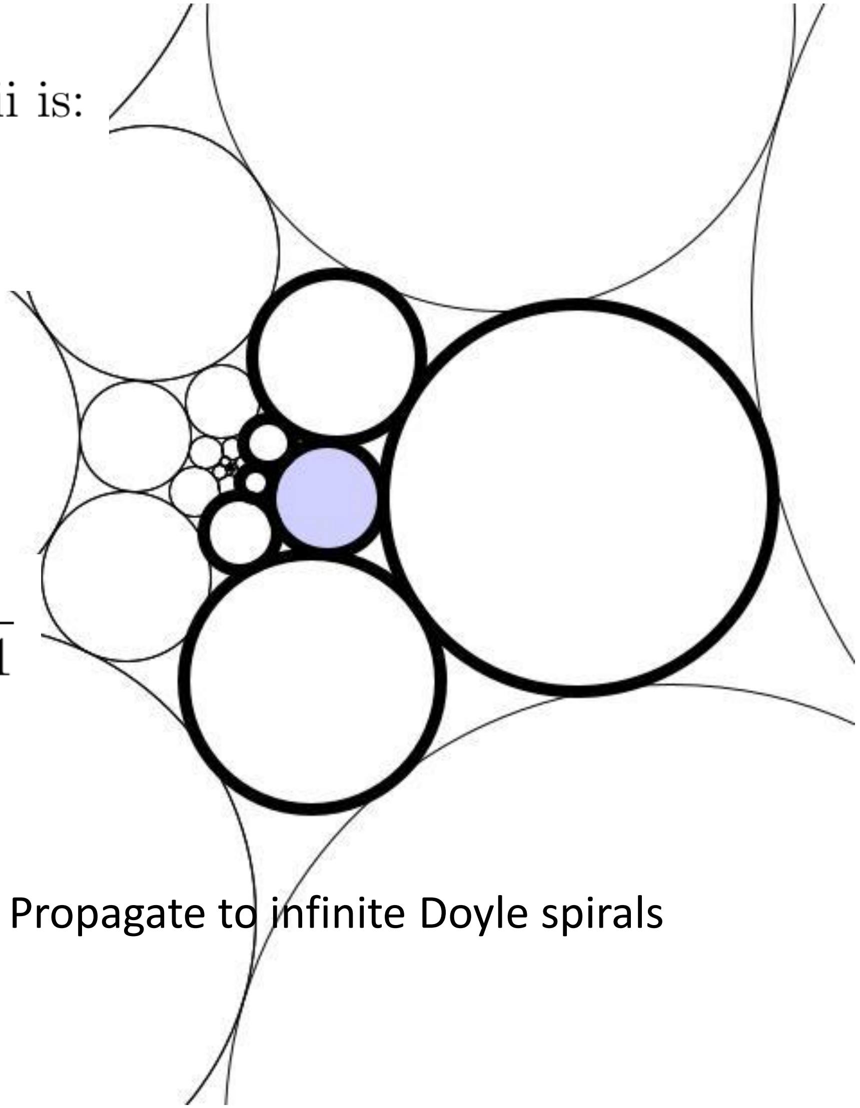
Radii: $$\{a,\ b,\ \frac{b}{a},\ \frac{1}{a},\ \frac{1}{b},\ \frac{a}{b}\}$$

Pairing similar faces, get repeat pattern of edge labels:

Edge labels: $$\{u_1, u_2, u_3,\quad u_1, u_2, u_3\}$$

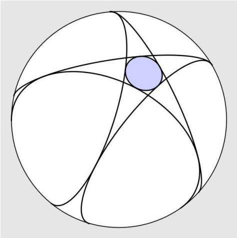
$$u_1 = \mathfrak{U}_6(u_1, u_2, u_3) = \frac{u_1 u_2}{3u_1u_2u_3 - u_1 - u_3} \implies u_3 = \frac{u_1 + u_2}{3u_1u_2 - 1}$$

So this is also a two-parameter family: $u_1 > 0, u_2 > 0$.



Propagate to infinite Doyle spirals

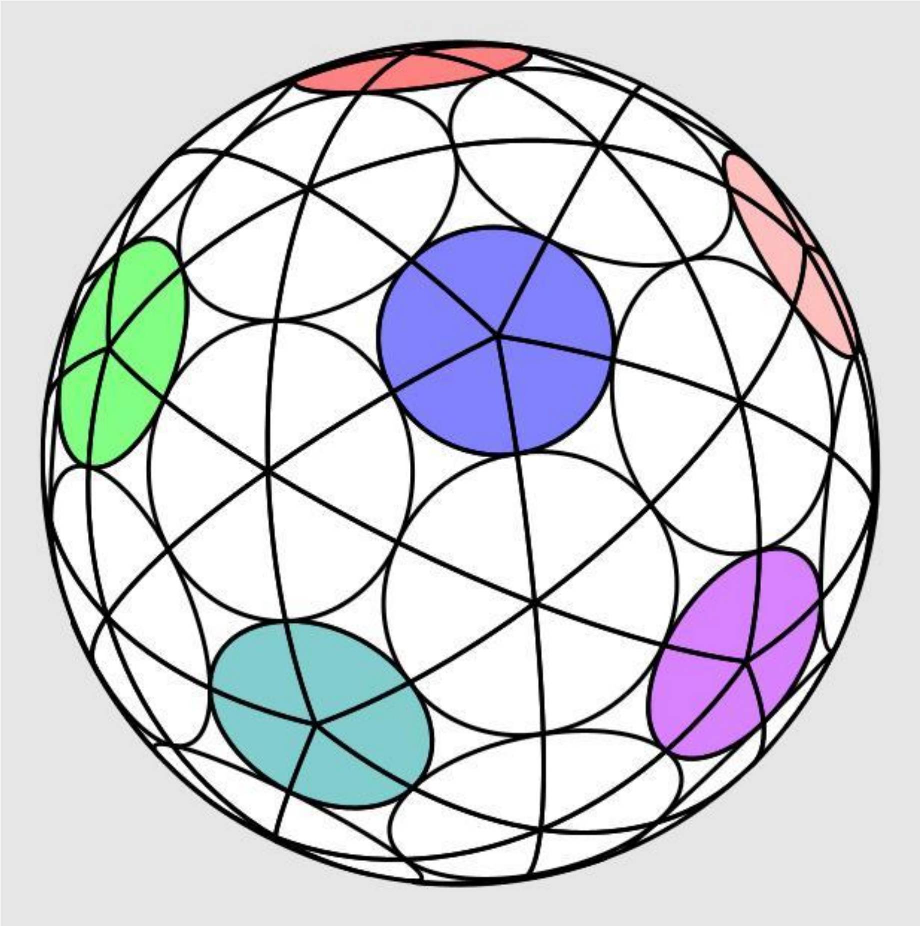
Are there other local patterns that propagate ad infinitum?
Perhaps in square-grid combinatorics?



Soccerball Flowers

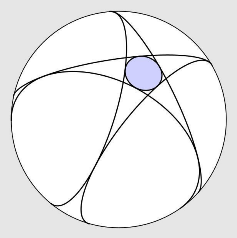
K and its max packing P_K have dodecahedral symmetry: 42 verts, 12 degree 5, 30 degree 6.

- By symmetry, $\exists$ just 2 schwarzians, s and s'.
- 5-flowers are uniform, so $u = 1 - s = \frac{2}{\sqrt{3}}\cos(\pi/5)$.
- 6-flowers have alternating pattern $\{u, u', u, u', u, u'\}$, thus $u' = \frac{uu'}{3uu'u - u - u'}$, which yields $uu' = 1$.



Therefore, for P_K:

$$s = 1 - \frac{2}{\sqrt{3}}\cos(\pi/5) \text{ and } s' = 1 - \frac{\sqrt{3}}{2}\sec(\pi/5).$$



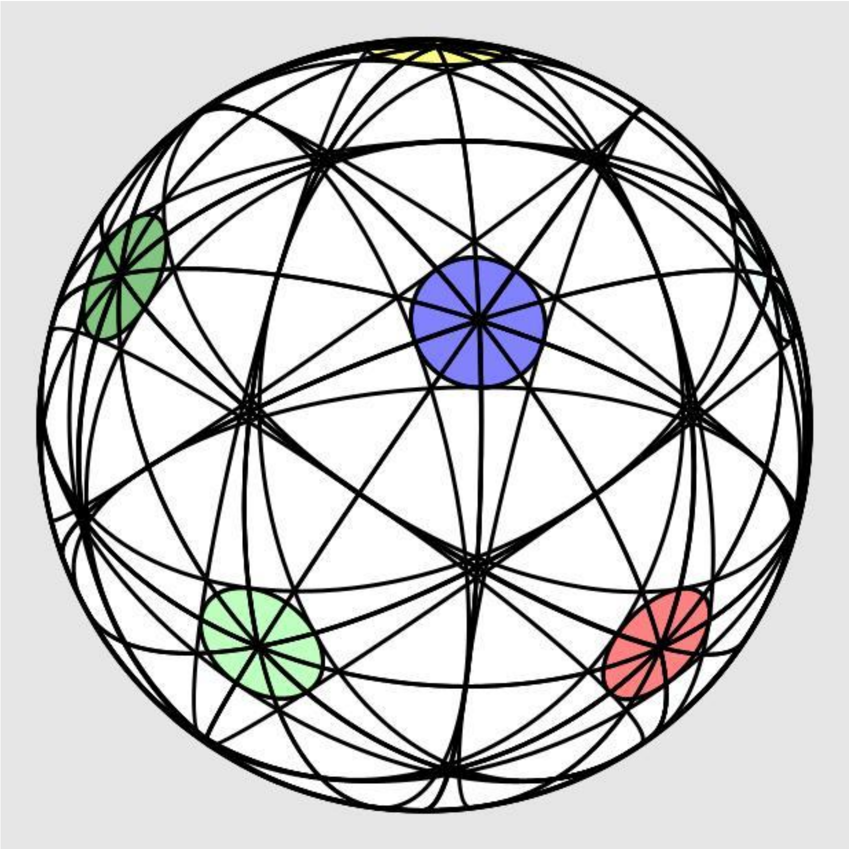
Branched Soccerball Flowers

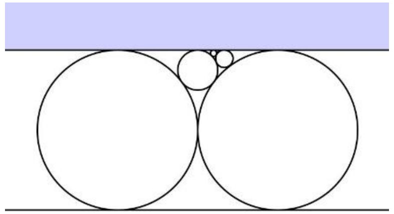
The same analysis applies here: there are just two schwarzians, s and s' and $(1-s)(1-s') = 1$. However, now the petals in each 5-degree flower wrap twice around the center, implying

$$s = 1 - \frac{2}{\sqrt{3}} \cos(2\pi/5) \text{ and } s' = 1 - \frac{\sqrt{3}}{2} \sec(2\pi/5).$$

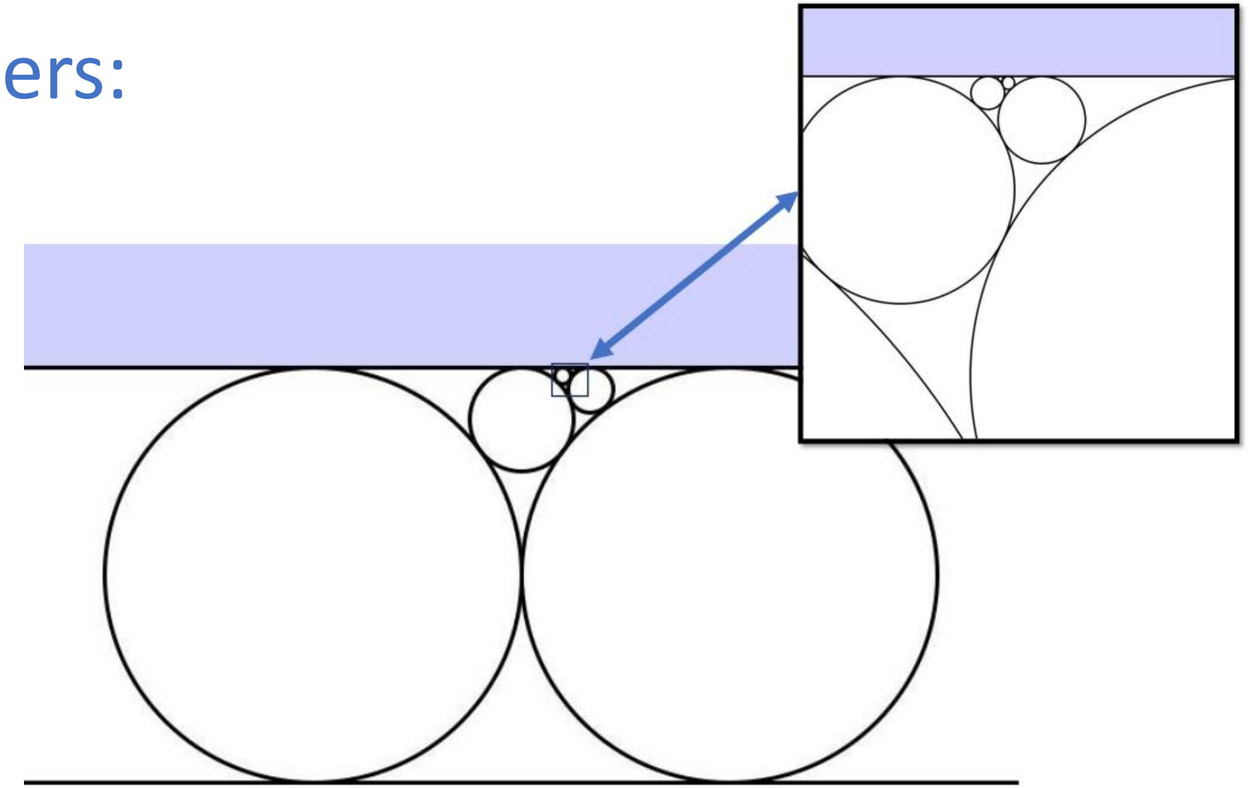
There are infinitely many pairs s, s' satisfying $(1-s)(1-s') = 1$ which can be used to lay out circle packings for this K.

In general, these generate **topological spheres** with **projective structures**, each having 12 **cone points**.





Ring Lemma Flowers:



The extremal flowers for the Rodin-Sullivan "ring" lemma have fascinating geometry, with connections to the **Descartes Circle Theorem**, **Farey arithmetic**, **Apollonian packings**, the **golden ratio,** etc.

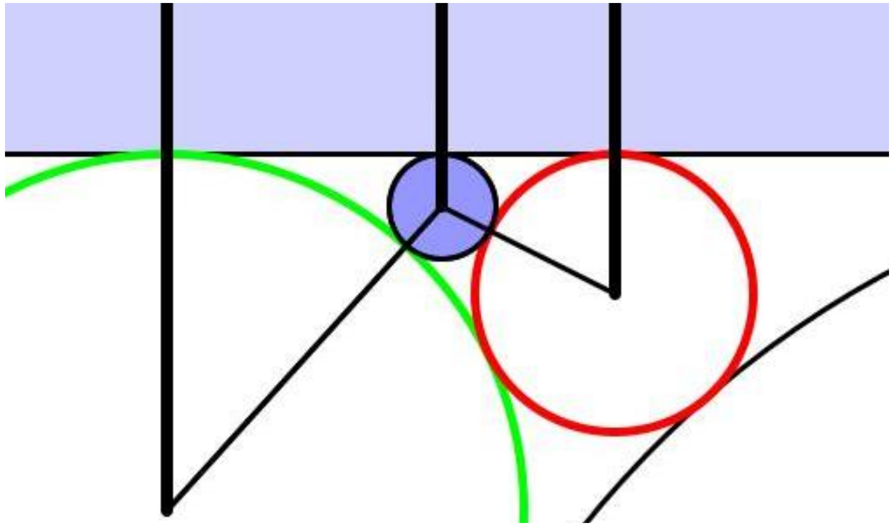
<u>Procedure</u>: Starting with 3 petals:

- Add a new petal in the smallest interstice,
- Repeat
- Repeat
- .....

If we stop when we have an *n*-flower, here is the sequence of labels:

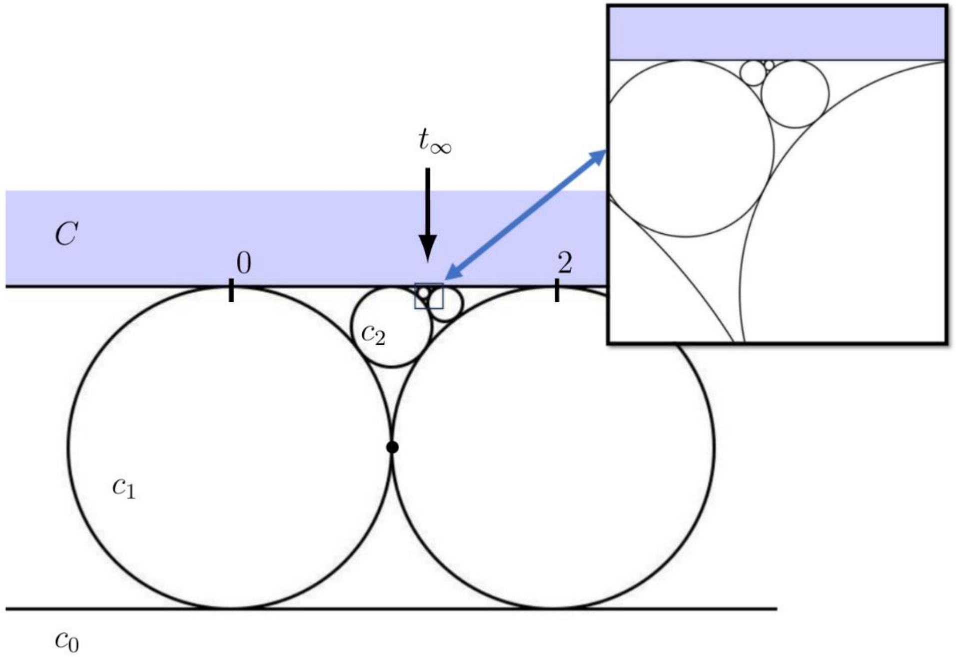
$$\{\sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{1}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ \sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{1}{\sqrt{3}}\}$$

Ring Lemma Steps:



$$\{\sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{1}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ \sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{1}{\sqrt{3}}\}$$

$$\{\sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{2}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \sqrt{3},\ \cdots\cdots,\ \sqrt{3},\ \frac{1}{\sqrt{3}}\}$$



$$\frac{1}{\sqrt{r_{j+1}}} = \frac{1}{\sqrt{r_j}} + \frac{1}{\sqrt{r_{j-1}}}$$

$$\Longrightarrow \quad \frac{r}{r'} \longrightarrow \left(\frac{1+\sqrt{5}}{2}\right)^2 = \tau^2$$

$$\Longrightarrow \quad t_\infty = 2/\tau$$

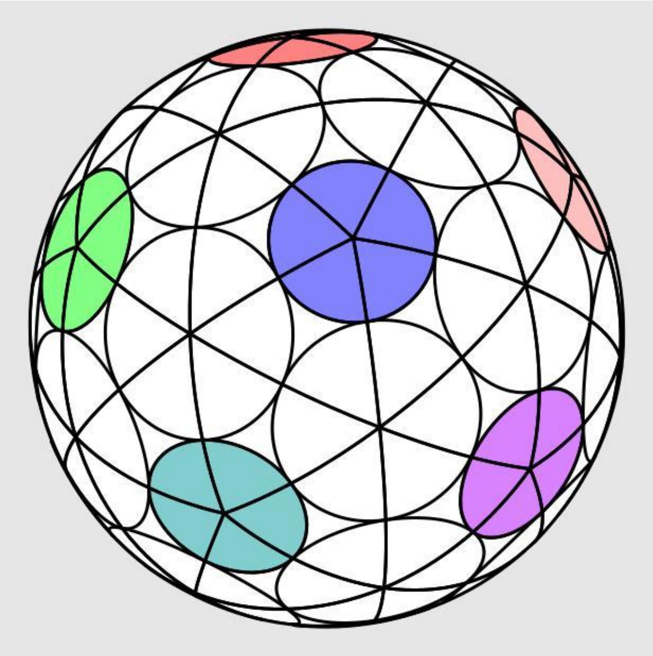
Conclusion: aren’t circles grand?

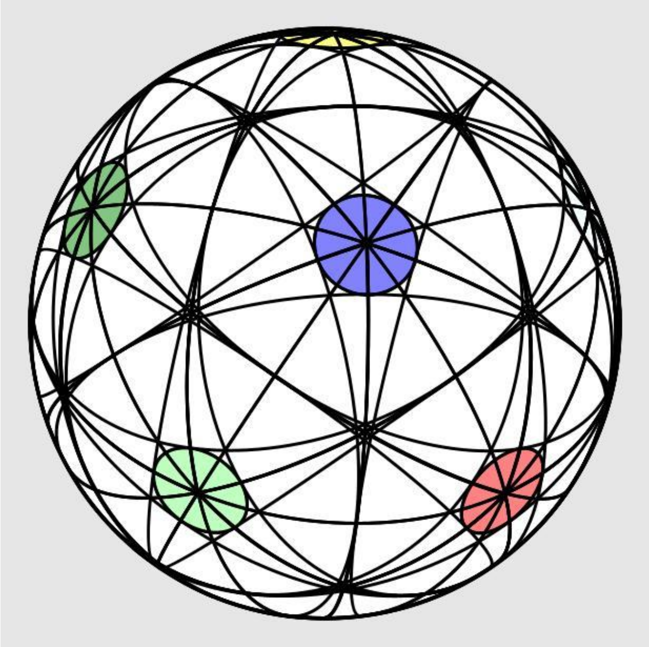
Concluding Comments:

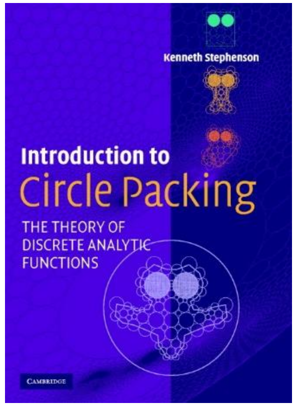
Main Question: How does one compute packing edge labels S for a complex K? The soccerball example relies on extreme symmetry.

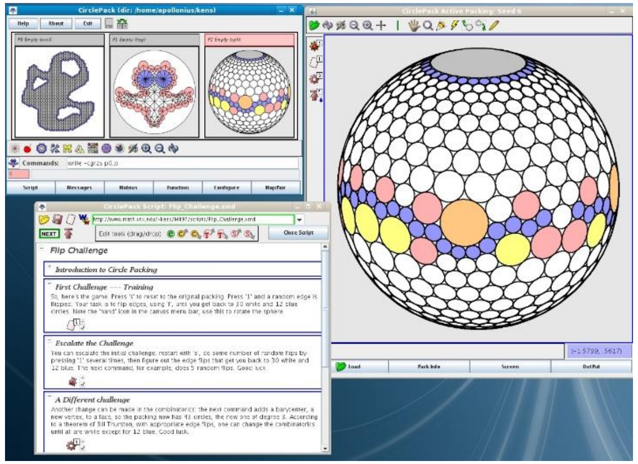
Is there a packing algorithm using schwarzians, which lack angle sums, radii, and their associated monotonicities?

In the meantime, there are some beautiful visual, geometric, and numerical results to pursue.

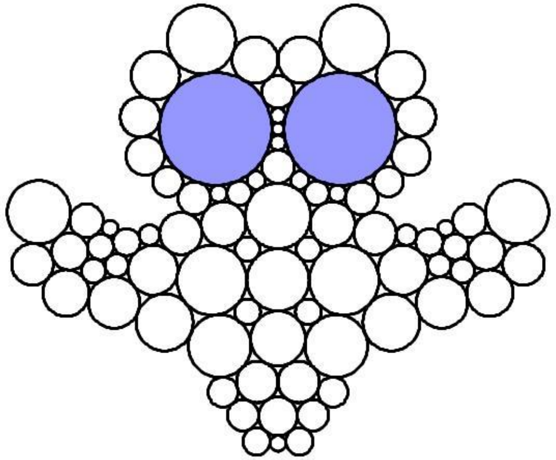








Thanks for your attention



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