Update on Bianchi Modular Form and Elliptic Curve data in the LMFDB

John Cremona

University of Warwick

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Topics

- Bianchi Modular Forms (BMF)
  `https://www.lmfdb.org/ModularForm/GL2/ImaginaryQuadratic/`
- Elliptic Curves over Number Fields (ECNF)
  `https://www.lmfdb.org/EllipticCurve/`
- Elliptic Curves over $\mathbb{Q}$ (ECQ)
  `https://www.lmfdb.org/EllipticCurve/Q/`

BMF data update

- Then (2023):
  - BMF data for 32 imaginary quadratic fields (IQFs) K (all with $|D_K| \leq 100$ and $\mathbb{Q}(\sqrt{-163})$), all levels of norm ≤ 1000
  - for each level: old and new dimension, rational newform data with Hecke and Atkin-Lehner eigenvalues

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- Now (2025)
  - BMF data for 763 fields, all with $|D_K| \leq 2500$
  - Levels of norm up to 1000 ($|D_K| \leq 120$), 100 ($|D_K| \leq 1000$), 15 ($|D_K| \leq 2100$), 11 ($|D_K| \leq 2500$)
  - 456188 newform spaces, 233333 rational newforms
  - At least 25 $a_{\mathfrak{p}}$ for each (100 for $|D_K| \leq 250$)
  - 244 larger fields have no rational newforms at small levels

ECNF data update

- More elliptic curves over IQFs, found from their BMFs
- The last few curves over totally real sextic fields have been found from their Hilbert modular forms (HMFs), so now all rational weight 2 HMFs in the database have matching elliptic curves

Details on next two slides...

ECNF curves from BMFs

- 455002 curves in 233944 isogeny classes
- (including 975 CM curves not from Bianchi cuspforms)
- 376 rational BMFs have no elliptic curve: 305 do not exist, 71 unknown as yet
- Where there is no elliptic curve in the database, the BMF homepage displays either “No elliptic curve” or “Isogeny class . . . missing”

ECNF curves from HMFs

- In 2023 we had elliptic curves matching all rational weight 2 HMFs with 5 missing (up to Galois conjugacy and quadratic twist)
- Since then, these have been found thanks a joint effort by myself, Benjamin Matschke and John Voight, using Black Box Galois Representation methods to find the 2-division fields, and S-unit equation solving over these to find the j-invariants
- e.g. HMF 6.6.1292517.1-53.1-c
  $\leftrightarrow$ ECNF 6.6.1292517.1-53.1-c1.

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- (write-up in preparation)

ECNF and ECQ home pages

- More consistency in data display ECNF/ECQ
- ECNF BSD data to higher and more consistent precision
- BSD formula displayed in detail for both ECQ and ECNF
- ECQ: more Galois representation data (adelic image, Serre invariants)
- ECQ: Faltings height, abc quality, Szpiro ratio

ECQ expansion plans

- Work is under way to add all elliptic curves with conductor $500000 \leq N \leq 1000000$
- Combined effort and methods: modular symbols (JC), binary cubics and Thue-Mahler solving (Bennett, Gherga), ternary quadratic forms (Hein, Tornaria, Voight)

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Watch this space!