# High-order semi-implicit schemes for evolutionary partial differential equations with higher order derivatives

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# Outline

- Introduction:
  1. convection-diffusion equations
  2. convection-dispersion equations
  3. Convection-biharmonic type equations
- Semi-Implicit (SI) strategy in an implicit-explicit (IMEX) Runge-Kutta (RK) setting
- Concluding remarks
- Work in progress: Semi-Implicit (SI) Rosenbrock-type and SI-Linear Multistep (LM) setting

# Introduction

We are interested in proposing a time discretization strategy for solving evolutionary PDEs containing high order spatial derivatives, such as:

- convection-diffusion equation

$$u_t + f(u)_x - (a(u)u_x)_x = 0,$$

  where $a(u) \geq 0$;

- convection-dispersion equation

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x = 0,$$

  where $r(u)$ and $g(u)$ are arbitrary (smooth) functions;

- the convection-biharmonic type equation

$$u_t + f(u)_x + (a(u_x)u_{xx})_{xx} = 0$$

We will use MOL (methods of lines), namely we first discretize the spatial derivatives to obtain an very large ODE system

$$\frac{dU}{dt} = F(U) + G(U),$$

where $F$ and $G$ are derived from the spatial discretization of the two parts.
The term $F(u)$ usually is non linear and $G$ contains the stiffness due to the discretization of the high-order space derivatives.

The spatial discretization could be:

- finite different method;
- finite volume method;
- finite element method, including discontinuous Galerkin method;
- spectral methods and so on ...

We assume that the spatial discretization is stable, that is, we assume that the solution to the methods of lines ODE satisfies

$$||U(t)|| \leq ||U(0)|| \tag{1}$$

(strong stability) or

$$||U(t)|| \leq C(t)||U(0)|| \tag{2}$$

for some constant $C(t)$ depending on $t$ (regular stability), for some norm, semi-norm, or convex functional $||\cdot||$, (e.g. $L^2$ norm, $L^\infty$ norm, total variation (TV) semi-norm, entropy, ....)

Our aim is to maintain the strong stability or the regular stability property with high accuracy time discretization under a reasonable time restriction.

Explicit time discretization, e.g. explicit Runge-Kutta or multistep methods: stability can be achieved under a severe time step restriction $\Delta t = \mathcal{O}(\Delta x^k)$, where $\Delta t$ is the time step and $\Delta x$ the spatial mesh size, for the $k$-th ($k \geq 2$) order PDEs.

Those problems for which the step size has to be restricted drastically to keep the computation stable are called: stiff.

Fully implicit methods, which can often be designed to be unconditionally stable, but with a large computational cost, as it would need to solve a large, nonlinear algebraic system for every time step.

The ODEs system

$$\frac{dU}{dt} = F(U) + G(U),$$

suggests the use of an IMplicit-EXplicit (IMEX) method.

The idea of the IMEX method is the following.

$F(U)$ is the nonlinear term corresponding to convection and lower order terms, then we treat it explicit, $G(U)$ is the stiff term corresponding to the highest order derivative and we treat implicit.

# IMEX-RK schemes

IMEX R-K schemes and the coefficients of the method are usually represented in a double Butcher tableau as

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}.$$

where $\tilde{A} = (\tilde{a}_{ij})$, is a $s \times s$ matrix for the explicit scheme, with $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (\tilde{a}_{ij})$ is a $s \times s$ matrix for the implicit one.

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, \quad c_i = \sum_{j=1}^{i} a_{ij}.$$

For the implicit part, diagonally implicit R-K (DIRK) schemes are often employed.

Usually it is useful to characterize the different IMEX-RK methods presented in the literature in two main types:

- We call an IMEX RK method of type I (or type A) if the matrix $A \in \mathbb{R}^{s\times s}$ is invertible, or equivalently $a_{ii} \neq 0$ for all $i$;
- We call an IMEX RK method of type II (or type CK) if the matrix $A \in \mathbb{R}^{s\times s}$ can be written

$$\begin{pmatrix} 0 & 0 \\ a & \hat{A} \end{pmatrix},$$

with $a = (a_{21}, a_{31}, ..., a_{s1})^{\top} \in \mathbb{R}^{s-1}$ and the sub-matrix $\hat{A} \in \mathbb{R}^{(s-1)\times(s-1)}$ is invertible.

In the special case $a = 0,\ b_1 = 0$, the method is said to be of type ARS, and the DIRK method is reducible to a method using $s - 1$ stages.

# Linear Stability results

If the leading high order spatial derivative term is linear, e.g. convection-diffusion equation

$$u_t + f(u)_x = du_{xx},$$

or the convection-dispersion equation

$$u_t + f(u)_x = -du_{xxx},$$

or the convection-biharmonic type equation

$$u_t + f(u)_x = -du_{xxxx},$$

with $d > 0$.

Theoretical stability results:

- for convection-diffusion an convection-biharmonic equations, the IMEX scheme is unconditionally stable when $d$ is large enough; if $d$ is very small and $\Delta x$ is not too small, stability holds under the usual CFL condition $\Delta t \leq C\Delta x$, (see: Wang, Shu, and Zhang, SINUM 2015; AMC 2016)
- for the convection-dispersion type equation the IMEX scheme is stable under the CFL condition, $\Delta t \leq C\Delta x$. (See: Tan, Cheng and Shu, IJNAM 2021).

- If the highest order space derivative term is nonlinear, then a straight-forward application of the IMEX schemes above may not be efficient, as we still must solve a nonlinear algebraic system per implicit stage.

  Therefore we have to adopt another strategy in order to avoid nonlinear algebraic systems.

- Explicit-implicit-null time-marching (M. Tan, J. Cheng, C. W. Shu, JCP 2022) . This method consists to add and subtract two equal sufficiently large linear highest derivative terms with constant coefficients $a_0$ (to be determined, $a_0 \geq 0.54$) on one side of the considered equation.
  After that, the linear highest derivative term is treated implicitly, while the remaining term is treated explicitly using an IMEX R-K setting.

- An alternative approach. The idea is to propose the strategy called *semi-implicit* (SI) introduced in S.B., F. Filbet, G. Russo JSC 2016 based on IMEX R-K schemes (SI-IMEX-RK), for the solution of different types of equations.

# Nonlinear diffusion equation

In the following we take the nonlinear diffusion equation

$$u_t = (a(u)u_x)_x, \quad a(u) \geq 0, \tag{3}$$

as an example to introduce the SI approach in details.

Assume that the semi-discrete formulation of (3) can be written as

$$\frac{dU(t)}{dt} = \frac{1}{\Delta x^2}\mathcal{B}(U(t))U(t), \tag{4}$$

where $U(t) = (U_1(t), U_2(t), ..., U_M(t))^T$, with $U_i(t) \approx u(x_i, t)$, for $i = 1, ..., M$, and $\Delta x_i = x_{i+1} - x_i$, $\mathcal{B} \in R^{M\times M}$ is a tridiagonal matrix arising from the discretization.

In the term $\mathcal{B}(U(t))U(t)$ the occurrence of the solution $U$ within $\mathcal{B}(U(t))$ is considered as non-stiff, while that of the factor $U(t)$ as stiff.

Thus the implicit treatment is applied only to the linear factor $U$, while $\mathcal{B}(U(t))$ is treated explicitly. This approach avoids the solution of nonlinear systems.

# First order SI scheme

$$U^{n+1} = U^n + \frac{\Delta t}{\Delta x^2} \mathcal{B}(U^n) U^{n+1},$$

$$U^{n+1} = \left( I - \frac{\Delta t}{\Delta x^2} \mathcal{B}(U^n) \right)^{-1} U^n.$$

This scheme is *strong stable* in the sense that $||U^n|| \leq ||U^0||$ for all $n$ and for any positive time-step $\Delta t > 0$, if

$$\rho \left( \left( I - \frac{\Delta t}{\Delta x^2} \mathcal{B}(U^n) \right)^{-1} \right) < 1.$$

$$[(a(u)u_x)_x]_{x_j} \to \mathcal{B}(U(t)) =$$
$$\frac{1}{\Delta x^2} \left( a_{j+1/2} U_{j+1}(t) - (a_{j+1/2} + a_{j-1/2}) U_i(t) + a_{j-1/2} U_{j-1}(t) \right).$$

with

$$a_{j+1/2} = \frac{a(U_{j+1}(t)) + a(U_j(t))}{2}.$$

# High order semi-implicit R-K (SI-RK) temporal discretization

Inspired by the approach outlined in S.B., F. Filbet, G. Russo, JSC 2016, and by (4), we start to consider a more general class of autonomous problems of the form

$$u' = F(u,u), \quad u(t_0) = u_0. \tag{5}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently differentiable and the right-hand side has a stiff dependence only on the last argument.

Introducing an auxiliary variable $u^*$, (5) can rewrite system as a partitioned one

$$\begin{cases} \dfrac{du^*}{dt} = F(u^*, u), \\ \dfrac{du}{dt} = F(u^*, u), \end{cases}$$

with initial conditions $u^*(t_0) = u(t_0) = u_0$. Note that in the continuous setting, we have that: $u(t) = u^*(t)$.

- The diffusion and convection-diffusion equations:

$$F(u^*, u) = (a(u^*)u_x)_x, \quad F(u^*, u) = -f(u^*)_x + (a(u^*)u_x)_x,$$

- Case nonlinear dispersive equation with third derivatives

$$F(u^*, u) = -((u^*)^m)_x - (u^*(a(u^*)u_x)_x)_x$$

with $a(u) = nu^{n-1}$.

- Fourth order diffusion equation

$$F(u^*, u) = -f(u^*)_x - (a(u^*_x)u_{xx})_{xx}$$

In practice, after discretizing the spatial operators of Eqs. on a chosen space grid with $U(t) = (U_1(t), U_2(t), ...., U_s(t))^\top$ and $U_i(t) \approx u(x_i, t)$, for $i = 1, ..., M$ , they are converted into a semi-discrete form. Specifically,

$$F(U^*(t), U(t)) = \mathcal{B}(U^*(t))U(t),$$

and

$$F(U^*(t), U(t)) = \mathcal{F}(U^*(t)) + \mathcal{B}(U^*(t))U(t),$$

where $\mathcal{F} : \mathbb{R}^M \to \mathbb{R}^M$ and $\mathcal{B}(U^*(t))$ a $M \times M$ matrix. Then the resulting SI-IMEX-RK scheme can be applied to:

$$\begin{cases} \dfrac{dU(t)^*}{dt} = F(U^*(t), U(t)), \\ \dfrac{dU(t)}{dt} = F(U^*(t), U(t)), \end{cases}$$

$$\begin{cases} \dfrac{dU(t)^*}{dt} = F(U^*(t), U(t)), \\ \dfrac{dU(t)}{dt} = F(U^*(t), U(t)), \end{cases}$$

The SI-IMEX-RK method is implemented as follows.
First we set $U_n^* = U_n$ and compute for $i = 1, ..., s$, the RK fluxes $K_i$ as basic unknowns

$$K_i = F(U_i^*, \bar{U}_i + \Delta t a_{ii} K_i), \quad i = 1, ..., s$$

where $U_i^* = U_n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} K_j$ and $\bar{U}_i = U_n + \Delta t \sum_{j=1}^{i-1} a_{ij} K_j$, with the numerical solution

$$U_{n+1} = U_n + \Delta t \sum_{i=1}^{s} b_i K_i,$$

where we set $\tilde{b}_i = b_i$, for $i = 1, ..., s$. Let us point out that the duplication of the unknowns in the system does not occur when the coefficients $\tilde{b}_i = b_i$ are chosen in the IMEX-RK scheme.

For instance, the nonlinear diffusion system can be expressed as linear equations in terms of $K_i$:

$$K_i = \mathcal{B}(U_i^*)\left(\bar{U}_i + \Delta t a_{ii} K_i\right),$$

where $\bar{U}_i = U_n + \Delta t \sum_{j=1}^{i-1} a_{ij} K_j$ and the matrix $\mathcal{B}(U_i^*)$ is computed explicitly.

Semi-implicit approach:

- This approach avoids the solution of nonlinear algebraic systems typically associated with implicit methods, and stringent time stepping constraint usually required by an explicit method.
- This SI strategy is really convenient and useful in the case in which we have a linearly implicit evaluation for the unknown variable in the term involving high order spatial derivatives.

# Semi-implicit approach

This SI-IMEX-RK strategy have been already used for solving:

- Relaxation problems containing degenerate and/or fully nonlinear diffusion terms (S.B., F. G. LeFloch G. Russo SIAM JSC 2014);
- A class of degenerate convection-diffusion problems (S.B., R. Bürger, P. Mulet, G. Russo, L.M. Villada, SIAM JSC 2015),
- Surface diffusion of graph, (S.B., F. Filbet, G. Russo JSC 2016).
- For All-Mach Full Euler System of Gas Dynamics (S.B. J. Qiu, G. Russo, T. Xiong, SIAM JSC, 2022);
- Shallow water equations with Non-Flat Bottom Topography and Manning Friction Term (S.B., G. Huang, S. Boscarino, T. Xiong, Computational Methods in Applied Mathematics, 2025);

# The Spatial discretization

- SI strategy is combined with *finite difference schemes.*
- We choose the finite difference schemes because of its simplicity in design and coding. Furthermore, it is it is straightforward to extend to higher-dimensional equations.
- However, other suitable space discretization can be considered as: *Finite volume, local discontinuous Galerkin schemes*.

# The Spatial discretization for the nonlinear diffusion equation

The following formula provides a fourth order approximation to $(a(u)u_x)_x$, with a five-point stencil

$$(a(u)u_x)_x|_{x_j} \approx \frac{1}{\Delta x^2}(a_{j-2}, a_{j-1}, a_j, a_{j+1}, a_{j+2}) \begin{pmatrix} -25/144 & 1/3 & -1/4 & 1/9 & -1/48 \\ 1/6 & 5/9 & -1 & 1/3 & -1/18 \\ 0 & 0 & 0 & 0 & 0 \\ -1/18 & 1/3 & -1 & 5/9 & 1/6 \\ -1/48 & 1/9 & -1/4 & 1/3 & -25/144 \end{pmatrix} \begin{pmatrix} u_{j-2} \\ u_{j-1} \\ u_j \\ u_{j+1} \\ u_{j+2} \end{pmatrix}$$

Note that in the case $a(u) = 1$ the formula becomes the classical fourth order central finite difference scheme, i.e.,

$$\mathcal{D}_x^2 u_j = \frac{1}{12\Delta x^2}(-u_{j-2} + 16u_{j-1} - 30u_j + 16u_{j+1} - u_{j+2}). \qquad (6)$$

# High order space discretization for the nonlinear dispersive equation.

$$u_t + (u^m)_x + (u(u^n_{xx}))_x = 0, \quad m > 1, \quad m = n + 1.$$

For the nonlinear term $(u(u^n_{xx}))_x$, we consider again the equivalent nonlinear term:

$$(u(a(u)u_x)_x)_x, \tag{7}$$

with $a(u) = nu^{n-1}$. Then, to approximate the term (7) we use the following fourth order finite difference spacial approximation,

$$\mathcal{D}_x \left(u_j^* \mathcal{D}_x(a(u_j^*)\mathcal{D}_x(u_j))\right),$$

where

$$\mathcal{D}_x u_j = \frac{-(u_{j+2} - u_{j-2}) + 8(u_{j+1} - u_{j-1})}{12\Delta x}, \tag{8}$$

is a fourth order central approximation for $u_x(x_j)$.

# The spatial discretization for the biharmonic-type equation

The fourth order finite difference scheme for the fourth order spatial derivative in the biharmonic-type equation

$$u_t + (a(u_x)u_{xx})_{xx} = 0,$$

can be written as

$$\mathcal{D}_x^2 \left( a(\mathcal{D}_x u_j^*)\mathcal{D}_x^2 u_j \right), \tag{9}$$

where $\mathcal{D}_x^2 u_j$ is a fourth order centered difference approximation to $u_{xx}(x_j)$ defined as (6). Note that when $a(u_x) = 1$ we get from (9), $\mathcal{D}_x^2\mathcal{D}_x^2 u_j$, i.e., the central difference approximation for the fourth order derivative $u_{xxxx}$.

# Numerical Results

- In each of the numerical tests, we consider the third order IMEX-RK schemes for the time discretization and we provide the time step $\Delta t$ that will produce stable solutions across the entire solution domain.
- The list of third-order IMEX-RK schemes presented are:
  1. schemes of type II-ARS: ARS(3,4,3) and ARS(4,4,3) with $a_{11} = 0$, $\tilde{c}_i = c_i$, for $i = 2, ..., s$.
  2. scheme of type I: SSP-DIRK3(4,3,3) and I-IMEX(3,4,3) have $a_{11} \neq 0$, and $\tilde{c}_i = c_i$, for $i = 2, .., s$.

# The second order diffusion equations

**Test 1.** Nonlinear diffusion equation

$$u_t = (a(u)u_x)_x + f(x,t), \quad x \in (-\pi, \pi),$$

with $a(u) = u^2 + 1$, initial condition $u(x,0) = \sin(x)$, source term $f(x,t)$ is chosen such that the exact solution is $u(x,t) = \sin(x-t)$.

Time step $\Delta t = C\Delta x$, with $C = 1, 10$ and $\Delta x = 2\pi/N$, periodic boundary conditions and final time $T = 10$.

| Scheme | N | $L^2$-error | order | $L^1$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|---|---|
| ARS(3,4,3) | 80 | 1.4115e-05 | - | 1.3084e-05 | - | 1.8073e-05 | - |
| | 160 | 1.7174e-06 | 3.03 | 1.5352e-06 | 3.09 | 2.2652e-06 | 3.00 |
| | 320 | 2.2800e-07 | 2.91 | 2.0045e-07 | 2.94 | 3.0734e-07 | 2.88 |
| | 640 | 2.8506e-08 | 3.00 | 2.4834e-08 | 3.01 | 3.8601e-08 | 2.99 |
| | 1280 | 3.5450e-09 | 3.00 | 3.0949e-09 | 3.00 | 4.8778e-09 | 2.98 |
| ARS(4,4,3) | 80 | 9.1092e-06 | - | 7.8450e-06 | - | 1.1315e-05 | - |
| | 160 | 1.1875e-06 | 2.93 | 1.1106e-06 | 2.82 | 1.3913e-06 | 3.02 |
| | 320 | 1.6670e-07 | 2.83 | 1.5803e-07 | 2.81 | 2.0969e-07 | 2.73 |
| | 640 | 2.2076e-08 | 2.91 | 2.0831e-08 | 2.92 | 2.8563e-08 | 2.88 |
| | 1280 | 2.8841e-09 | 2.93 | 2.6930e-09 | 2.95 | 3.8139e-09 | 2.90 |
| SSP-DIRK3(4,3,3) | 80 | 4.6382e-05 | - | 6.4241e-05 | - | 7.6849e-05 | - |
| | 160 | 9.1835e-06 | 2.34 | 1.0143e-05 | 2.66 | 1.2152e-05 | 2.66 |
| | 320 | 1.2390e-06 | 2.89 | 1.6282e-06 | 2.64 | 2.0006e-06 | 2.60 |
| | 640 | 1.6372e-07 | 2.92 | 2.1761e-07 | 2.90 | 2.6973e-07 | 2.89 |
| | 1280 | 2.1516e-08 | 2.93 | 2.9577e-08 | 2.87 | 3.7484e-08 | 2.85 |
| I-IMEX(3,4,3) | 80 | 7.9110e-05 | - | 7.5889e-05 | - | 8.8279e-05 | - |
| | 160 | 1.1497e-05 | 2.78 | 1.0928e-05 | 2.80 | 1.5516e-05 | 2.50 |
| | 320 | 1.8343e-06 | 2.65 | 1.7342e-06 | 2.66 | 2.6851e-06 | 2.53 |
| | 640 | 2.4884e-07 | 2.89 | 2.3356e-07 | 2.89 | 3.6901e-07 | 2.86 |
| | 1280 | 3.4581e-08 | 2.85 | 3.2183e-08 | 2.86 | 5.2026e-08 | 2.83 |

Table: The $L^2$, $L^1$, $L^\infty$ errors and orders of accuracy and $C = 1$.

| Scheme | N | $L^2$-error | order | $L^1$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|---|---|
| I-IMEX(3,4,3) | 80 | 5.0659e-02 | - | 5.3755e-02 | - | 4.9010e-02 | - |
| | 160 | 7.4379e-03 | 2.76 | 7.5014e-03 | 2.84 | 8.4932e-03 | 2.52 |
| | 320 | 1.3316e-03 | 2.48 | 1.3237e-03 | 2.50 | 1.5276e-03 | 2.48 |
| | 640 | 1.6673e-04 | 3.00 | 1.6038e-04 | 3.04 | 2.1040e-03 | 2.86 |
| | 1280 | 2.1751e-05 | 2.93 | 2.0595e-05 | 2.96 | 2.8863e-05 | 2.87 |
| ARS(3,4,3) | 80 | 3.5769e-03 | - | 3.4828e-03 | - | 4.6953e-03 | - |
| | 160 | 4.2474e-04 | 3.07 | 4.3510e-04 | 3.00 | 4.0581e-04 | 2.50 |
| | 320 | 1.2740e-04 | 1.73 | 1.1228e-04 | 1.95 | 1.7068e-04 | 1.25 |
| | 640 | 1.9314e-05 | 2.72 | 1.6766e-05 | 2.74 | 2.5721e-05 | 2.73 |
| | 1280 | 2.7838e-06 | 2.79 | 2.4141e-06 | 2.80 | 1.1315e-05 | 2.81 |

Table: The $L^2$, $L^1$, $L^\infty$ errors and orders of accuracy and $C = 10$.

We consider the porous media equation (PME)

$$u_t = (u^m)_{xx}, \quad m > 1.$$

It can be written as

$$u_t = (a(u)u_x)_x, \quad a(u) = mu^{m-1}.$$

with $F(u^*, u) = (a(u^*)u_x)_x$.
One of the famous solution for PME is the *weak Barenblatt solution*.
The boundary condition is $u(\pm 6, t) = 0$ for $t > 1$ and we use uniform mesh with $N = 160$ points and time step $\Delta t = \Delta x$.
In this simulation we adopted a not suitable space discretization for the PME equation, in the literature several optimal discretization are introduced for PME equation, for instance, in Y. Liu, C-W Shu, M. Zhang, SIAM J.Sci.Comput. 2011 an adaptation of the WENO technique has been proposed in oder to maintain conservation, accuracy and non-oscillatory performance.

In figure we plot the numerical results for $m = 2, 3, 5$ at $t = 2$.

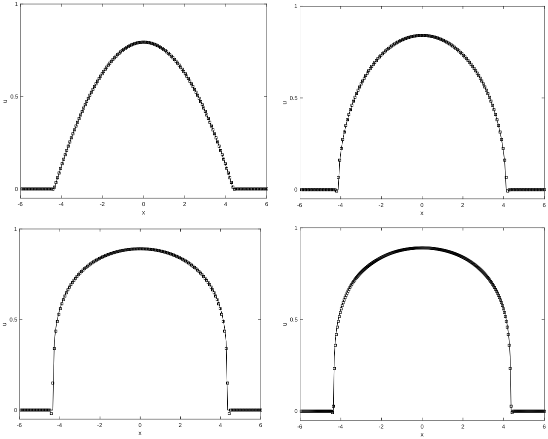


Figure: Numerical results of the Barenblatt solution for the PME.

Undershoot is reduced when we take more mesh points, for example $N = 320$.

An example of a strongly degenerate parabolic convection-diffusion equation:

$$u_t + f(u)_x = \epsilon(a(u)u_x)_x, \quad a(u) \geq 0.$$

$\epsilon = 0.1$, $f(u) = u^2$ and

$$a(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25. \end{cases}$$

The equation is of hyperbolic nature when $u \in [-0.25, 0.25]$ and parabolic elsewhere.
The initial data

$$u(x,0) = \begin{cases} 1, & -\frac{1}{\sqrt{2}} - 0.4 < x < -\frac{1}{\sqrt{2}} + 0.4 \\ -1, & \frac{1}{\sqrt{2}} - 0.4 < x < \frac{1}{\sqrt{2}} + 0.4 \\ 0, & \text{otherwise} \end{cases}$$

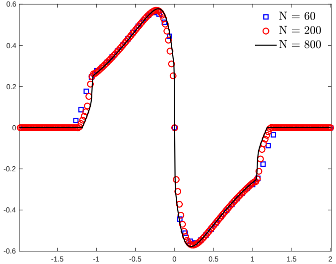


Figure: Final Time: T = 0.7.

To set $\Delta t$, we consider the classical hyperbolic CFL condition

$$\max_u |f'(u)| \frac{\Delta t}{\Delta x} = 1.0.$$

The scheme provides the high resolution of discontinuities and the accurate transition between the hyperbolic and parabolic regions.

# Third order dispersive equation

We consider the general KdV equation

$$u_t + (u^3)_x + (u(u^2)_{xx})_x = 0, \quad x \in \left(-\frac{3}{2}\pi, \frac{5}{2}\pi\right),$$

with initial condition $u(x,0) = \sqrt{2\lambda}\cos(x/2)$ and exact solution

$$u(x,t) = \sqrt{2\lambda}\cos((x-\lambda t)/2).$$

We compute to $T = \pi$, with $\lambda = 0.1$. We choose $\Delta t = \Delta x$ for scheme of type I and $\Delta t = 0.5\Delta x$ for scheme of type II, as ARS.

| Scheme | N | $L^2$-error | order | $L^1$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|---|---|
| ARS(3,4,3) | 80 | 8.3725e-04 | - | 7.4281e-04 | - | 1.3917e-03 | - |
| | 160 | 1.0635e-04 | 2.98 | 9.2883e-05 | 3.00 | 2.6707e-04 | 2.38 |
| | 320 | 1.3409e-05 | 2.99 | 1.1537e-05 | 3.00 | 4.4911e-05 | 2.57 |
| | 640 | 1.6916e-06 | 2.99 | 1.4396e-06 | 3.00 | 7.3264e-06 | 2.64 |
| | 1280 | 2.1616e-07 | 2.97 | 1.8187e-07 | 2.98 | 1.1467e-06 | 2.68 |
| SSP-DIRK3(4,3,3) | 80 | 8.3727e-04 | - | 7.4283e-04 | - | 1.3918e-03 | - |
| | 160 | 1.0635e-04 | 2.97 | 9.2883e-05 | 3.00 | 2.6706e-04 | 2.38 |
| | 320 | 1.3409e-05 | 2.98 | 1.1537e-05 | 3.00 | 4.4912e-05 | 2.57 |
| | 640 | 1.6916e-06 | 2.98 | 1.4397e-06 | 3.00 | 7.3266e-06 | 2.62 |
| | 1280 | 2.1307e-07 | 2.99 | 1.7959e-07 | 3.00 | 1.1467e-06 | 2.68 |
| I-IMEX(3,4,3) | 80 | 8.3745e-04 | - | 7.4297e-04 | - | 1.3924e-03 | |
| | 160 | 1.0637e-04 | 2.98 | 9.2901e-05 | 3.00 | 2.6718e-04 | 2.38 |
| | 320 | 1.3412e-05 | 2.99 | 1.1539e-05 | 3.00 | 4.4927e-05 | 2.57 |
| | 640 | 1.6919e-06 | 2.99 | 1.4399e-06 | 3.00 | 7.3288e-06 | 2.62 |
| | 1280 | 2.1311e-07 | 2.99 | 1.7963e-07 | 3.00 | 1.1470e-06 | 2.68 |

Table: The $L^2$, $L^1$, $L^\infty$ errors and orders of accuracy for general KdV equation.

Note that, since $\lambda$ represents the velocity of the traveling wave, selecting a larger value of $\lambda$ requires a careful choice of the time step to ensure the stability of the method. In such cases, we set the time step as

$$\Delta t = CFL \frac{\Delta x}{\max_u |f'(u)|}$$

As an example, we take $\lambda = 10$ and $CFL = 0.5$.

| Scheme | N | $L^2$-error | order | $L^1$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|---|---|
| ARS(3,4,3) | 80 | 1.0829e-01 | - | 1.0793e-01 | - | 1.1449e-01 | - |
| | 160 | 1.7007e-02 | 2.67 | 1.6973e-02 | 2.66 | 1.8247e-02 | 2.64 |
| | 320 | 2.3665e-03 | 2.84 | 2.3598e-03 | 2.84 | 2.4543e-03 | 2.89 |
| | 640 | 3.1044e-04 | 2.93 | 3.1027e-04 | 2.92 | 3.1433e-04 | 2.96 |
| I-IMEX(3,4,3) | 80 | 1.0829e-01 | - | 1.0793e-01 | - | 1.1449e-01 | - |
| | 160 | 1.7009e-02 | 2.67 | 1.6974e-02 | 2.66 | 1.8246e-02 | 2.64 |
| | 320 | 2.3667e-03 | 2.84 | 2.3600e-03 | 2.84 | 2.4550e-03 | 2.89 |
| | 640 | 3.1046e-04 | 2.93 | 3.1029e-04 | 2.92 | 3.1437e-04 | 2.96 |

Table: The $L^2$, $L^1$ and $L^\infty$ errors and orders of accuracy for general KdV equation.

We consider nonlinear dispersive Eq.

$$u_t + (u^m)_x + (u(u^n_{xx}))_x = 0, \quad m > 1, \quad m = n + 1$$

with $m = 2$ and $n = 1$. Initial data

$$u(x,0) = \begin{cases} 3\cos^2(x/4) & |x| \leq 2\pi, \\ 0, & |x| > 2\pi \end{cases}.$$

$N = 200$ and CFL condition

$$\Delta t = 0.5 \frac{\Delta x}{\max_u |f'(u)|}.$$

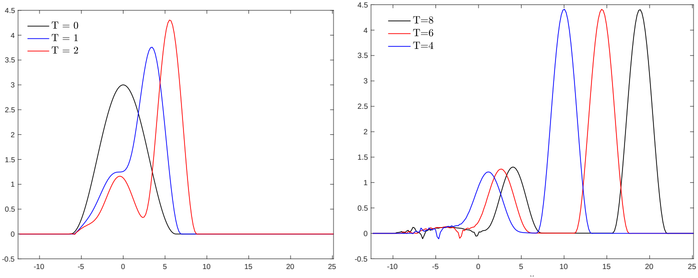


Figure: Solution with $m = 2$ and $n = 1$, $N = 200$ and time $T = 0, 1, 2, 4, 6, 8$.

The results of the SI approach coupled with WENO discretization suffer from spurious oscillations in the tail for $T = 4, 6, 8$.
Note that our approach does not solve correctly the oscillations in the tail but it is able with a large time step $\Delta t$ to control the oscillations and ensure that the solution does not blow up for a long time.

# Fourth order diffusion equation

We consider the biharmonic-type equation with a flux term:

$$u_t + (u^3)_x + (u^2 u_{xx})_{xx} = f(x,t), \quad x \in (-\pi, \pi).$$

initial condition $u(x,0) = \sin(x)$ and the source term $f(x,t)$ is chosen such that the exact solution is

$$u(x,t) = e^{-2t}\sin(x).$$

We compute the solution to $T = 1$ with the time step $\Delta t = \Delta x$.

| Scheme | N | $L^2$-error | order | $L^1$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|---|---|
| SSP-DIRK(4,3,3) | 80 | 8.9164e-04 | - | 8.351e-04 | - | 1.3096e-03 | - |
| | 160 | 1.1018e-04 | 3.01 | 1.0114e-04 | 3.04 | 1.6713e-04 | 2.97 |
| | 320 | 1.3285e-05 | 3.05 | 1.2028e-05 | 3.07 | 2.0485e-05 | 3.02 |
| | 640 | 1.5942e-06 | 3.05 | 1.4288e-06 | 3.07 | 2.4813e-06 | 3.04 |
| | 1280 | 1.9294e-07 | 3.04 | 1.6872e-07 | 3.08 | 3.2165e-07 | 2.95 |
| I-IMEX(3,4,3) | 80 | 4.8321e-04 | - | 4.7076e-04 | - | 5.5627e-04 | - |
| | 160 | 5.9002e-05 | 3.03 | 5.4596e-05 | 3.10 | 8.298e-05 | 2.74 |
| | 320 | 7.4111e-06 | 2.99 | 6.5299e-06 | 3.06 | 1.2265e-05 | 2.76 |
| | 640 | 9.0655e-07 | 3.03 | 7.7402e-07 | 3.07 | 1.6229e-06 | 2.92 |
| | 1280 | 1.1112e-07 | 3.02 | 9.6049e-08 | 3.01 | 2.0855e-07 | 2.96 |

Table: The $L^2$, $L^1$, $L^\infty$ errors and orders of accuracy for fourth order diffusion equation.

# Conclusions for SI-IMEX-RK

- We have developed a SI strategy based on IMEX-RK methods coupled with high order finite difference schemes for solving high order dissipative, dispersive and special biharmonic-type equations in one dimension.
- The SI IMEX-RK schemes so designed for high order time-dependent PDEs does not need any nonlinear iterative solver that usually one has using implicit methods, and not require any time step restriction that usually one has using explicit methods.
- Numerical experiments show that the schemes are stable and achieve the aspected orders of accuracy for large time step.
- We considered only classical finite difference spatial discretization because of its simplicity in design and coding and it is straightforward to extend to higher-dimensional equations.
- Other types of suitable space discretization can be used.

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