Kodaira dimension of Hilbert modular threefolds

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Upcoming program

In fall 2026 there will be a semester program at ICERM entitled "Computations on K3 surfaces and related varieties".

You can read about it at `https://tinyurl.com/icerm-k3`. Web pages for individual weeklong workshops coming soon.

Elliptic and modular curves are off-topic, but if you do not restrict yourself to dimension ≤ 1 we would be very happy to receive your application.

You can apply for the whole program (September 9 to December 11) or any part of it.

Organizers include Kiran and me; please feel free to speak to us or to contact the others with questions.

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Modular curves

Modular curves, parametrizing elliptic curves with additional information about the torsion subgroup, have been studied for over 100 years.

They can be seen as quotients of the upper half-plane $\mathcal{H}$ by a finite-index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$.

It is natural to want to replace $\mathbb{Z}$ with the maximal order of another number field. However, if we embed $\mathrm{PSL}_2(\mathcal{O}_K)$ into $\mathrm{PSL}_2(\mathbb{R})$, we obtain a dense subgroup, and the topology of the quotient is unmanageable.

Hilbert modular varieties

Hilbert realized that if instead of choosing a single embedding we work with all of them together, things are much better.

Let K be a totally real number field of degree d. Then $\mathrm{PSL}_2(\mathcal{O}_K)$ can be embedded into $\mathrm{PSL}_2(\mathbb{R})^d$ by means of the d distinct embeddings $K \hookrightarrow \mathbb{R}$.

The image is discrete, and $\mathrm{PSL}_2(\mathbb{R})^d$ acts on $\mathcal{H}^d$ component by component.

The action of $\mathrm{PSL}_2(\mathcal{O}_K)$ is properly discontinuous and has finite stabilizers, so the quotient has a complex structure. Denote it $\mathbb{H}_K$.

$\mathbb{H}_K$ is compactified by adjoining cusps. The interior points parametrize abelian d-folds with an action of $\mathcal{O}_K$.

Hilbert modular forms

We have the following definition, very much analogous to the classical case:

Let subscripts denote embeddings $K \hookrightarrow \mathbb{R}$ and let Mz be the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$. Let U be an open subset of $\mathcal{H}^d \cup \{\text{cusps}\}$ and let f be a holomorphic function on U. If

$$f(M_1 z_1, \ldots, M_d z_d) = \prod_{i=1}^{d} (c_i z_i + d_i)^k f(z_1, \ldots, z_d)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, where Γ is commensurable with $\mathrm{SL}_2(\mathcal{O}_K)$, then f is a *Hilbert modular form* on U of weight k for Γ.

q-expansion

Let L be the lattice in K such that $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \in P\Gamma$ if and only if $\lambda \in L$. (This is $\mathcal{O}_K$ if $\Gamma = \mathrm{SL}_2$.)

A Hilbert modular form is periodic with respect to L, so it has a Fourier expansion $\sum_{\ell \in L_+^\vee} c_\ell e^{2\pi i \ell z}$, where ℓz is short for $\sum_{i=1}^{[K:\mathbb{Q}]} \ell_i z_i$.

In addition, let U be the subgroup of units u with $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \in P\Gamma$. (This is $\mathcal{O}_K^{\times 2}$ if $\Gamma = \mathrm{SL}_2$.) Then $c_\ell = c_{u\ell}$ for $u \in U$.

Simplifications

In this talk we will make a few simplifying assumptions:

- We assume that K has narrow class number 1 (so only one cusp).
- We do not consider nontrivial level structure.
- We only consider asymptotic results on the dimension of $\mathrm{H}^0(K^{\otimes n})$, not results on individual powers.
- We do not look at the q-expansions of individual modular forms.

The first three assumptions are relaxed in the paper. The additional information coming from q-expansions will be the focus of a future paper with Assaf, Costa, and Schiavone.

Main result

Our main result is as follows:

Theorem

Let K be a totally real cubic field such that $p_g(\mathbb{H}_K) \leq 1$ and $\operatorname{disc} \mathcal{O}_K \geq 473$. If $\operatorname{disc} \mathcal{O}_K \neq 697$, then $\mathbb{H}_K$ is of general type.

(This covers 35 fields.)

I expect that the result also holds when $p_g(\mathbb{H}_K) > 1$, but in those cases we know already that $\kappa > 0$ (where κ is the Kodaira dimension, defined on the next slide). It could be proved by a finite calculation.

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Kodaira dimension (short version)

Let V be a smooth variety of dimension d. Then we can define the *canonical divisor class* K_V and try to use its multiples to define maps $\phi_{|nK_V|}$ to projective space.

If some positive multiple gives a birational map to its image, then V is *of general type*. Such varieties are "very far from rational". If there is a dominant map $\mathbb{P}^d \to V$, then no positive multiple of K_V has nonzero sections.

An equivalent condition is that $\mathrm{h}^0(K_V^{\otimes n}) > cn^d$ for some $c > 0$, for all n sufficiently large and divisible.

Kodaira dimension of Hilbert modular varieties

Most Hilbert modular varieties are of general type.

Theorem

(Tsuyumine) There are only finitely many totally real fields K*, and none of degree greater than* 6*, for which* $\mathbb{H}_K$ *is not of general type.*

For small degree the problem has been studied in more detail.

Theorem

(Hirzebruch, van der Geer, Zagier) If K *is a real quadratic field of discriminant* > 105 *then* $\mathbb{H}_K$ *is of general type.*

Theorem

(Grundman) If K *is a real cubic field of discriminant* $> 2.77 \cdot 10^8$*, then* K *is of general type.*

Sections of powers of the canonical

Sections of n times the canonical bundle of $\mathbb{H}_K$ are differential n-forms. Such forms satisfy the same identity as Hilbert modular forms of weight $2n$. In other words, $f\,(dz_1\,\ldots\,dz_d)^{\otimes n}$ is Γ-invariant if and only if f is a Hilbert modular form of weight $2n$.

Note that I am not saying that every Hilbert modular form of weight $2n$ is a global section of nK.

The criterion

Theorem

(Knöller) A Hilbert modular form of weight $2n$ extends to a section of nK over ∞ if and only if $c_n = 0$ for $n = 0$ and for all $n \gg 0 \in \mathfrak{d}_K^{-1}$ for which there exists $x \gg 0 \in \mathcal{O}_K$ with $\operatorname{tr} nx < k$.

(There can also be conditions at elliptic points, but the issue does not arise in this talk.)

Method for proving that $\mathbb{H}_K$ is of general type

We have a linear map $M_{2m}(K) \to \mathbb{C}\langle S_m/U \rangle$, where S_m is the set $\{n \gg 0 \in \mathfrak{d}_K^{-1} : \exists x \gg 0 \in \mathcal{O}_K, \operatorname{tr} nx < m\}$ of the last slide.

The kernel consists of sections of mK and its dimension is at least $\dim M_{2m}(K) - \#S_m$.

If the difference is at least cm^d, then $\mathbb{H}_K$ is of general type.

Previous and new work

For some special cubic fields, Thomas-Vasquez showed that if $\operatorname{tr} nx < k$ then $\operatorname{tr} n < k$. Grundman used this to prove that for 3 cubic fields K_1, K_2, K_3 one has $p_g(\mathbb{H}_{K_i}) = 0$ (so one might hope that $\mathbb{H}_{K_i}$ is rational) but $\kappa(\mathbb{H}_{K_i}) > 0$.

In fact she proved that $\mathbb{H}_{K_2}, \mathbb{H}_{K_3}$ are of general type.

We will study the condition $\operatorname{tr} nx < k$ for general cubic fields.

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Setting up

In this section we describe our approach to counting $n \gg 0 \in \mathfrak{d}_K^{-1}$ up to units with an integral multiple of trace $< k$.

Throughout this section, we fix a totally real number field K of degree d.

Definition

Let $x \in K$. If for all totally positive units $u \in \mathcal{O}_K^\times$ we have $\operatorname{tr} ux \geq \operatorname{tr} x$, then x is *trace-minimal*.

(Exercise: if x is trace-minimal, then $x = 0$ or $x \gg 0$. Every $x \gg 0$ is ${\mathcal{O}_{K,+}}^\times$-equivalent to some trace-minimal element, and mostly it is unique, so we get correct asymptotics by counting these.)

Balance

Definition

Let $x \gg 0 \in K$ and let $b > 0$. If $\max_i x_i / \min_j x_j \leq b$, then x is *b-balanced*.

Proposition

There is a constant b_K such that every trace-minimal $x \in K$ is b_K-balanced.

Proposition

There is a finite set U of units such that if x is not trace-minimal then $\operatorname{tr} xu < \operatorname{tr} x$ for some $u \in U$.

Corollary

In the coordinates for $\mathbb{R} \otimes K$ given by the embeddings of K, the trace-minimal elements of $\mathbb{R}^n$ constitute a rational polyhedral cone.

Reducers (1)

Let $r \gg 0 \in \mathcal{O}_K$. We say that r is a *reducer* if there exists $x \in K$ trace-minimal with $\operatorname{tr} xr < \operatorname{tr} x$.

Example

Let $K = \mathbb{Q}(\sqrt{7})$. Then $3 + \sqrt{7}$ is trace-minimal and $(3 - \sqrt{7})(3 + \sqrt{7}) = 2$, so $3 - \sqrt{7}$ is a reducer; so is $3 + \sqrt{7}$.

The special resolutions used by Knöller, Thomas-Vasquez, and Grundman depend on the fields in question having no reducers.

Reducers (2)

Lemma

Let $x_1, \ldots, x_t$ generate the rays of the trace-minimal cone. Then r is a reducer if and only if $\operatorname{tr} rx_i < \operatorname{tr} x_i$ for some i.

For each x_i, we can determine the r that reduce it (they are lattice points inside a finite polyhedron). So we have a good algorithm for listing the reducers.

A spiky region

Let $r_1, \ldots, r_s$ be the reducers and let $r_0 = 1$ (which is not a reducer). Let $\mathcal{C}$ be the trace-minimal cone, and for $0 \leq i \leq s$ let $\mathcal{C}_i = \mathcal{C} \cap (\operatorname{tr} r_i x \leq 1)$. Let $\mathcal{R} = \cup_i \mathcal{C}_i$, and for each integer $n > 0$ let $\mathcal{R}(n)$ be $\mathcal{R}$ scaled up by n.

Proposition

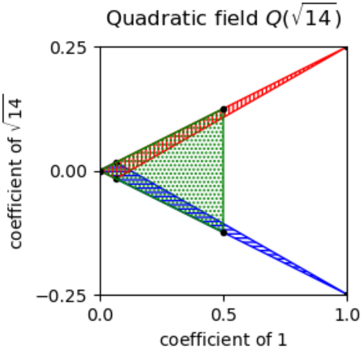
The number of totally nonnegative elements of $\mathfrak{d}_K^{-1}$ with a totally positive integral multiple of trace less than n up to totally positive units is asymptotically equal to the number of points of the lattice $\mathfrak{d}_K^{-1}$ in $\mathcal{R}(n)$.

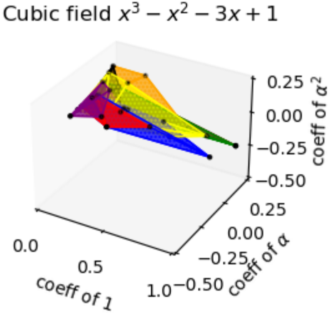
Corollary

The number is asymptotic to $D_K \operatorname{vol} \mathcal{R}\, n^d$.

One computes $\operatorname{vol} \mathcal{R}$ by a standard algorithm.

Picture





(See github for an interactive version of the second one.)

Putting it all together

Let K be a totally real number field of class number 1.

The dimension of the space of Hilbert cusp forms of weight $2m$ is asymptotic to $|2\zeta_K(-1)|m^d$.

So if $|2\zeta_K(-1)| > \operatorname{vol}\mathcal{R}$, then $\mathbb{H}_K$ is of general type.

We just do this calculation for each of the fields with $p_g(\mathbb{H}_K) \leq 1$, obtaining the result. Here is what it looks like.

Table (1)

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	h^+	r	n	t	t'
473	10/3	79/24	1	19	22	3.040	2.730
564	6	1021/180	1	227	84	23.150	19.970
568	20/3	1141/180	1	477	60	24.230	17.870
621	20/3	413/72	1	219	95	30.670	27.130
697*	16/3	67/12	2	0	0	0.200	0.000
733	8	221/36	1	251	75	27.590	21.470
756	26/3	469/72	1	297	102	43.500	31.160
761	20/3	13/2	2	0	0	0.290	0.010
785	22/3	29/6	1	71	76	19.470	18.050
788	28/3	113/12	2	50	16	6.800	2.730
837	32/3	1381/180	1	1154	95	59.040	39.400

Table (2)

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^{\times}}$	h^+	r	n	t	t'
892	$40/3$	$1091/90$	2	141	54	16.950	13.220
940	$44/3$	$703/72$	1	1209	194	105.260	93.150
985	$28/3$	$70/9$	2	0	0	0.300	0.000
993	$34/3$	$2377/360$	1	120	68	23.570	19.650
1076	$44/3$	$719/60$	2	28	11	2.740	1.970
1257	16	$101/9$	2	15	15	2.820	2.370
1300	18	$629/72$	1	742	94	63.660	43.120
1345	$46/3$	$779/120$	1	95	143	48.540	47.140
1396	$64/3$	$535/36$	2	45	19	6.340	4.140
1489	16	$134/15$	2	0	0	0.360	0.000
1593	$64/3$	$253/20$	2	7	8	1.910	1.270

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A couple of questions

There seems to be further structure that I have not found.

The volumes of the individual polyhedra in the spiky regions are much less smooth than the volume of the union.

Also, let $S_n = \{n \gg 0 \in \mathfrak{d}_K^{-1} : \exists x \gg 0 \in \mathcal{O}_K, \mathrm{tr}\, nx < k\}$ and $s_n = \# S_n$. Then we know that $\sum_{i=0}^{\infty} s_n x^n$ is a rational function. However, its denominator has degree 7, which is surprisingly small.

Why?

Thank you

Thank you for your attention.

Are there any questions?