Uniform accuracy of implicit-explicit methods for stiff hyperbolic relaxation systems and kinetic equations

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Overview

- Stiff PDEs, IMEX methods, uniform accuracy vs. order reduction
- Basic methodology of energy estimates
- Uniform accuracy of IMEX-BDF (J. Hu-RS 21')
- Uniform accuracy of IMEX-RK (J. Hu-RS 25')

Model problems

- Stiff kinetic equations

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{Q}(f)$$

- $f = f(t, x, v)$ particle density function, ε Knudsen number

- $\mathcal{Q}(f)$: collision operator (Boltzmann, BGK, Landau, …)

- Hyperbolic relaxation systems (S. Jin-Z. Xin 95')

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = \frac{1}{\varepsilon}(F(u) - v), \end{cases}$$

$$|F'(u)| < 1$$

sub-characteristic condition

Asymptotic limits

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon}\mathcal{Q}(f)$$

$\varepsilon \to 0$ $\qquad f(v) = M_{\rho,u,T}(v)$

Taking moments and using the local equilibrium,

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + pI) = 0, \\ \partial_t E + \nabla_x \cdot ((E+p)u) = 0, \end{cases}$$

$E = \frac{1}{2}\rho u^2 + \frac{d_v}{2}\rho T \qquad p = \rho T$

next order expansion gives compressible Navier-Stokes

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = \frac{1}{\varepsilon}(F(u) - v), \end{cases}$$

$\varepsilon \to 0$ $\qquad v = F(u)$

Using the local equilibrium in the first equation,

$$\partial_t u + \partial_x F(u) = 0$$

next order expansion gives convection-diffusion

IMEX methods

- Implicit-explicit Runge-Kutta methods (IMEX-RK): U. Ascher-S. Ruuth-R. Spiteri 97’, L. Pareschi-G. Russo 05', G. Dimarco-L. Pareschi 13’,…

$$f^{(i)} = f^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} v \cdot \nabla_x f^{(j)} + \frac{\Delta t}{\varepsilon} \sum_{j=1}^{i} a_{ij} \mathcal{Q}(f^{(j)}), \quad i = 1, \ldots, s$$

$$f^{n+1} = f^n - \Delta t \sum_{j=1}^{s} \tilde{b}_j v \cdot \nabla_x f^{(j)} + \frac{\Delta t}{\varepsilon} \sum_{j=1}^{s} b_j \mathcal{Q}(f^{(j)})$$

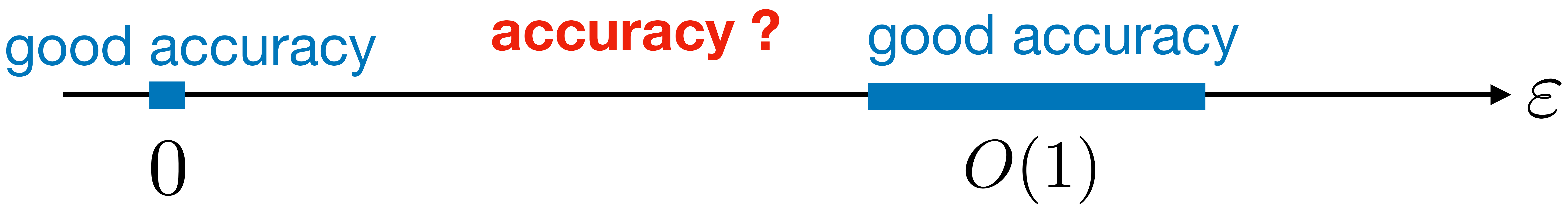
- Implicit-explicit multistep methods: U. Ascher-S. Ruuth-B. Wetton 95’, G. Dimarco-L. Pareschi 17’, … One example is IMEX-BDF:

$$\sum_{i=0}^{q} \alpha_i f^{n+i} + \Delta t \sum_{i=0}^{q-1} \gamma_i v \cdot \nabla_x f^{n+i} = \frac{\beta \Delta t}{\varepsilon} \mathcal{Q}(f^{n+q})$$

Asymptotic properties

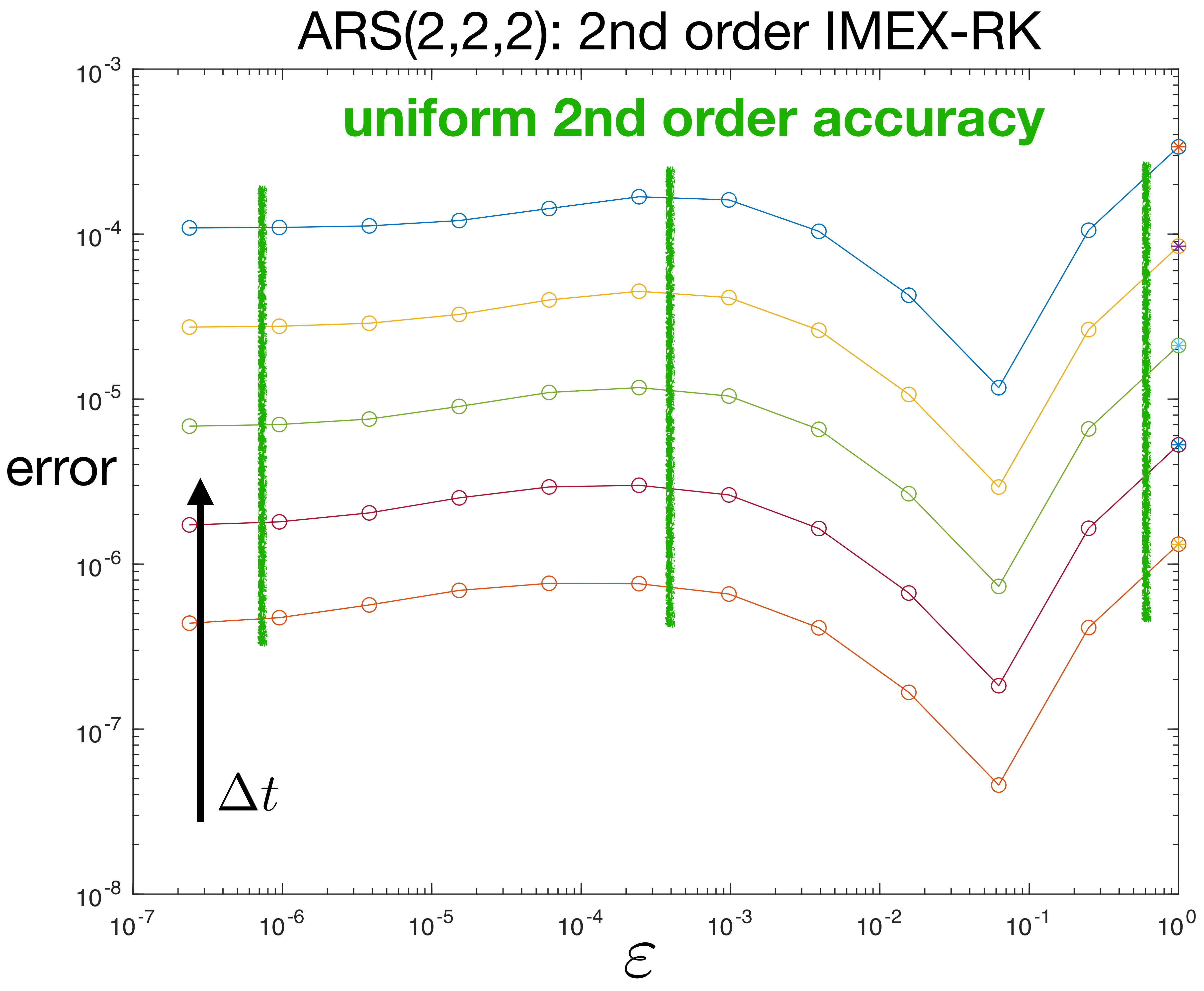
- Asymptotic-preserving (AP) schemes (S. Jin 99'): allow the time step Δt to be chosen independent of ε. Automatically captures the asymptotic limit as $\varepsilon \to 0$

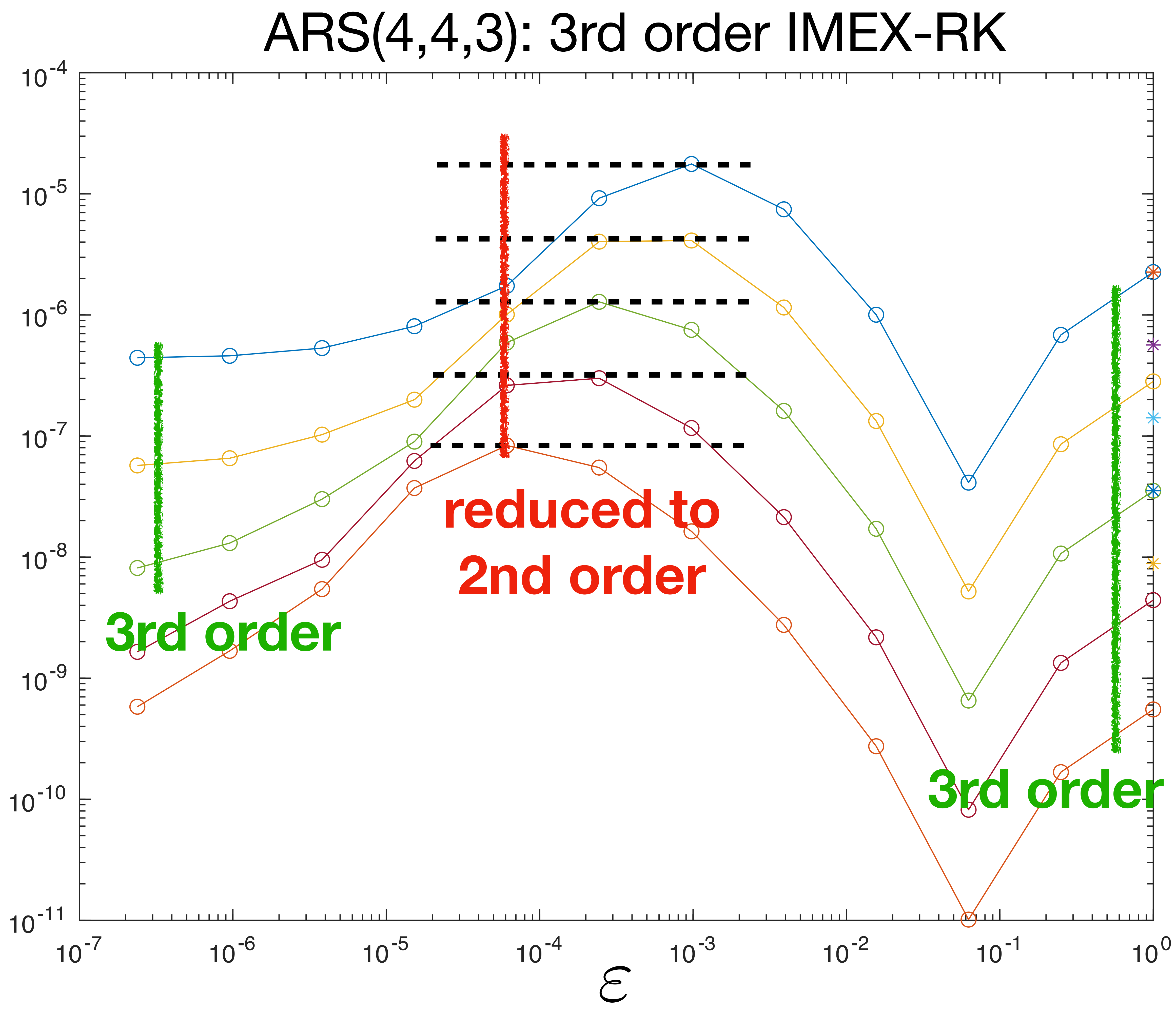
- Accuracy is guaranteed for kinetic regime $\varepsilon = O(1)$ and fluid regime $\varepsilon \to 0$
What happens for intermediate regime?



- **Uniform accuracy** of order q: $\mathrm{error} = O(\Delta t^q)$ uniformly in ε

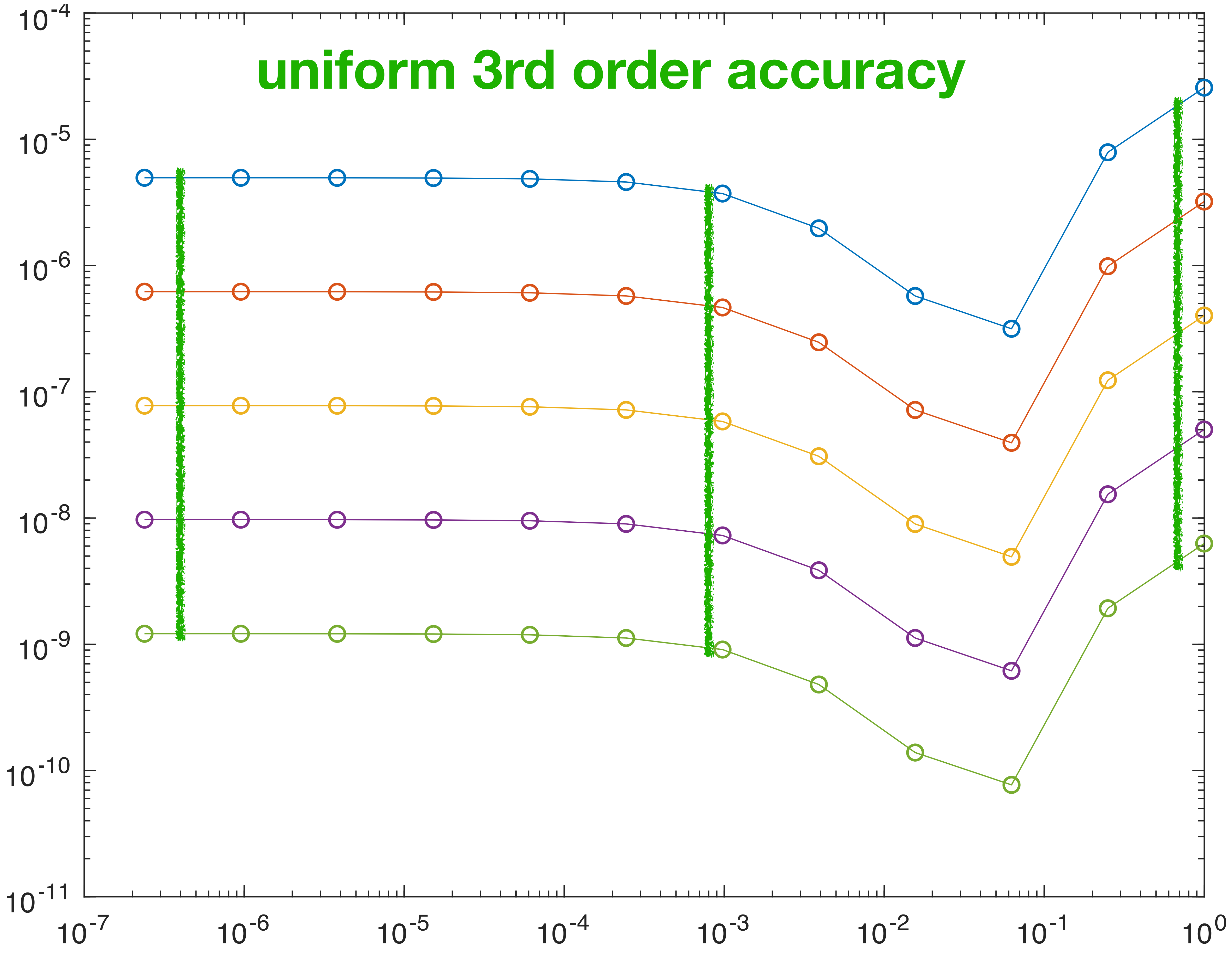
Uniform accuracy vs. order reduction



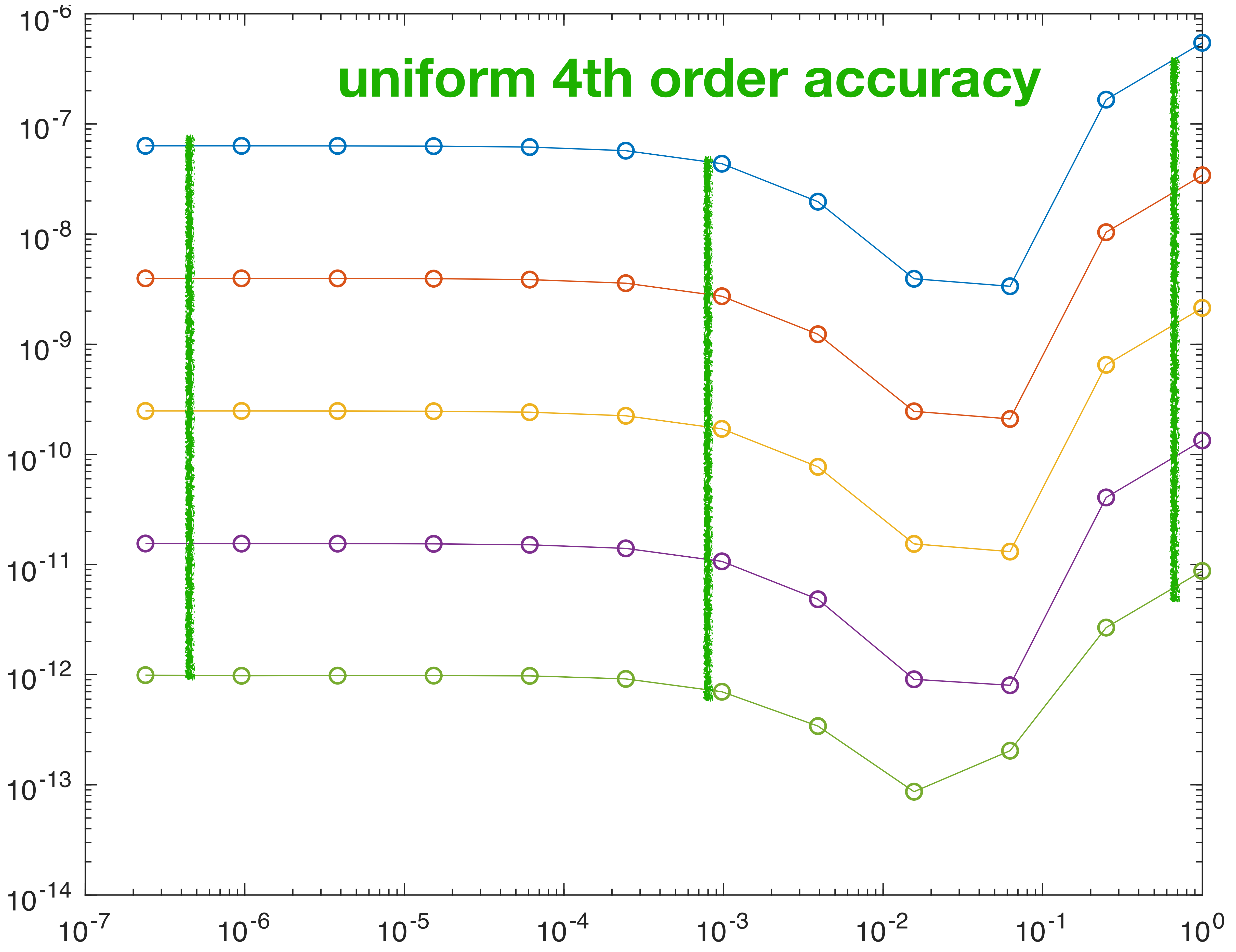


schemes from U. Ascher-S. Ruuth-R. Spiteri 97’

Uniform accuracy of IMEX-BDF



IMEX-BDF3

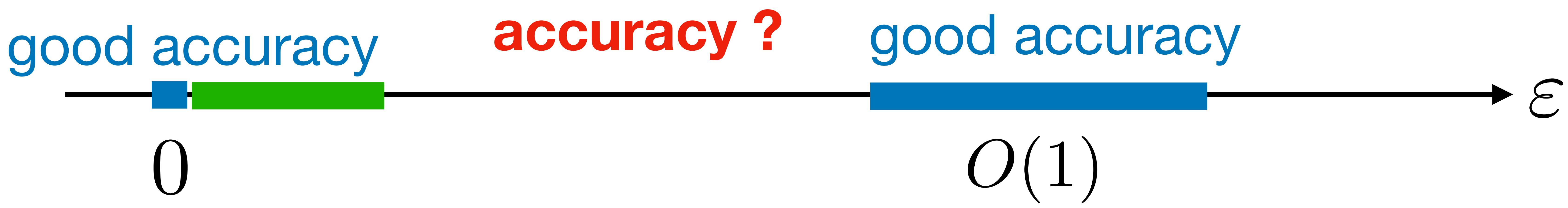


IMEX-BDF4

Uniform accuracy vs. order reduction

- One way to understand order reduction for IMEX-RK is to design schemes which captures the next order expansion (S. Boscarino 07', S. Boscarino-G. Russo 09', S. Boscarino-L. Pareschi 17', J. Hu-X. Zhang 17')

- However, this may not be sufficient to give uniform accuracy



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The model problem

- Linear hyperbolic relaxation equations

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = \dfrac{1}{\varepsilon}(bu - v), \end{cases} \qquad |b| < 1$$

- Change of variable $w = v - bu$

$$\begin{cases} \partial_t u + \partial_x (bu + w) = 0, \\ \partial_t w + \partial_x ((1 - b^2)u - bw) = -\dfrac{1}{\varepsilon} w. \end{cases}$$

- Asymptotic limit

$$\partial_t u + \partial_x (bu) = 0$$

Lemma (J. Hu-RS 21'): Assume initial data is smooth. Then, after an initial layer, we have the regularity estimate

$$\|\partial_t^{r_1} \partial_x^{r_2} u(t, \cdot)\|^2 \leq C$$

$$\|\partial_t^{r_1} \partial_x^{r_2} w(t, \cdot)\|^2 \leq C\epsilon$$

$\| \cdot \|$ denotes L^2 norm in x

Fully discretized scheme

- The spatial domain is discretized by Fourier-Galerkin method:

$$\mathbb{P}_N = \text{span}\{e^{ikx} | -N \leq k \leq N\},$$

$$\mathcal{P}_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx} \in \mathbb{P}_N, \quad \hat{f}_k = \langle f, e^{ikx} \rangle.$$

- Fully discretized Backward-forward Euler method (1st order)

$$\begin{cases} U_N^{n+1} - U_N^n + \Delta t \partial_x (b U_N^n + W_N^n) = 0, \\ W_N^{n+1} - W_N^n + \Delta t \partial_x ((1-b^2) U_N^n - b W_N^n) = -\dfrac{\Delta t}{\varepsilon} W_N^{n+1}. \end{cases}$$

- Subscript N will be omitted later

How to do energy estimate?

multiplier technique

$$2U^{n+1}\times \quad U^{n+1} - U^n + \Delta t \partial_x (bU^n + W^n) = 0$$

$$2W^{n+1}\times \quad W^{n+1} - W^n + \Delta t \partial_x ((1-b^2)U^n - bW^n) = -\frac{\Delta t}{\varepsilon} W^{n+1}$$

$$\|U^{n+1}\|^2 - \|U^n\|^2 + \underline{\|U^{n+1} - U^n\|^2} + 2\Delta t \int \underline{U^{n+1}\partial_x(bU^n} + W^n)dx = 0$$

good terms

$$\|W^{n+1}\|^2 - \|W^n\|^2 + \underline{\|W^{n+1} - W^n\|^2} + 2\Delta t \int W^{n+1}\partial_x((1-b^2)U^n - bW^n)dx = \underline{-2\frac{\Delta t}{\varepsilon}\|W^{n+1}\|^2}$$

$$\Delta t \left| \int U^{n+1}\partial_x U^n dx \right| = \Delta t \left| \int (U^{n+1} - U^n)\partial_x U^n dx \right| \leq c\|U^{n+1} - U^n\|^2 + C\Delta t^2 \|\partial_x U^n\|^2$$

$$\leq c\|U^{n+1} - U^n\|^2 + C\Delta t \|U^n\|^2$$ under a CFL condition $\Delta t \leq c_{CFL}/N^2$

$$\|U^{n+1}\|^2 - \|U^n\|^2 + \|U^{n+1} - U^n\|^2 + 2\Delta t \int U^{n+1} \partial_x (bU^n + W^n) dx = 0$$

$$\|W^{n+1}\|^2 - \|W^n\|^2 + \|W^{n+1} - W^n\|^2 + 2\Delta t \int W^{n+1} \partial_x ((1-b^2)U^n - bW^n) dx = -2\frac{\Delta t}{\varepsilon}\|W^{n+1}\|^2$$

- Combine to get

$$(1-b^2)\|U^{n+1}\|^2 + \|W^{n+1}\|^2 \leq (1 + C\Delta t)((1-b^2)\|U^n\|^2 + \|W^n\|^2)$$

- Use Gronwall to get uniform stability

$$(1-b^2)\|U^n\|^2 + \|W^n\|^2 \leq C(T)((1-b^2)\|u_{in}\|^2 + \|w_{in}\|^2)$$

- Uniform accuracy follows from the same procedure applied on the evolution of error, with the local truncation error taken into consideration

$$(1-b^2)\|U_e^n\|^2 + \|W_e^n\|^2 \leq C(T)\Delta t^2$$

here, first order uniform accuracy

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Uniform accuracy of IMEX-BDF

$$\begin{cases} \sum_{i=0}^{q} \alpha_i U_N^{n+i} + \Delta t \sum_{i=0}^{q-1} \gamma_i \partial_x (b U_N^{n+i} + W_N^{n+i}) = 0, \\ \sum_{i=0}^{q} \alpha_i W_N^{n+i} + \Delta t \sum_{i=0}^{q-1} \gamma_i \partial_x ((1-b^2) U_N^{n+i} - b W_N^{n+i}) = -\frac{\beta \Delta t}{\varepsilon} W_N^{n+q}. \end{cases}$$

- **Theorem** (J. Hu-RS 21'): for q=1,2,3,4, assume the initial data is "consistent", then under CFL condition $\Delta t \leq c_{CFL}/N^2$, IMEX-BDF-q for linear hyperbolic relaxation system has uniform q-th order accuracy:

$$\|u(t_n) - U_N^n\|^2 + \|w(t_n) - W_N^n\|^2 \leq C(T) \left(\Delta t^{2q} + \frac{1}{N^{4q}} \right)$$

temporal error ↑   ↖ spatial error

- Accuracy follows from stability due to stage order

- Generalized to linear hyperbolic systems by Z. Ma-J. Huang-W. Yong 25'

$$2W^{n+1} \times \quad W^{n+1} - W^n + \Delta t \partial_x((1-b^2)U^n - bW^n) = -\frac{\Delta t}{\varepsilon} W^{n+1}$$

$$\|W^{n+1}\|^2 - \|W^n\|^2 + \underline{\|W^{n+1} - W^n\|^2} + 2\Delta t \int W^{n+1} \partial_x((1-b^2)U^n - bW^n) dx = \underline{-2\frac{\Delta t}{\varepsilon}\|W^{n+1}\|^2}$$

good terms

$$\left(W^{n+q} - \sum_{i=1}^{q-1} \eta_i W^{n+i}\right) \times \sum_{i=0}^{q} \alpha_i W^{n+i} + \Delta t \sum_{i=0}^{q-1} \gamma_i \partial_x((1-b^2)U^{n+i} - bW^{n+i}) = -\frac{\beta \Delta t}{\varepsilon} W^{n+q}$$

$$G(W^{n+1}, \dots, W^{n+q}) - G(W^n, \dots, W^{n+q-1}) \underline{+ \|\cdot\|^2} + \text{transport terms}$$

$$= -A(W^{n+2}, \dots, W^{n+q}) + A(W^{n+1}, \dots, W^{n+q-1}) \underline{- \|\cdot\|^2}$$

G, A positive definite quadratic forms

$$\left(W^{n+q} - \sum_{i=1}^{q-1} \eta_i W^{n+i}\right) \times \sum_{i=0}^{q} \alpha_i W^{n+i} + \Delta t \sum_{i=0}^{q-1} \gamma_i \partial_x ((1-b^2)U^{n+i} - bW^{n+i}) = -\frac{\beta \Delta t}{\varepsilon} W^{n+q}$$

$$G(W^{n+1}, \ldots, W^{n+q}) - G(W^n, \ldots, W^{n+q-1}) \underline{+ \|\cdot\|^2} + \text{transport terms}$$

good term 1

$$= -A(W^{n+2}, \ldots, W^{n+q}) + A(W^{n+1}, \ldots, W^{n+q-1}) \underline{- \|\cdot\|^2}$$

good term 2

- Requirements on the multiplier:
  - Gives G with good term 1 (Nevanlinna-Odeh 81' up to q=5)
  - Gives A with good term 2
  - Good term 1 controls the transport terms

For $q = 1$, the coefficients are given by

$$g_{11} = \frac{1}{2}, \quad d_1 = \frac{1}{2}, \quad c_1 = 1. \tag{3.32}$$

For $q = 2$, the coefficients are given by

$$\eta_1 = 0, \quad g_{11} = \frac{1}{6}, \quad g_{22} = \frac{5}{6}, \quad g_{12} = -\frac{1}{3}, \quad d_1 = \frac{1}{6}, \quad d_2 = \frac{3}{2}, \quad a_{11} = 0, \quad c_2 = 1, \quad c_1 = 0. \tag{3.33}$$

For $q = 3$, the coefficients g_{ij}, η_i, d_i are given by New multiplier!

$$\begin{aligned}
&g_{11} = \frac{\sqrt{30}}{187} + \frac{8}{187}, \quad g_{22} = \frac{\sqrt{30}}{34} + \frac{95}{187}, \quad g_{33} = \frac{\sqrt{30}}{22} + \frac{7}{11}, \\
&g_{12} = -\frac{3\sqrt{30}}{187} - \frac{24}{187}, \quad g_{13} = \frac{3\sqrt{30}}{187} + \frac{24}{187}, \quad g_{23} = -\frac{6\sqrt{30}}{187} - \frac{9}{17}, \\
&\eta_2 = -\frac{2\sqrt{30}}{17} + \frac{18}{17}, \quad \eta_1 = \frac{\sqrt{30}}{17} - \frac{9}{17}, \quad d_1 = -\frac{\sqrt{30}}{22} + \frac{4}{11}, \quad d_2 = \frac{11\sqrt{30}}{102} + \frac{44}{51}.
\end{aligned} \tag{3.34}$$

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IMEX-RK

$$U^{(i)} = U^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij}(b\partial_x U^{(j)} + \partial_x W^{(j)})$$

$$W^{(i)} = W^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij}((1-b^2)\partial_x U^{(j)} - b\partial_x W^{(j)}) - \frac{\Delta t}{\varepsilon} \sum_{j=1}^{i} a_{ij} W^{(j)}$$

$$U^{n+1} = U^n - \Delta t \sum_{j=1}^{s} \tilde{b}_j(b\partial_x U^{(j)} + \partial_x W^{(j)})$$

$$W^{n+1} = W^n - \Delta t \sum_{j=1}^{s} \tilde{b}_j((1-b^2)\partial_x U^{(j)} - b\partial_x W^{(j)}) - \frac{\Delta t}{\varepsilon} \sum_{j=1}^{s} a_{sj} W^{(j)}$$

Butcher tables

explicit | implicit

$$
\begin{array}{c|cccc|cccc}
c_1 = 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_2 & \tilde{a}_{21} & 0 & 0 & 0 & a_{21} & a_{22} & 0 & 0 \\
c_3 & \tilde{a}_{31} & \tilde{a}_{32} & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 \\
c_4 & \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & 0 & a_{41} & a_{42} & a_{43} & a_{44} \\
\hline
 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & \tilde{b}_4 & b_1 & b_2 & b_3 & b_4
\end{array}
$$

type CK

$$c_i = \sum_j a_{ij} = \sum_j \tilde{a}_{ij}$$

Implicitly Stiffly Accurate (ISA):

$$b_j = a_{sj}$$

Globally Stiffly Accurate (GSA):

$$b_j = a_{sj},\ \tilde{b}_j = \tilde{a}_{sj}$$

ARS(2,2,2): 2nd order IMEX-RK

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\gamma & \gamma & 0 & 0 \\
1 & \delta & 1-\delta & 0 \\
\hline
 & \delta & 1-\delta & 0
\end{array}
\qquad
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\gamma & 0 & \gamma & 0 \\
1 & 0 & 1-\gamma & \gamma \\
\hline
 & 0 & 1-\gamma & \gamma
\end{array}
$$

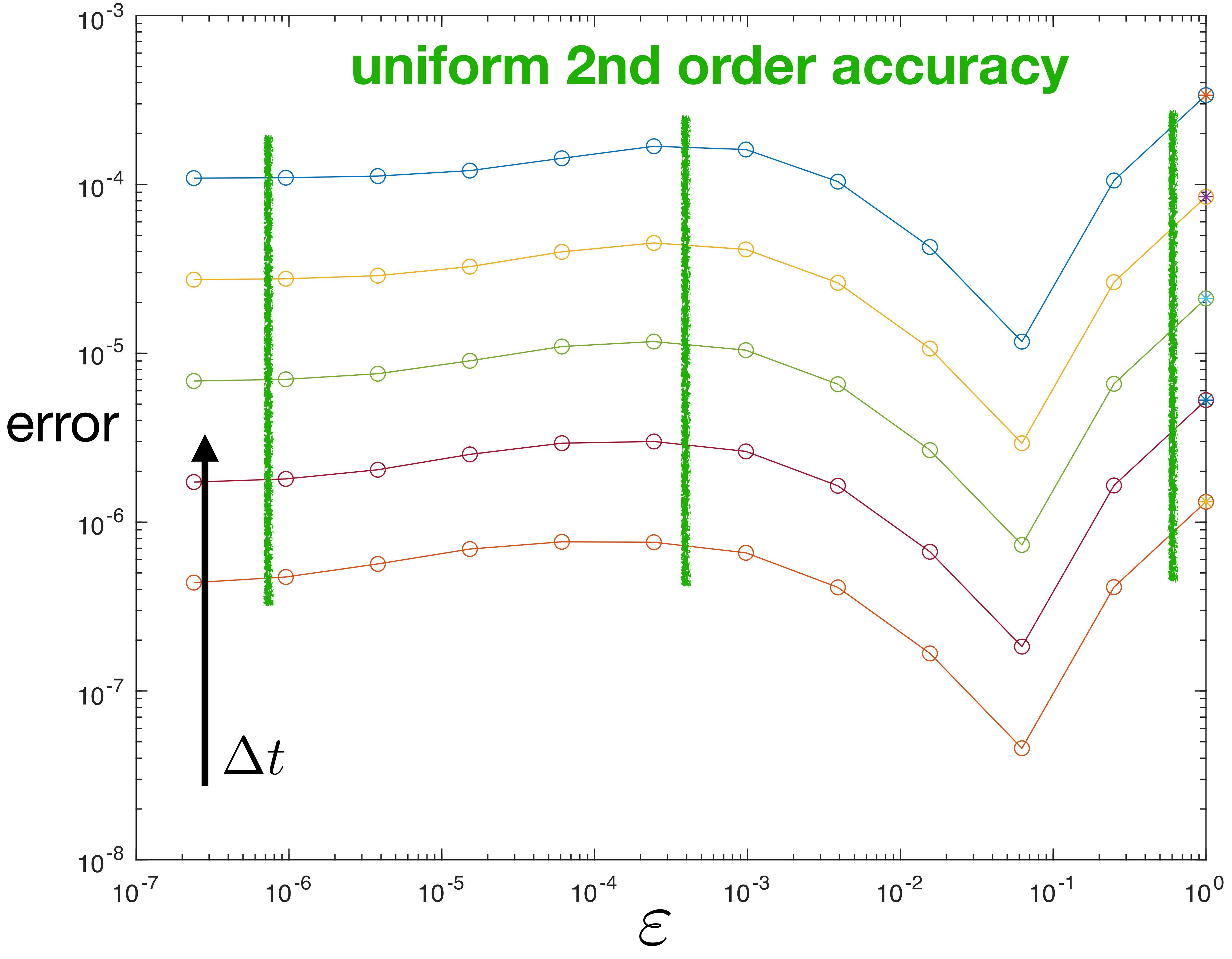
type ARS

$$\gamma = 1 - \frac{\sqrt{2}}{2},\ \delta = 1 - \frac{1}{2\gamma}$$

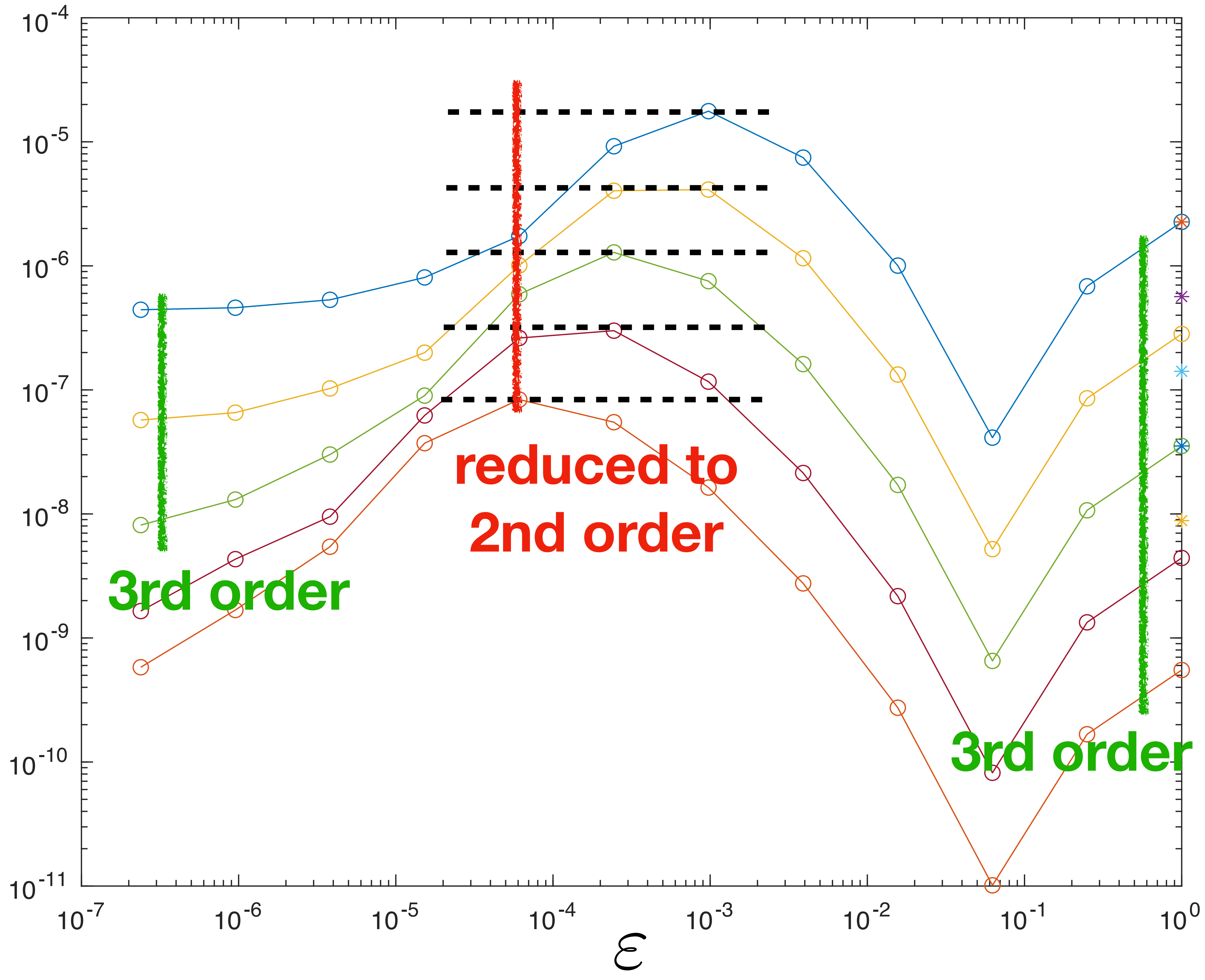
ARS(4,4,3): 3rd order IMEX-RK

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 \\
2/3 & 11/18 & 1/18 & 0 & 0 & 0 \\
1/2 & 5/6 & -5/6 & 1/2 & 0 & 0 \\
1 & 1/4 & 7/4 & 3/4 & -7/4 & 0 \\
\hline
 & 1/4 & 7/4 & 3/4 & -7/4 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 & 0 \\
2/3 & 0 & 1/6 & 1/2 & 0 & 0 \\
1/2 & 0 & -1/2 & 1/2 & 1/2 & 0 \\
1 & 0 & 3/2 & -3/2 & 1/2 & 1/2 \\
\hline
 & 0 & 3/2 & -3/2 & 1/2 & 1/2
\end{array}
$$

Uniform accuracy vs. order reduction



ARS(2,2,2): 2nd order IMEX-RK



ARS(4,4,3): 3rd order IMEX-RK

Second order uniform accuracy

- **Theorem** (J. Hu-RS 25'): assume the initial data is "consistent" and under CFL condition $\Delta t \leq c_{CFL}/N^2$. IMEX-RK (type CK, ISA) for linear hyperbolic relaxation system has second order uniform accuracy:

$$\|u(t_n) - U_N^n\|^2 + \|w(t_n) - W_N^n\|^2 \leq C(T)\left(\Delta t^4 + \frac{1}{N^8}\right)$$

if the following are true:

1. $c_i = \sum_j a_{ij} = \sum_j \tilde{a}_{ij}$
2. Usual second order conditions
3. Existence of a multiplier matrix M
4. Last component of $\vec{v}$ is 0 (← null space of A) (related to the L-stability)

- This applies to ARS(2,2,2) and ARS(4,4,3)

An IMEX-RK with third order uniform accuracy

- S. Boscarino-G. Russo 09’: third order IMEX-RK of type CK, ISA

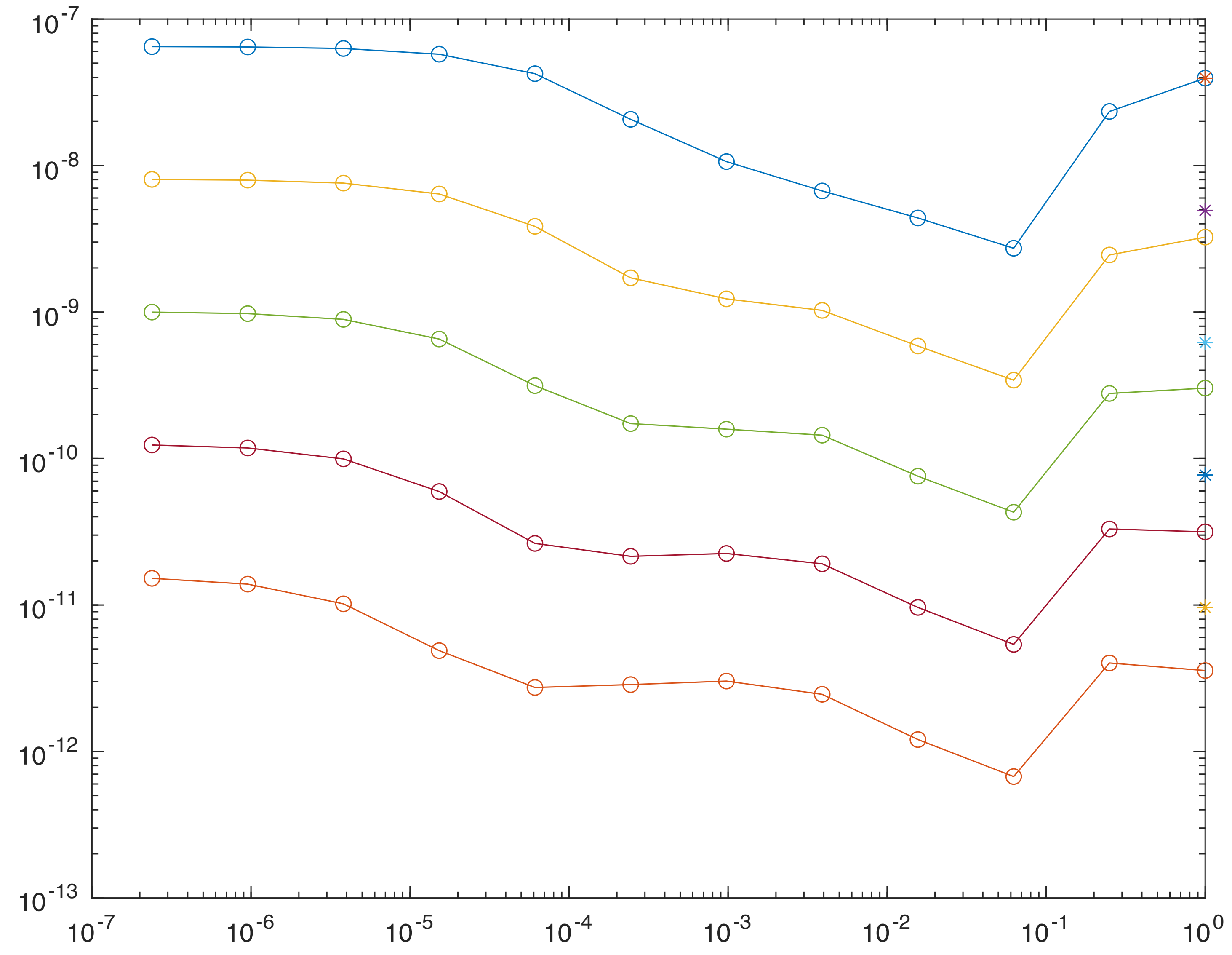
0	0	0	0	0	0	0	0	0	0	0
0.8717	0.8717	0	0	0	0	0.4359	0.4359	0	0	0
0.8717	0.4359	0.4359	0	0	0	0.4359	0	0.4359	0	0
1.5000	0.2095	0	1.2905	0	0	0.5236	0	0.5405	0.4359	0
1.0000	0.3177	-0.3629	1.1960	-0.1508	0	0.3694	0	0.3629	-0.1681	0.4359
	0.3694	0	0.3629	-0.1681	0.4359	0.3694	0	0.3629	-0.1681	0.4359

stage order 1

stage order 2

the ‘less accurate’ stage
is not used directly

- Third order uniform accuracy is observed



Third order uniform accuracy

- **Theorem** (J. Hu-RS 25'): further assume

1. Usual third order conditions

2. Stages (3)-(s) have stage order 2

3. $\tilde{b}_2 = 0$ $\quad a_{i,2} = 0,\ i = 3, 4, \ldots, s$

Then we have third order uniform accuracy

$$\|u(t_n) - U_N^n\|^2 + \|w(t_n) - W_N^n\|^2 \leq C(T)\left(\Delta t^6 + \frac{1}{N^{12}}\right)$$

Proof of stability

$$U^{(i)} = U^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij}(b\partial_x U^{(j)} + \partial_x W^{(j)})$$

$$W^{(i)} = W^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij}((1-b^2)\partial_x U^{(j)} - b\partial_x W^{(j)}) - \frac{\Delta t}{\varepsilon} \sum_{j=1}^{i} a_{ij} W^{(j)}$$

- Vector form

$$\vec{U} = U^{(1)}\vec{e} - \Delta t \tilde{A}(b\partial_x \vec{U} + \partial_x \vec{W})$$

$$\vec{W} = W^{(1)}\vec{e} - \Delta t \tilde{A}((1-b^2)\partial_x \vec{U} - b\partial_x \vec{W}) - \frac{\Delta t}{\varepsilon} A\vec{W}$$

- Use a multiplier to do energy estimate?

Proof of stability

$$\color{blue}{\vec{U}^\top M \times} \quad \vec{U} = U^{(1)}\vec{e} - \Delta t \tilde{A}(b\partial_x \vec{U} + \partial_x \vec{W})$$

$$\color{blue}{\vec{W}^\top M \times} \quad \vec{W} = W^{(1)}\vec{e} - \Delta t \tilde{A}((1-b^2)\partial_x \vec{U} - b\partial_x \vec{W}) - \frac{\Delta t}{\varepsilon} A\vec{W}$$

$$(W^{(s)})^2 = (W^n)^2 - \underline{\vec{W}^\top M_* \vec{W}} - \underline{\frac{\Delta t}{\varepsilon}\vec{W}^\top M A \vec{W}} - \Delta t \vec{W}^\top M \tilde{A}((1-b^2)\partial_x \vec{U} - b\partial_x \vec{W})$$

good terms (hopefully)

$$M_* := M \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

- Requirements on the multiplier matrix M:

$\vec{W}^\top M_* \vec{W}$, $\vec{W}^\top M A \vec{W}$ are semi-positive-definite

- For all the previously mentioned schemes, we can construct such M

Proof of accuracy

- Proof of accuracy is NOT obvious because the stage order is lower than the desired uniform accuracy order
- Say, to prove second order uniform accuracy, the evolution of error is

$$\vec{U}_e = U_e^{(1)}\vec{e} - \Delta t \tilde{A}(b\partial_x \vec{U}_e + \partial_x \vec{W}_e) + \vec{E}_U$$

$$\vec{W}_e = W_e^{(1)}\vec{e} - \Delta t \tilde{A}((1-b^2)\partial_x \vec{U}_e - b\partial_x \vec{W}_e) - \frac{\Delta t}{\varepsilon} A\vec{W}_e + \vec{E}_W$$

L.T.E., $O(\Delta t^2)$ for some stages

- Directly applying multiplier, local truncation error (L.T.E.) will pollute the estimate and only gives first order uniform accuracy

Proof of accuracy

$$\vec{U}_e = U_e^{(1)}\vec{e} - \Delta t\tilde{A}(b\partial_x\vec{U}_e + \partial_x\vec{W}_e) + \vec{E}_U$$

$$\vec{W}_e = W_e^{(1)}\vec{e} - \Delta t\tilde{A}((1-b^2)\partial_x\vec{U}_e - b\partial_x\vec{W}_e) - \frac{\Delta t}{\varepsilon}A\vec{W}_e + \vec{E}_W$$

- Use a change of variables to absorb the L.T.E.

$$\vec{U}_{e*} := \vec{U}_e - \vec{E}_U, \quad \left(I + \frac{\Delta t}{\varepsilon}A\right)\vec{W}_{e*} := \left(I + \frac{\Delta t}{\varepsilon}A\right)\vec{W}_e - \vec{E}_W$$

$$\vec{U}_{e*} = U_e^{(1)}\vec{e} - \Delta t\tilde{A}(b\partial_x\vec{U}_{e*} + \partial_x\vec{W}_{e*}) - \Delta t\tilde{A}\vec{F}_U$$

$$\vec{W}_{e*} = W_e^{(1)}\vec{e} - \Delta t\tilde{A}((1-b^2)\partial_x\vec{U}_{e*} - b\partial_x\vec{W}_{e*}) - \frac{\Delta t}{\varepsilon}A\vec{W}_{e*} - \Delta t\tilde{A}\vec{F}_W$$

some linear combination of L.T.E.

- Notice that now the L.T.E. terms has a factor Δt on it!

Proof of accuracy

- Then the multiplier techniques work as before. But finally we need to change back to the original variables…

$$U_e^{n+1} = U_{e*}^{(s)} - \Delta t(\tilde{\vec{b}}^\top - \tilde{A}_s)(b\partial_x \vec{U}_{e*} + \partial_x \vec{W}_{e*}) - \Delta t(\tilde{\vec{b}}^\top - \tilde{A}_s)\vec{F}_U + E_U^{n+1}$$

$$W_e^{n+1} = W_{e*}^{(s)} - \Delta t(\tilde{\vec{b}}^\top - \tilde{A}_s)((1-b^2)\partial_x \vec{U}_{e*} - b\partial_x \vec{W}_{e*}) - \Delta t(\tilde{\vec{b}}^\top - \tilde{A}_s)\vec{F}_W - \underline{\frac{\Delta t}{\varepsilon} A_s (I + \frac{\Delta t}{\varepsilon} A)^{-1} \vec{E}_W} + E_W^{n+1}$$

all the difficulties are here

- Then one needs a careful study of the matrix $\frac{\Delta t}{\varepsilon} A(I + \frac{\Delta t}{\varepsilon} A)^{-1}$
- This gives second order uniform accuracy by some energy estimate
- For third order, one needs to do the change of variables twice…
- Generalized to linear hyperbolic systems by Z. Ma-J. Huang 25’