Census of genus 6 curves over $\mathbf{F}_2$

Joint work with Kiran S. Kedlaya and Jun Bo Lau

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Census-taking of ~~population~~curves[1] over finite fields

Census-taking of both population and curves over finite fields can be

1. expensive: cost of human resources v. computational resources, and
2. difficult to verify completeness of data.

	Population	**Curves**
Why?	districting, macroeconomic analysis, etc.	?
How?	census survey	?
What?	age, income, household characteristics,. . . ,	# of $\mathbf{F}_{q^n}$-points (zeta function), #Aut,. . .
Completeness	statistics	?

[1] smooth, projective, geometrically integral

Motivation 1: Project Genesis

At LuCaNT I, Kedlaya submitted and presented a paper that resolves the last few remaining cases of the relative class number one problem for function fields, which relies on making a *partial* census of genus 6 and 7 curves over $\mathbf{F}_2$.

> ### A full census of genus-6 and genus-7 curves
>
> It would be desirable to have a full census of genus-g curves over $\mathbb{F}_2$ for $g = 6, 7$. This would provide a valuable consistency check, and also serve as a rich resource for future investigation (ideally as part of LMFDB).
>
> A further consistency check[16] would be provided by computing[17] $\#M_g(\mathbb{F}_2)$ using explicit generators/relations for the Chow ring. For $g = 6$, this has been achieved using very recent work of Canning–H. Larson.[18]
>
> It should be possible to upgrade our existing code to remove the filtering on zeta functions to achieve a full census. For $g = 6$, this is work in progress with Jun Bo Lau, but extra help would be welcome.
>
> [16] Such a count can even be used to **certify** the validity of a census: it is easy to compute automorphism groups and check pairwise nonisomorphism for an explicit list of curves, this providing a concrete lower bound on stacky $\#M_g(\mathbb{F}_2)$.
>
> [17] This point count is **stacky**: the isomorphism class of a curve C has weight $\frac{1}{\#\operatorname{Aut}(C)}$.
>
> [18] Odd coincidence: Hannah is also lecturing in Providence at this hour!
>
> Kiran S. Kedlaya (UC San Diego) | Relative class number 1 for function fields | ICERM, July 13, 2023 | 27 / 28

Motivation 2: Motivation 1 v.2

Let $\mathcal{M}_g$ denote the moduli space of curves, and $\mathcal{A}_g$ denote the moduli space of g-dimensional principally polarized abelian varieties. We have the Torelli map

$$\mathcal{T} : \mathcal{M}_g \to \mathcal{A}_g, X \mapsto \mathrm{Jac}(X).$$

By the Honda-Tate theorem, we can enumerate isogeny classes of abelian varieties over $\mathbf{F}_q$ by enumerating q-Weil polynomials; such was the origin of the Abelian Varieties over Finite Fields section of the LMFDB (Dupuy-Kedlaya-Roe-Vincent).

Cont.

The Torelli morphism is an embedding. Define the *open Torelli locus* $\mathcal{T}_g^\circ$ to be the image of $\mathcal{T}$. For $g = 1, 2, 3$, $\mathcal{T}_g^\circ$ is dense in $\mathcal{A}_g$.

For $g \geq 2$, $\mathcal{M}_g$ is a smooth Deligne-Mumford (DM) stack of relative dimension $3g - 3$ over $\mathbb{Z}$, whereas $\mathcal{A}_g$ has relative dimension $g(g+1)/2$.

Question 1: Which p.p.a.v. arises as the Jacobian of a curve?

Question 2: Taking the numerator of the zeta function defines a map $\mathcal{M}_g \to \{\text{degree } 2g \ q\text{-Weil polynomials}\}$. How does one compute preimages of this map?

Most generally, questions of this flavor can only be answered by having a (partial) census of curves over finite fields.

Motivation 2.1: Filling missing data in the LMFDB

There are 235,942 isogeny classes of 5 dimensional abelian varieties over $\mathbf{F}_3$ for which we do not know whether they contain Jacobians.

Abelian variety isogeny class 5.3.ae_d_c_t_acq over $\mathbb{F}_3$

Invariants

Base field:	$\mathbb{F}_3$
Dimension:	5
L-polynomial:	$(1 - x - 2x^2 - 3x^3 + 9x^4)(1 - 3x + 2x^2 + x^3 + 6x^4 - 27x^5 + 27x^6)$
	$1 - 4x + 3x^2 + 2x^3 + 19x^4 - 68x^5 + 57x^6 + 18x^7 + 81x^8 - 324x^9 + 243x^{10}$
Frobenius angles:	$\pm 0.0487092080293, \pm 0.0734519173280, \pm 0.286461628719, \pm 0.740118583995, \pm 0.767045968880$
Angle rank:	4 (numerical)

This isogeny class is not simple, primitive, ordinary, and not supersingular. It is principally polarizable.

Jacobians and polarizations

This isogeny class is principally polarizable, but it is unknown whether it contains a Jacobian.

We'd like to thank David Roe for adding our data of 6-dimensional Jacobians over $\mathbf{F}_2$ to the LMFDB.

Motivation 3: Point-counting on Moduli

For k a field, we have a bijection of sets

$$\mathcal{M}_g(k) \longleftrightarrow \{\text{isomorphism classes of curves of genus } g \text{ over } k\}.$$

In particular, if k is a finite field, $\mathcal{M}_g(k)$ is also finite; for $g \geq 2$, we let

$$\#\mathcal{M}_g(k) := \sum_{C \in \mathcal{M}_g(k)} \frac{1}{\# \operatorname{Aut}_k(C)}.$$

More surprisingly, $\#\mathcal{M}_g(k) \in \mathbb{Z}$.

Moduli Space of Curves

For X a DM stack, we have a Grothendieck-Lefschetz trace formula due to Behrend:

$$\#X(\mathbf{F}_q) = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Tr}(\operatorname{Frob}_q^* \,|H_c^i(X, \mathbf{Q}_\ell))$$

for any $(\ell, q) = 1$.
When there exists a polynomial $P(q) \in \mathbb{Z}[q]$ such that $\#X(\mathbf{F}_q) = P(q)$ for all prime powers q, we say X has a **polynomial point count**.

Canning–Larson–Payne–Willwacher: The DM stack $\mathcal{M}_g$ has a polynomial point count if and only if $g \leq 8$.
The polynomials $P(q)$ is known explicitly for $g \leq 6$.

Cont.

We have

- $\#\mathcal{M}_2(\mathbf{F}_q) = q^3$, and $\#\mathcal{M}_3(\mathbf{F}_q) = q^6 + q^5 + 1$ (Looijenga).
- $\#\mathcal{M}_4(\mathbf{F}_q) = q^9 + q^8 + q^7 - q^6$ (Bergström–Faber–Payne).
- $\#\mathcal{M}_5(\mathbf{F}_q) = q^{12} + q^{11} + q^{10} - q^8 + 1$
- $\#\mathcal{M}_6(\mathbf{F}_q) = q^{15} + q^{14} + 2q^{13} + q^{12} - q^{10} + q^3 - 1$ (Bergstrom–Canning–Petersen–Schmitt)

These polynomial point-count formulas provide an *independent* consistency check of the completeness of our computational result.

Cont.

The polynomial point-count formulas for $\mathcal{M}_7$ and $\mathcal{M}_8$ are currently not known.

For example, to derive the formula for $\#\mathcal{M}_7$, one needs to analyze $\overline{\mathcal{M}}_7 \backslash \mathcal{M}_7$ and obtain the polynomial point count for, e.g., $\mathcal{M}_{6,1}, \mathcal{M}_{6,2}/S_2,$ and $\mathcal{M}_{5,4}/S_4$.

As a byproduct of our table of genus 6 curves over $\mathbf{F}_2$, we have obtained counts for, e.g., $\#\mathcal{M}_{6,1}(\mathbf{F}_2), \#\mathcal{M}_{6,2}(\mathbf{F}_2)/S_2, \#\mathcal{M}_{6,3}(\mathbf{F}_2)$. This provides one additional linear constraint for computing explicit formulas for $\mathcal{M}_{6,1}, \mathcal{M}_{6,2}/S_2$.

Remark: We are currently working on making a census of genus 5 curves over $\mathbf{F}_3$ with the hope of helping Canning and his collaborators with computing the polynomial point count for $\mathcal{M}_{5,4}/S_4$ and subsequently $\mathcal{M}_7$.

The Census Problem

Probelm: Given q and g, find one curve representing each isomorphism class in $\mathcal{M}_g(\mathbf{F}_q)$.

General strategy:

1. Find some covering set S' for $\mathcal{M}_g(\mathbf{F}_q)$.
2. Use MAGMA to sieve out redundancies in S', and denote the resulting set by S.

We have discovered a few bugs in MAGMA that have all been fixed in the latest release.

We obtain a **lower bound**

$$\#S := \sum_{C \in S} \frac{1}{\#\operatorname{Aut}_k(C)} \leq \#\mathcal{M}_g(\mathbf{F}_q).$$

In general, it is hard to determine whether we have enumerated all of $\mathcal{M}_g(\mathbf{F}_q)$ unless we have an independent upper bound for $\#\mathcal{M}_g(\mathbf{F}_q)$.

Overview of Past and Current Work

There is a complete census in the following cases:

- $g \leq 3$ and various q: Sutherland (and hyperelliptic case by Howe).
- $g = 4, q = 2$: Xarles.
- $g = 4, q = 3$: Bergström–Faber–Payne
- $g = 5, q = 2$: Dragutinović.
- $g = 6, q = 2$: H.–Kedlaya–Lau.

All data can be found in the Abelian Varieties over Finite Fields section of the LMFDB.

We are currently working on the following two cases:

- $g = 5, q = 3$ (hyperelliptic case was handled by Howe), and
- $g = 7, q = 2$.

The census in the non-generic strata in both cases is completed.

Our Results

We have known that there are $164,937$ isogeny classes of abelian varieties of dimension 6 over $\mathbf{F}_2$.

Our new results

- $38,327$ out of the $164,937$ isogeny classes contain the Jacobian of curves of genus 6.
- All 20 of the possible Newton polygons appear, and the maximum number of Jacobians in a single isogeny class is 20.
- There are $72,227$ isomorphism classes of curves of genus 6 over $\mathbf{F}_2$.
- We have $\#\mathcal{M}_6(\mathbf{F}_2) = 68,615 = 2^{15} + 2^{14} + 2 \cdot 2^{13} + 2^{12} - 2^{10} + 2^3 - 1.$

How to make the census: Classification of Curves

We briefly recall some algebraic geometry of curves.
For $g \geq 2$, the canonical divisor defines a map $\phi_K : C \to \mathbf{P}^{g-1}$, which is an embedding if and only if C is not hyperelliptic, in which case we refer to ϕ_K as the **canonical embedding**.

Petri: $\phi_K(C)$ is cut out by quadrics unless C is trigonal or $g = 6$ and C is a smooth plane quintic.

For $g \leq 5$, we have the usual classification of curves:

- $g = 2$: C is hyperelliptic.
- $g = 3$: C is hyperelliptic or a plane quartic
- $g = 4$: C is hyperelliptic, or a complete intersection of a quadric and a cubic hypersurface in $\mathbf{P}^3$.
- $g = 5$: C is hyperelliptic, trigonal, or a complete intersection of three quadrics in $\mathbf{P}^4$.

Notably, this classification holds in any characteristic and for finite fields.

Brill-Noether Stratification of Genus 6 Curves

From Petri's theorem plus work of Mukai, we have that for $g = 6$, C is exactly one of the following:

- Hyperelliptic.
- Bielliptic.
- Smooth plane quintic in $\mathbf{P}^2_k$.
- Trigonal of Maroni invariant 0: a (3,4) in $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$.
- Trigonal of Maroni invariant 2: a $(2,1) \cap (1,3)$ in $\mathbf{P}^1_k \times_k \mathbf{P}^2_k$.
- Brill-Noether-general: a $(1)^4 \cap (2) \cap \mathrm{Gr}(2,5)$ in $\mathbf{P}^9_k$.

Remark: Work of Kedlaya descends Mukai's result from $\overline{k}$ to k.

Beyond Genus 6 and Beyond Curves?

- There exists a similar Brill-Noether stratification for $g = 7$, which is what we are using in our on-going work.
- In genus 7, the *rational* and *irrational* g^2_6 strata both have non-integral stacky counts, but together they add up to an integer.
- It is theoretically possible to also do a census in $g = 8, 9$ over $\mathbf{F}_2$ analogously, as Mukai has similar descriptions of Brill-Noether generic curves in these genera. This is not as computationally tractable, and one probably needs a better reason to do a census in these cases.
- It is also of interest to have a census of quartic K3 surfaces and cubic fourfolds in low characteristics (Kedlaya–Sutherland, Auel–Kulkarni–Petok–Weinbaum).
- What other mathematical problems can be solved or at least benefit from having a census of varieties in low characteristics?